# Lecture 12 - State Feedback

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*Theme: Using the state for control.*

# Introduction

- The simple design method becomes very cumbersome for systems of high order
- The PID controller predicts based on linear extrapolation
- Can we do something better
- Where to look for inspiration?
- *State* a number of variables that summarizes the past that is useful for prediction
- The future behavior can be predicted from the state
- Can we find a general controller?
- The state is an ideal basis for control
- We will focus on the predictive part, reference values and integral action will be dealt with later

# State Feedback

Assume that the process is described by

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

The general linear controller is: $u = -Lx + l_r r$

The closed loop system then becomes

$$\frac{dx}{dt} = Ax + Bu = Ax + B(-Lx + l_r r) = (A - BL)x + Bl_r r$$

The closed loop system has the characteristic equation

$$\det(sI - A + BL) = 0$$

Can we choose L so that this equation has specified poles?

# The Mathematical Tool - Matrices

Please refresh your knowledge of matrices.

- What is a matrix
- Matrix algebra: addition, multiplication, notice $AB \neq BA$ (a very useful property), transpose
- Matlab
- Linear equations and inverses $Ax = B$, $x = A^{-1}B$
- Eigenvalues and eigenvectors

$$\begin{aligned}Ae &= \lambda e\\ (A - \lambda I)e &= 0\\ \det A - \lambda I &= 0\end{aligned}$$

## Simple Examples

Consider the following matrices and their transpose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

- Which matrices can be added?
- Which matrices can be multiplied?

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Characteristic polynomial $\det(sI - A)$

$$\det \begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix} = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}$$

## Example - The Car

Process model

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = Cx = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Control law

$$u = -l_1 x_1 - l_2 x_2 + l_r r$$

Closed loop system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -l_1 & -l_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ l_r \end{pmatrix} r$$

Characteristic equation

$$s^2 + l_1 s + l_2 = 0$$

Choosing

$$l_1 = 2\zeta\omega_0, \quad l_2 = \omega_0^2$$

gives the characteristic polynomial $s^2 + 2\zeta\omega_0 s + \omega_0^2$.
The closed loop system is

$$Y(s) = \frac{l_r}{s^2 + 2\zeta\omega_0 s + \omega_0^2} R(s)$$

Choosing $l_r = \omega_0^2$ gives the correct steady state value.

$$Y(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} R(s)$$

## Example

Try to do same thing for

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = Cx = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Control law $u = -l_1 x_1 - l_2 x_2 + l_r r$. Closed loop system

$$\frac{dx}{dt} = \begin{pmatrix} -l_1 & 1 - l_2 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} l_r \\ 0 \end{pmatrix} r$$

Characteristic equation

$$s(s + l_1) = s^2 + l_1 s = 0$$

We cannot obtain a desired characteristic polynomial. Why? The control signal does not influence the second state! Some conditions are required!

## Controllability

$$\frac{dx}{dt} = Ax + Bu$$

Find control signal that moves the system from $x(0) = a$ to $x(t) = b$. Hence

$$b = x(t) = e^{At}a + \int_0^t e^{A(t-s)} Bu(s) ds$$

This implies

$$\int_0^t e^{-As} Bu(s) ds = -a + be^{-At}$$

## Controllability ...

$$\int_0^t e^{-As}Bu(s)ds = -a + be^{-At}$$

It follows from Cayley-Hamiltons theorem (a matrix satisfies it own characteristic equation) that

$$\sum_{k=1}^{n} A^{k-1}B \int_0^t \alpha_k(s)u(s)ds = -a + be^{-At}$$

Left side a combination of vectors $B$, $AB$, $\ldots$, $A^{n-1}B$. Equation can be solved if the matrix

$$W_c = \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}$$

can be inverted. **Controllability!**

## Examples

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

we have

$$W_c = (B, AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{Controllable!}$$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

we have

$$W_c = (B, AB) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Not Controllable!}$$

## A Special Case

The system

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \ldots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

is controllable

$$W_c = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & \ldots & \\ 0 & 1 & -a_1 & \ldots & \\ 0 & 0 & 1 & \ldots & \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

## The Characteristic Equation

The system

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

has the characteristic polynomial

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n$$

## A Special Case which is Easy

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

The feedback $u = -l_1 z_1 - l_2 z_2 - \dots - l_n z_n + l_r r$ gives

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 - l_1 & -a_2 - l_2 & \dots & -a_n - l_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} l_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r$$

## A Special Case

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

The feedback

$$u = -l_1 z_1 - l_2 z_2 - \dots - l_n z_n + l_r r$$
$$l_1 = p_1 - a_1, \quad l_2 = p_2 - a_2, \dots, l_n = p_n - a_n$$

gives the characteristic polynomial

$$s^n + p_1 s^{n-1} + p_2 s^{n-2} + \dots + p_n$$

## The General Case

Problem solved for special case. Can we transform a given system to this case? Start with

$$\frac{dx}{dt} = Ax + Bu$$

with controllability matrix

$$W_c = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

Change coordinates $z = Tx$

$$\frac{dz}{dt} = T\frac{dx}{dt} = TAx + TBu = TAT^{-1}z + TBu = \tilde{A}z + \tilde{B}u$$

This system has the controllability matrix

$$\tilde{W}_c = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix} = T\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = TW_c$$

## The General Case

A system

$$\frac{dx}{dt} = Ax + Bu$$

which is controllable can be transformed to

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

It is easy to find a state feedback $u = -\tilde{L}z$ for this system which gives a specified characteristic equation.

$$\tilde{L} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \dots & p_n - a_n \end{pmatrix}$$

## An Algorithm

Transform the system $\frac{dx}{dt} = Ax + Bu$ to controllable canonical form $\frac{dz}{dt} = \tilde{A}z + \tilde{B}u$ using the transformation $z = Tx = \tilde{W}_c W_c^{-1} x$.

Feedback for transformed system $u = -\tilde{z}$ where

$$\tilde{L} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \dots & p_n - a_n \end{pmatrix}$$

We have $u = -Lx = -LT^{-1}z = -\tilde{L}z$, hence

$$L = \tilde{L}T = \tilde{L}\tilde{W}_c W_c^{-1}$$

## How to Compute the Feedback Gain

- For simple problems we just write the characteristic equation, compute the characteristic polynomial and match coefficients. (Typical exam problem)
- Ackermann's formula summarizes the derivation given above

$$\begin{aligned} L &= \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}^{-1} P(A) \\ e_n &= \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} \\ W_c &= \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \\ L &= e_n W_c^{-1} P(A) \end{aligned}$$

  where $P(s)$ is the desired characteristic polynomial This formula does not have good numerical properties
- There are efficient numerical routines `acker` and `place` in Matlab

## The Inverted Pendulum

$$\frac{dx}{dt} = Ax + Bu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

The system is controllable. The control law

$$u = -l_1 x_1 - l_2 x_2$$

gives the closed loop system

$$\frac{dx}{dt} = (A - BL)x = \begin{pmatrix} 0 & 1 \\ 1 - l_1 & -l_2 \end{pmatrix} x$$

The characteristic polynomial is

$$\det(sI - A + BL) = \det \begin{pmatrix} s & -1 \\ l_1 - 1 & s + l_2 \end{pmatrix} = s^2 + l_2 s + l_1 - 1$$

## Using Matlab

There are several command is Matlab for design of state feedback. The command `help toolbox/control` tells you what command are available

```
ACKER  Pole placement using Ackermann's formula.
   K = ACKER(A,B,P)  calculates the feedback gain
   matrix K such that the single input system
           .
           x = Ax + Bu

   with a feedback law of  u = -Kx  has closed loop
   poles at the values specified in vector P,
   i.e.,  P = eig(A-B*K).  See also  PLACE.
```

## Integral Action - A Fix

Process model

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Introduce the integral of the error as an extra state

$$I = \int (r - y)dt, \qquad \frac{dI}{dt} = r - y = r - Cx$$

Form the augmented system

$$\frac{d}{dt}\begin{pmatrix} x \\ I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} x \\ I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

Looks as original system with one extra state!

## Integral Action - A Fix ...

$$\frac{d}{dt}\begin{pmatrix} x \\ I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} x \\ I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

Control law

$$u = -\begin{pmatrix} L & l_I \end{pmatrix}\begin{pmatrix} x \\ I \end{pmatrix} + l_r r$$

Closed loop system

$$\frac{d}{dt}\begin{pmatrix} x \\ I \end{pmatrix} = \begin{pmatrix} A - BL & -BL_i \\ -C & 0 \end{pmatrix}\begin{pmatrix} x \\ I \end{pmatrix} + \begin{pmatrix} Bl_r \\ 1 \end{pmatrix} r$$

If closed loop system stable there is no steady state error

$$Cx_0 = r$$

## Is the Augmented System Controllable?

$$\frac{d}{dt}\begin{pmatrix} x \\ I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} x \\ I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

Controllability matrix

$$W_c = \begin{pmatrix} B & AB & \dots & A^n B \\ 0 & -CB & \dots & -CA^{n-1}B \end{pmatrix}$$

The system is controllable if the original system is controllable and if $CA^{n-1}B \neq 0$. Controllability of cascaded systems.

## Summary

- A nice way to use process dynamics to predict
- A nice alternative to pole placement
- Actually works for systems with many inputs and outputs
- Nice computational methods matrix calculations
- Works for high order systems
- The idea of augmentation
- Integral action still ad hoc will be dealt with later
- Reference values will be done later
- The control problem is thus easy if all state variables are measured
- What to do if all states are not measured? (Stay tuned for next lecture!)