Lectures 5 - Frequency Response

K. J. Åström

1. Introduction
2. Frequency response
3. Nyquist curves and stability
4. Bode plots
5. The concept of minimum phase
6. Summary

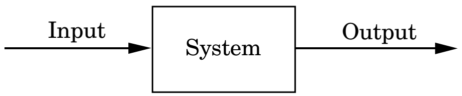
Theme: The input-output view of dynamical systems. Fourier's idea: sinusoidal inputs. Graphical representations of frequency response. Nyquist and Bode plots. The concepts of minimum and non-minimum phase.

Introduction

- How to describe dynamics
- The Giant Table - One way to view dynamics
- The heritage of electrical engineering
- Fits block diagrams
- Makes it possible to deal with systems having a large number of states.

  Bode: Electronic feedback amplifiers are much more complex than steam engines, systems have orders 50-100 rather than 2-4. (A lot of capacitors!)
- Synonyms: input-output models, external descriptions, black boxes.
- Experimental determination of dynamics

The Idea of Black Boxes



Consider a system as a black box. Forget about the internal details and focus on the input-output behavior of the system.

- Make a Giant Table over all pairs of inputs and outputs
- A stroke of luck: A few entries suffice for linear time-invariant systems
- Steps (step response), reaction curve
- Impulses (impulse response)
- Sinusoids (Fourier, frequency responses)

Linear Time Invariant Systems

Linearity: Let $(u_1,\ y_1)$ och $(u_2,\ y_2)$ be input-output pairs. Then $(au_1 + bu_2,\ ay_1 + ay_2)$ is also an input-output pair, **superposition**.

Time-invariance: Let δ_t be an operator that shifts a signal t time units forward and let $(u,\ y)$ be an input-output pair. A system is **time-invariant** if $(\delta_t u,\ \delta_t y)$ is also an input-output pair.

Consequences: The table can be simplified drastically for linear time-invariant systems. It is enough to give one pair only. If all initial conditions are zero the output for all inputs is given by the transfer function of the system.

$$y(t) = \int_0^t g(t-s)u(s)ds, \quad Y(s) = G(s)U(s)$$

Frequency Response

- Fourier's idea: An LTI system is completely determined by its response to sinusoidal signals.
- Implications for the Table
- Transmission of sinusoids given by $G(i\omega)$
- The $j\omega$-method
- Analytic continuation
- The transfer function $G(s)$ uniquely given by its values on the imaginary axis
- Experimental determination of the frequency response

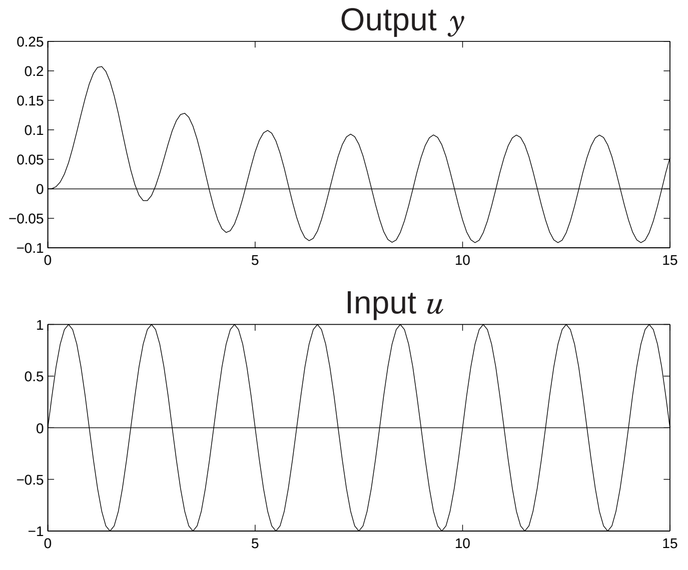
Interpretation of Frequency Responses

The complex number $G(i\omega)$ tells how a sinusoid propagates through the system *in steady state*. If the input is $u(t) = \sin \omega t$, then the output is

$$y(t) = |G(i\omega)| \sin\left(\omega t + \arg G(i\omega)\right)$$

The number $|G(i\omega)|$ is called gain ratio or simply gain and the number $\arg G(i\omega)$ is called phase of the transfer function.

Notice Steady State Responses



Proof

Consider a system with transfer function $G(s)$ having distinct stable poles α_k. Let the input be $u(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$. The Laplace transform of the input is $U(s) = \dfrac{1}{s - i\omega_0}$. The Laplace transform of the output is

$$Y(s) = G(s)\frac{1}{s - i\omega_0} = G(i\omega_0)\frac{1}{s - i\omega_0} + \sum \frac{R_k}{s - \alpha_k}\frac{1}{\alpha_k - i\omega_0}$$

This corresponds to the time function

$$y(t) = G(i\omega_0)e^{i\omega_0 t} + \sum \frac{R_k}{\alpha_k - i\omega_0}e^{\alpha_k t}$$

Since all α_k are negative the first terms go to zero and $y(t) \to G(i\omega_0)e^{i\omega_0 t}$ for large t.

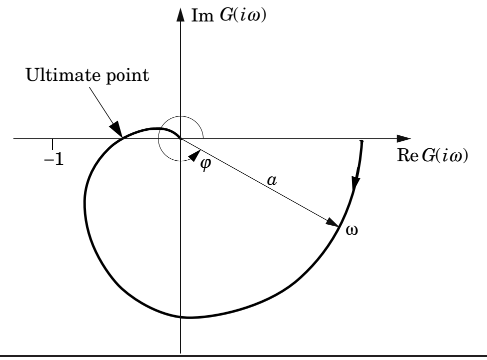
Nyquist’s Stability Theorem

- So far focus on the characteristic equation
- Difficult to see how the characteristic equation is influenced by controller.
- How to change the controller to make unstable system stable?
- Nyquist’s results was a major paradigm shift
- Investigate propagation of sinusoids around the loop
- Based on transfer functions (Always useful to have different ways to look at a problem!)
- Strong practical implications
- Possibilities to introduce stability margins.

The Nyquist Curve

H. Nyquist was born in Sweden. Emigrated to the US and made his career at Bell Laboratories. The Nyquist curve represents the transfer function by showing a graph of the complex number $G(i\omega)$ as a function of frequency.

Physical interpretation!



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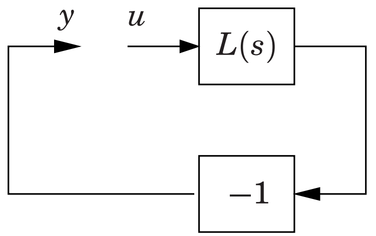
Conditions for Oscillations



Cut the loop. Let u be a sinusoid. If y is a sinusoid with the same amplitude and phase, then the loop can be closed and the oscillation will be maintained. The condition for this is

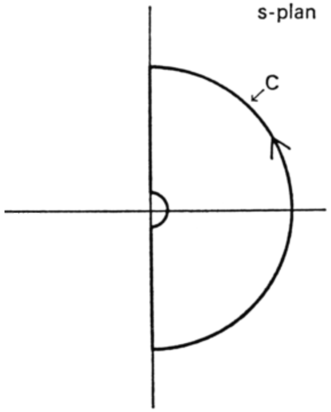
$$L(i\omega) = -1$$

where $L = PC$ is the loop transfer function. The condition implies that the Nyquist curve of L goes through the point -1 (the critical point)!

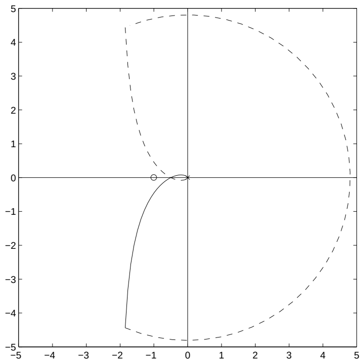
The Complete Nyquist Curve

The complete Nyquist curve is image of the contour C under the map $L(s)$

s-plane



$L(s)$-plane



Nyquist's Stability Theorem

In the special case when the loop transfer function does not have poles in the right half plane the closed loop system is stable if the complete Nyquist curve does not encircle the critical point.

There are more general result which also covers the case where the loop transfer function has poles in the RHP.

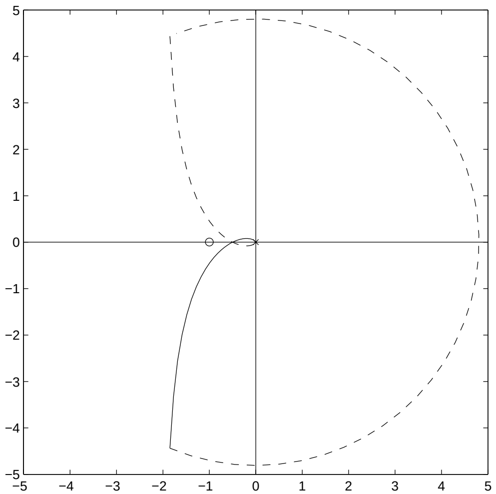
Use pencil and string to determine encirclements in tricky situations.

There is some really beautiful mathematics behind this!

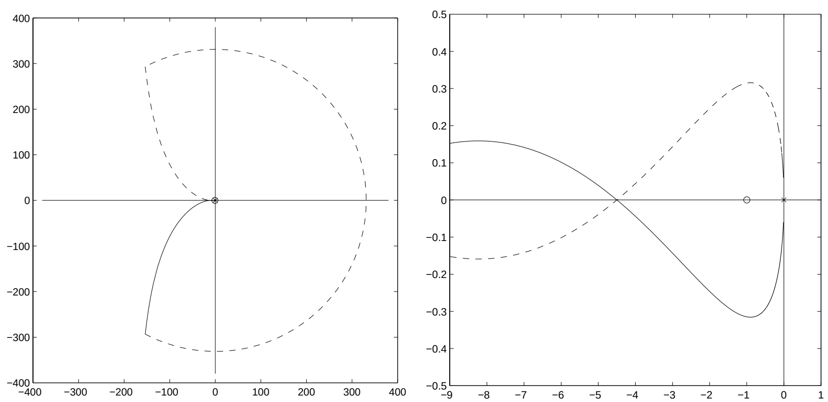
Example

$$L(s) = \frac{1}{s(s+1)^2}$$

L no poles in RHP.
No encirclements
Stable



Conditional Stability $L(s) = \dfrac{3(s+1)^2}{s(s+6)^2}$



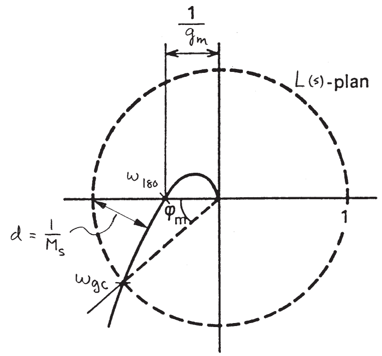
Loop transfer function has no zeros in the RHP. No encirclements. Closed loop system stable. Notice counter-intuitive.

Stability Margins

Stability as it has been defined is black and white. In practice there is often a need to have concepts like degrees of stability. Some useful concepts are

- Gain margin g_m (2-6)
- Phase margin φ_m (45°-60°)
- Shortest distance to critical point d (0.5-0.8)

Notice d is safe but only one of g_m or φ_m is not!



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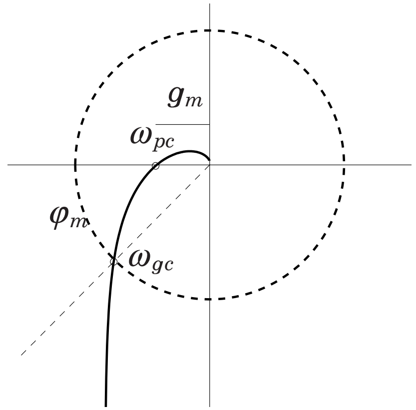
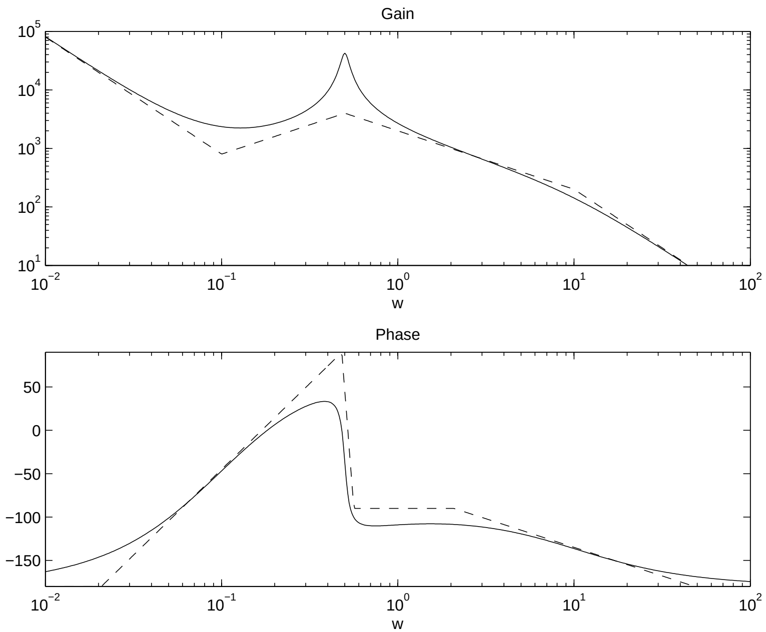


The Bode Plot

Bode was a researcher at Bell Laboratories. The complex function $G(s)$ can also be represented by two graphs, one for the gain curve, $|G(i\omega)|$, and one for the phase curve, $\arg G(i\omega)$. It is tradition to use logarithmic scales for frequency and gain and linear scales for the phase. A nice consequence of this is that the curves have asymptotes that are very easy to obtain! The gain curve is sometimes calibrated in dB (20 dB equals a factor of 10).

- Making the plot
- Interpreting the plot

  Extends the intuitive argument that small s correspond to large t to all frequencies. Gives a quick view of the behavior of the system for all frequencies.



Matlab

There are excellent tools in Matlab for drawing Nyquist curves and Bode plots in the Control System Toolbox in Matlab.

```
NYQUIST  Nyquist frequency response of LTI models.
BODE  Bode frequency response of LTI models.
```

Use the help function to figure out how the commands work. Try a few examples.

Notice that Matlab uses decibel (dB) as a standard unit for amplitude, where 20 dB corresponds to a factor 10. If you think dB is an unnecessary complication it is easy to make your own plots with other units.

Sketching Bode Plots

It is easy to sketch Bode plots *because with the right scales they have linear asymptotes*. This is useful in order to get a quick estimate of the behavior of a system. It is also a good way to check numerical calculations.

Consider first a transfer function which is a polynomial $G(s) = B(s)/A(s)$. We have

$$\log G(s) = \log B(s) - \log A(s)$$

Since a polynomial is a product of terms:

$$s, \quad s + a, \quad s^2 + 2\zeta a s + a^2$$

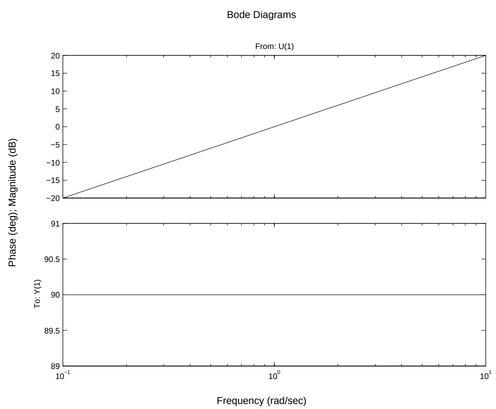
it suffices to be able to sketch Bode diagrams for these terms. The Bode plot of a complex system is then obtained by composition.

Differentiator

$$G(s) = s$$

We have $G(i\omega) = i\omega$

$$\log |G(i\omega)| = \log \omega$$
$$\arg G(i\omega) = \pi/2$$



Matlab:

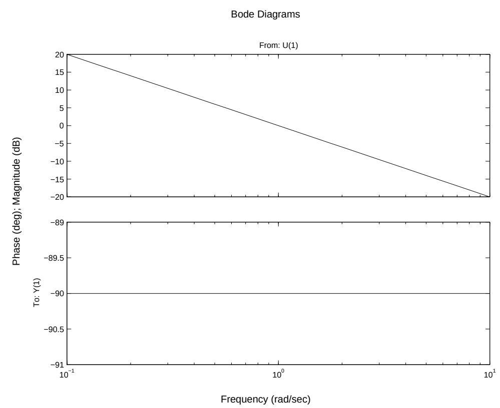
```
sys=tf([1 0],1)
bode(sys,{0.1,1})
```

Integrator

$$G(s) = \frac{1}{s}$$

We have $G(i\omega) = -i\frac{1}{\omega}$

$$\log |G(i\omega)| = -\log \omega$$
$$\arg G(i\omega) = -\pi/2$$



Matlab:

```
sys=tf(1,[1 0])
bode(sys,{0.1,1})
```

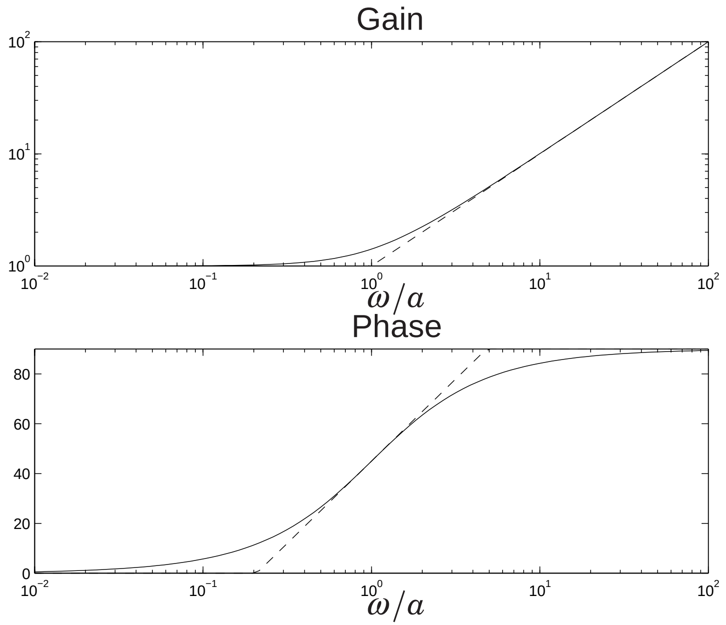
First Order System $G(s) = s + a$

We have $G(i\omega) = a + i\omega$, hence $|G(i\omega)| = \sqrt{\omega^2 + a^2}$ and $\arg G(i\omega) = \arctan \omega/a$, hence

$$\log|G(i\omega)| = \frac{1}{2}\log(\omega^2 + a^2), \quad \arg G(i\omega) = \arctan \omega/a$$

$$\log|G(i\omega)| \approx \begin{cases} \log a & \textit{if } \omega << a, \\ \log a + \log\sqrt{2} & \textit{if } \omega = a, \\ \log \omega & \textit{if } \omega >> a \end{cases},$$

$$\arg G(i\omega) \approx \begin{cases} 0 & \textit{if } \omega << a, \\ \dfrac{\pi}{4} + \dfrac{1}{2}\log\dfrac{\omega}{a} & \textit{if } \omega \approx a, \\ \dfrac{\pi}{2} & \textit{if } \omega >> a \end{cases},$$

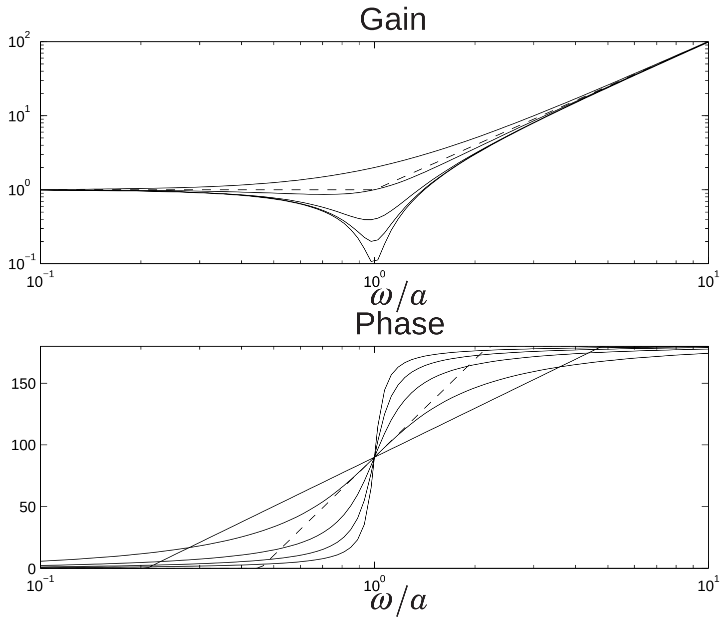


Second Order System $G(s) = s^2 + 2\zeta a + a^2$

$$G(i\omega) = a^2 - \omega^2 + 2i\zeta a\omega$$
$$\log|G(i\omega)| = \frac{1}{2}\log(\omega^4 + 2a^2\omega^2(2\zeta^2 - 1) + a^4)$$
$$\arg G(i\omega) = \arctan 2\zeta a\omega/(a^2 - \omega^2)$$

$$\log|G(i\omega)| \approx \begin{cases} 2\log a & \textit{if } \omega << a, \\ 2\log a + \log 2\zeta & \textit{if } \omega = a, \\ 2\log \omega & \textit{if } \omega >> a \end{cases},$$

$$\arg G(i\omega) \approx \begin{cases} 0 & \textit{if } \omega << a, \\ \dfrac{\pi}{2} + \dfrac{\omega - a}{a\zeta} & \textit{if } \omega = a, \\ \pi & \textit{if } \omega >> a \end{cases},$$

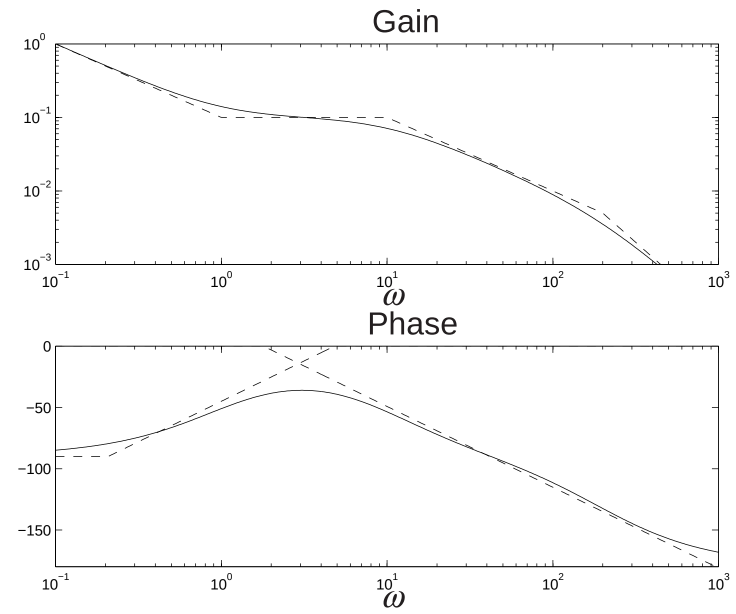


Sketching Bode Plots

System

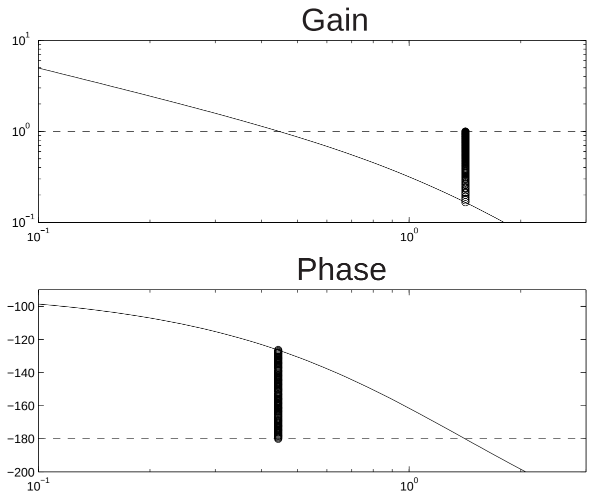
$$G(s) = \frac{200(s+1)}{s(s+10)(s+200)} = \frac{1+s}{10s(1+0.1s)(1+0.01s)}$$

- Determine break points (poles and zeros) sort them in increasing frequency
- Start with low frequencies ($G(s) \approx \dfrac{1}{10s}$)
- Draw the low frequency asymptote
- Go over all break points and note the slope changes
- A crude sketch of the phase curve is obtained by using the relation that, for systems with no RHP poles or zeros, one unit slope corresponds to a phase of 90°



Gain and Phase Margins in Bode Plots

Make a Bode plot of the loop transfer function $L = PC$



The Concept of Minimum Phase

A system is called a *minimum phase* system if all its poles and zeros are in the left half plane. Minimum phase systems are easy to control.

For minimum phase systems the phase curve is given by the gain curve and vice versa. An approximate relation is

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega},$$

A slope of one for the gain curve corresponds to 90° phase. The exact relations are called Bodes relations. Systems that are not minimum phase are called *non-minimum phase*. The property of non-minimum phase imposes severe limitations to what can be achieved by control.

Bode's Relations between Amplitude and Phase

let $G(s)$ be a transfer function with all poles and zeros in the left half plane. Introduce

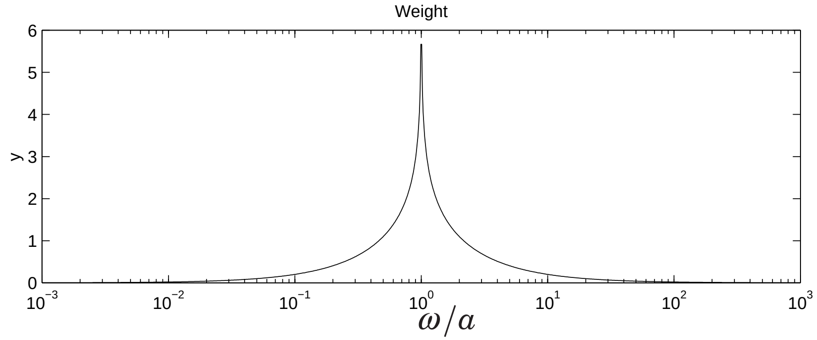
$$\begin{aligned}\arg G(i\omega_0) &= \frac{2\omega_0}{\pi}\int_0^\infty \frac{\log|G(i\omega)| - \log|G(i\omega_0)|}{\omega^2-\omega_0^2}d\omega \\ &= \frac{1}{\pi}\int_0^\infty \frac{d\log|G(i\omega)|}{d\log\omega}\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d\omega \approx \frac{\pi}{2}\frac{d\log|G(i\omega)|}{d\log\omega}\end{aligned}$$

$$\begin{aligned}\frac{\log|G(i\omega)|}{\log|G(i\omega_0)|} &= -\frac{2\omega_0^2}{\pi}\int_0^\infty \frac{\omega^{-1}\arg G(i\omega) - \omega_0^{-1}\arg G(i\omega_0)}{\omega^2-\omega_0^2}d\omega \\ &= -\frac{2\omega_0^2}{\pi}\int_0^\infty \frac{d\left(\omega^{-1}\arg G(i\omega)\right)}{d\omega}\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d\omega\end{aligned}$$

The Smoothing Kernel

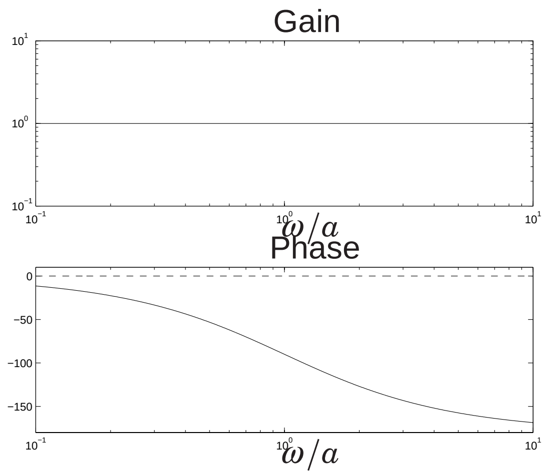
We have

$$\int_0^\infty \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|d\omega = \frac{\pi^2}{2}$$



RHP Zero $G(s) = \dfrac{a-s}{a+s}$

We have $|G(i\omega)| = 1$ and $\arg G(i\omega) = -2\arctan\dfrac{\omega}{a}$



Step Response for System with RHP Zero

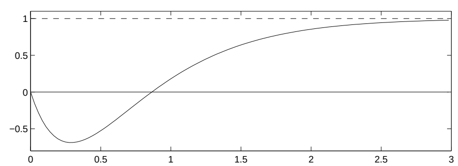
Laplace transform of the step response h is

$$\frac{G(s)}{s} = \int_0^\infty e^{-st}h(t)dt$$

If G has a RHP zero at $s = \alpha > 0$ we have

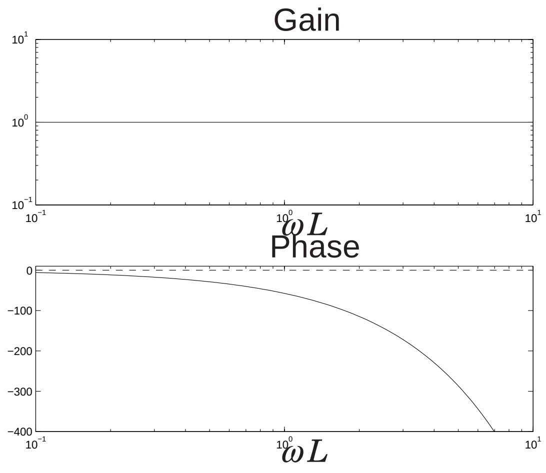
$$0 = \frac{G(\alpha)}{\alpha} = \int_0^\infty e^{-\alpha t}h(t)dt$$

Since the integral is zero the step response must assume both positive and negative values



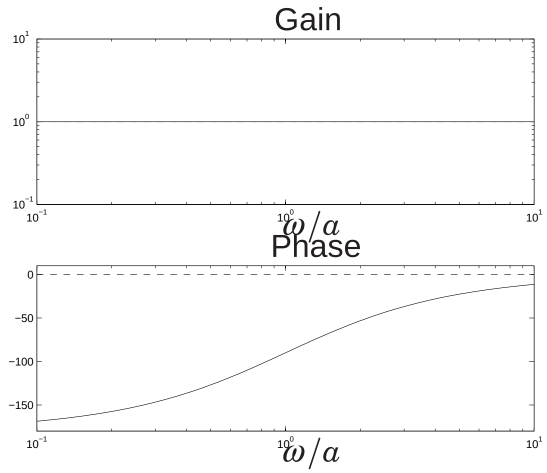
Time Delay $G(s) = e^{-sL}$

We have $|G(i\omega)| = 1$ and $\arg G(i\omega) = \omega L$



RHP Pole $G(s) = \frac{s-a}{s+a}$

We have $|G(i\omega)| = 1$ and $\arg G(i\omega) = -2\arctan\frac{b}{\omega} = -\pi + 2\arctan\frac{\omega}{b}$



Airplanes

The transfer function from elevon to height of an airplane with elevons in the rear are always non-minimum phase. The Wright brothers avoided this by an elevon in front.

Modern fighter planes have canards in the front and even jet thrusters to avoid the problem.

X-29 is an experimental aircraft. In one operating condition the system is approximately described by the transfer function

$$G_{nmp}(s) = \frac{s-26}{s-6}$$

One pole and one zero in the right half plane. This plane is difficult to control well.

Power Systems

- Level dynamics in boilers is non-minimum phase because of the shrink and swell phenomena
- The transfer function from tube opening to power is for a hydro electric power system

$$\frac{P(s)}{A(s)} = \frac{P_0}{A_0}\frac{1-2sT}{1+sT}$$

Bicycles

- Front wheel steering

$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ} \frac{s + \dfrac{V_0}{a}}{s^2 - \dfrac{mg\ell}{J}}$$

  Non-minimum phase because of the right half plane pole

- Rear wheel steering

$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ} \frac{-s + \dfrac{V_0}{a}}{s^2 - \dfrac{mg\ell}{J}}$$

  Both poles and zeros in the right half plane

Summary

- The input-output view of dynamical systems
- Describe a system by making a table of all input-output pairs
- Fourier's idea: look at steady state propagation of sinusoids
- Frequency response $G(i\omega)$
- Graphical representations, very useful for intuition
- Bode and Nyquist plots
- The concept of non-minimum phase
- Systems which are non-minimum phase have severe performance limitations