Lecture 11 Stability

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1. Introduction
2. Stability concepts
3. Linear Systems
4. Nonlinear Systems
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Theme: Nonlinear systems have much richer behavior. Stability of nonlinear systems is determined by their local linear approximation. The main reason why linera control theory works.

Introduction

- Risk for instability is the main drawback of feedback
- Instabilities were frequently encountered in early use of feedback
- Created a pressing need for theory
  - Understand mechanisms that create instability
  - Criteria for stability - beginning of control theory
  - Ways to avoid instabilities
- Linear systems are very special
- Stability problems appear in many different contexts, buckling, critical speed, oscillations in combustion, etc.

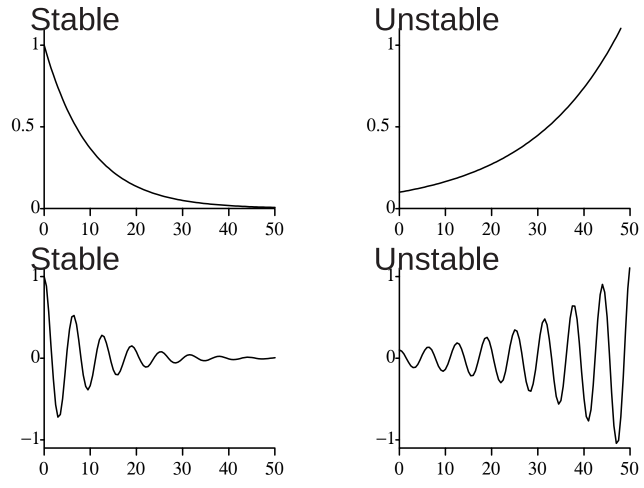
Maxwell's Observations 1868

"It will be seen that the motion of a machine with its governor consist in general of a uniform motion, combined with a disturbance which may be expressed as the sum of several component motions. These components may be of four different kind:

1. The disturbance may continually increase.
2. It may continually diminish.
3. It may be an oscillation of continually increasing amplitude.
4. It may be an oscillation of continually decreasing amplitude.

The first and third cases are evidently inconsistent with the stability of the motion: and the second and fourth alone are admissible in a good governor. *Stability is mathematically equivalent to the condition that all roots of an algebraic equation (the characteristic equation) are in the left half plane.*

Maxwell's Qualitative Observation



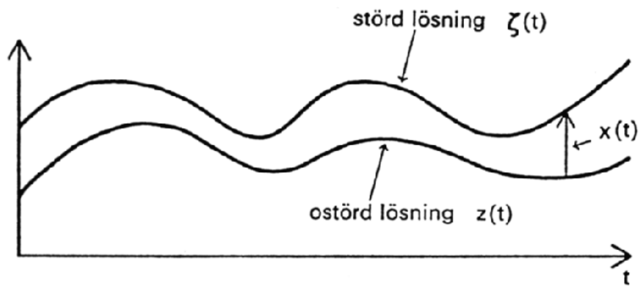
How to formalize the ideas?

Stability Concept

Consider a solution $x(t,a)$ to the differential equation

$$\frac{dx}{dt} = f(x)$$

with initial conditions $x(0,a) = a$. Investigate what happens to a solution $x(t,b)$ with initial condition $x(0,b) = b$ where b is close to a.



Lyapunov Stability

The solution $x(t,a)$ is called *stable* if $|x(t,a) - x(t,b)| < \varepsilon$ for all b such that $|a - b| < \delta$.

The solution is called **asymptotically stable** if it is stable and if in addition $|x(t,a) - x(t,b)|$ goes to zero as t increases towards ∞.

Notice that we can only talk about stability of a particular solution. One solution may be stable and another unstable. Example: the pendulum.

In control we will require asymptotic stability.

It is convenient to normalize so that the interesting solution is $x(t,a) = 0$.

Stability of Nonlinear Systems

Consider the nonlinear system

$$\frac{dx}{dt} = f(x)$$

Assume that it has an equilibrium $x = a$, i.e. $f(a) = 0$. The equilibrium is stable if the linearized equation

$$\frac{dx}{dt} = Ax$$

where $A = f'(a)$ is stable

Linear Systems

Consider the solutions $x(t,a)$ and $x(t,b)$ to the equations

$$\frac{dx}{dt} = Ax$$

with initial conditions $x(0,a) = a$ and $x(0,b) = b$. We have

$$x(t,a) = e^{At}a, \quad x(t,b) = e^{At}b$$

Hence $x(t,a) - x(t,b) = e^{At}(a - b)$

The solution $x(t,a)$ is stable if the matrix A has all eigenvalues in the left half plane or on the imaginary axis, and if the eigenvalues on the imaginary axis are simple. The solution is asymptotically stable if all eigenvalues of A are in the proper left half plane.

Linear Time Invariant Systems are Very Special

- In general we can only talk about the stability of a specific solution
- This means that some solutions may be stable and other unstable
- Linear time invariant system are very special because if one solution is stable all other solutions are also stable
- It is thus possible to talk about the stability of a solution
- This is a very unusual property for linear systems

The Characteristic Equation

The system $\dfrac{dx}{dt} = Ax$ has the characteristic equation

$$\det(sI - A) = 0$$

The system $\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_{n-1}\frac{dy}{dt} + a_n y = 0$

has the characteristic equation

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

so does the system

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \ldots + b_1 s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

Eigenvectors and Eigenvalues

An eigenvector v of a matrix A has the property

$$Av = \lambda v$$

where λ is the eigenvalue. This means that the equation

$$(A - \lambda I)v = 0$$

has a non-trivial solution, hence $\det(\lambda I - A) = 0$

Now consider the differential equation

$$\frac{dx}{dt} = Ax$$

The function $x(t) = e^{\lambda t}v$ is a solution with the initial condition $x(t) = v$

Example Inverted Pendulum

Linearize around $x_1 = 0$.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\det(sI - A) = s^2 - 1$$

Characteristic equation has roots $s = \pm 1$, solution is unstable!

Linearize around $x_1 = \pi$.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\det(sI - A) = s^2 + 1$$

Characteristic equation has roots $s = \pm i$, the solution is stable but not asymptotically stable

Using Matlab

Does the equation

$$s^3 + s^2 + s + k = 0$$

have roots in the right half plane for $k = 1$ or $k = 10$?

```
>> roots([1 3 2 1])
ans =
  -2.3247
  -0.3376 + 0.5623i
  -0.3376 - 0.5623i
```

```
>> roots([1 3 2 10])
ans =
  -3.3089
   0.1545 + 1.7316i
   0.1545 - 1.7316i
```

Algebraic stability conditions (Routh-Hurwitz) were been important historically, but are now less important because of computational tools like Matlab. The commands `roots` and `eigen` give numerical solutions. What do we mean by solution to a problem?

Algebraic Criteria Routh-Hurwitz

All zeros of polynomial $a_0 s + a_1$ are in left half plane if *all coefficients are positive*

All zeros of polynomial $a_0 s^2 + a_1 s + a_2$ are in left half plane if *all coefficients are positive*

All zeros of polynomial $a_0 s^3 + a_1 s^2 + a_2 s + a_3$ are in left half plane if *all coefficients are positive* and if

$$a_1 a_2 - a_0 a_3 > 0$$

Example: The polynomial $s^3 + 3s^2 + 2s + k$ has all zeros in LHP if $k < 6$ because

$$a_1 a_2 - a_0 a_2 = 3 \times 2 - 1 \times k = 6 - k$$

The Furuta Pendulum

Model

$$J_p(\ddot{\theta} - \omega^2 \sin\theta \cos\theta) - mg\ell \sin\theta = 0$$

θ tilt angle of pendulum
ω rate of rotation of arm

Stationary solutions for $\omega =$ constant.

$$-J_p\omega^2\Big(\cos\theta + \frac{mg\ell}{J_p\omega^2}\Big)CD \sin\theta = 0$$

The Furuta Pendulum

Stationary solutions $\omega =$ const.

$$-J_p\omega^2\Big(\cos\theta + \frac{mg\ell}{J_p\omega^2}\Big)CD \sin\theta = 0$$

Two solutions if $\omega < \sqrt{mg\ell/J_p}$

$$\theta = 0, \quad \theta = \pi$$

Four solutions if $\omega > \sqrt{mg\ell/J_p} \approx \sqrt{g/\ell} \approx$ 7rad/s

$$\theta = 0, \quad \theta = \pi, \quad \theta = \theta_0, \quad \theta = -\theta_0$$

where $\theta_0 = \arccos\left(-mg\ell/J_p\omega^2\right)$. Physical interpretation!

Linearization

Model

$$J_p(\ddot\theta - \omega^2 \sin\theta\cos\theta) - mg\ell\sin\theta = 0$$

Introduce $x_1 = \theta$ och $x_2 = \dot\theta$

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= \frac{mg\ell}{J_p}\sin x_1 + \frac{1}{2}\omega^2\sin 2x_1\end{aligned}$$

Linearize around $x = x_0$

$$A = \frac{\partial f}{\partial x}\Big|_{x=x_0} = \begin{pmatrix} 0 & 1\\ \frac{mg\ell}{J_p}\cos x_1 + \omega^2\cos 2x_1 & 0\end{pmatrix}\Bigg|_{x=x_0}$$

Stability of Stationary Solutions

Solution $x_1 = \theta = 0$ has

$$A = \begin{pmatrix} 0 & 1\\ \frac{mg\ell}{J_p} + \omega^2 & 0\end{pmatrix}$$

The matrix A has one eigenvalue in the RHP, unstable

Solution $x_1 = \theta = \pi$

$$A = \begin{pmatrix} 0 & 1\\ -\frac{mg\ell}{J_p} + \omega^2 & 0\end{pmatrix}$$

The matrix A has eigenvalues on the imaginary axis if $mg\ell > J_p\omega^2$ the solution is then stable. If $mg\ell < J_p\omega^2$ the matrix A has one eigenvalue in the RHP and the solution is unstable.

Stability of Stationary Solutions ...

Solutions $\theta = \pm\theta_0 = \arccos(-mg\ell/J_p\omega^2)$,

$$A = \begin{pmatrix} 0 & 1\\ \left(\frac{mg\ell}{J_p\omega}\right)^2 - \omega^2 & 0\end{pmatrix}$$

The matrix A has eigenvalues on the imaginary axis if

$$J_p\omega^2 > mg\ell$$

the solution is then stable. Physical interpretation!

Summary

- Stability important in control and in many other fields
  - Buckling, critical speeds, combustion instability, acoustics
- Stability concepts
- Stability of solutions and stability of systems
- Linear systems: Characteristic equation $\det(sI - A)$ tells all
- Nonlinear systems
  - Only stability of particular solutions
  - A solution is stable if linearized equation stable
  - The main reason why linear control theory is so useful