Lecture 2 - Our First Control Design

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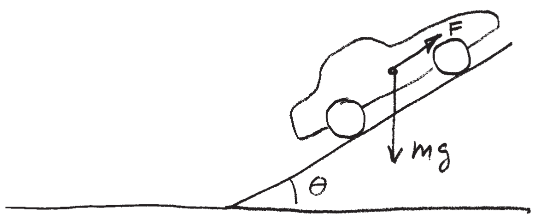
1. Review
2. Cruise Control
3. Standard Models
4. Summary

Themes: A simple control design. Use of standard models.

1. Review

- Block diagrams
- Feedback and feedforward (open or closed loop systems)
- Properties of feedback
  - + Reduce effects of process disturbances
  - + Make system insensitive to process variations
  - + Stabilize an unstable system
  - + Create well defined relations between output and reference
  - - Risk for instability
- PID control: $u = ke + k_i \int_0^t e(\tau)d\tau + k_d \dfrac{de}{dt}$

2. Cruise Control



- Process input or control variable: gas pedal (throttle) u
- Process output: velocity v
- Desired output or reference signal v_r
- Disturbances: slope θ

Construction of a Block Diagram

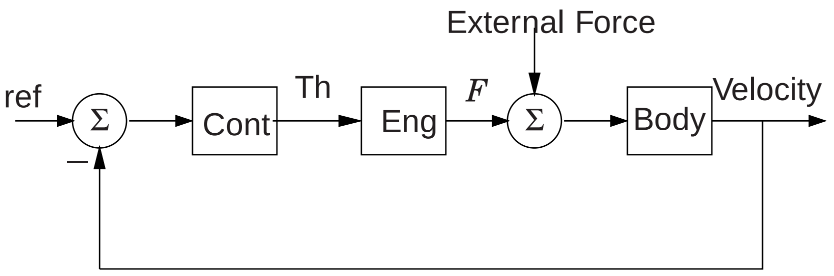
The block diagram gives an overview. To draw a block diagram:

- Understand how the system works.
- Identify the major components and the relevant signals.
- Key questions:

  Where is the essential dynamics?

  What are appropriate abstractions?
- Describe the dynamics of the blocks.

Block Diagram of Cruise Control



Simplifying assumptions

- Essential dynamics relates velocity and force
- The force responds very fast to a change in the throttle
- Assume that all relations are linear (small perturbations)

Process Model

A simplified mathematical model

$$m\frac{dv}{dt} + vd = F - mg\theta$$

With reasonable parameters

$$\frac{dv}{dt} + 0.02v = u - 10\theta$$

where

v [m/s] (10 m/s=36 km/h=22 miles/hour)

u normalized throttle $0 \le u \le 1$

θ slope in [rad/s]

The Closed Loop System

Process model: $\frac{dv}{dt} + 0.02v = u - 10\theta$

PI controller: $u = k(v_r - v) + k_i \int_0^t (v_r - v(\tau))d\tau$

The closed loop system is described by (differentiate both equations and add them) $e = v_r - v$

$$\frac{d^2e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

In steady state with constant θ and e we have $e = 0$.

No surprise the controller has integral action!

Stop and Think!!

How is the behavior of the equation

$$\frac{d^2e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

influenced by the parameters k and k_i?

What is the static gain of the system?

The Audience is Thinking ...

Behavior of the Equation

$$\frac{d^2e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

Compare with spring-mass-damper system

$$m\frac{d^2x}{dt^2} + d\frac{dx}{dt} + kx = 0$$

What are the effects of damping d and spring constant k?
Normalized parameters

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and $\zeta = 0.5\frac{d}{\sqrt{mk}}$.

How to Find Controller Parameters?

$$\frac{d^2e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

Comparison with the normalized mass-spring-damper system

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$$

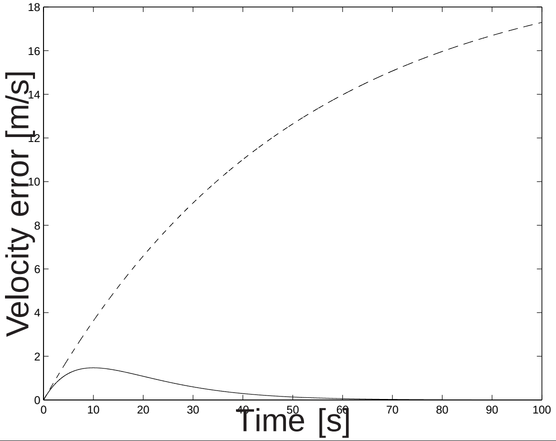
gives

$$k = 2\zeta\omega_0 - 0.02$$
$$k_i = \omega_0^2$$

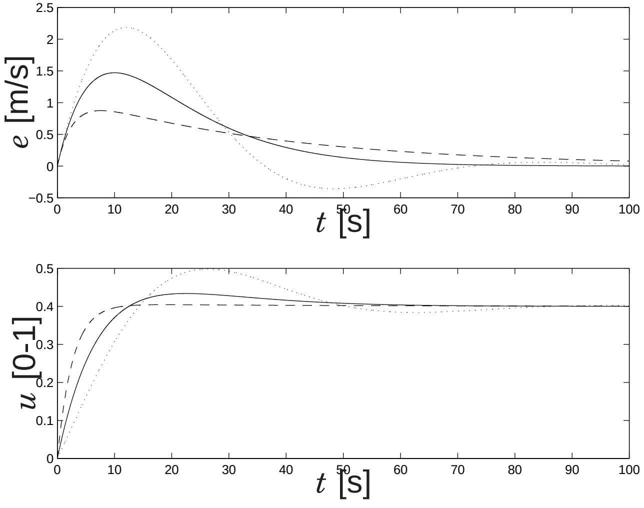
Parameter ω_0 gives response speed, and ζ gives shape of the response. Reasonable to choose $\zeta = 1$ critical damping. How to choose ω_0?

Comparison of Open Loop and Closed Loop?

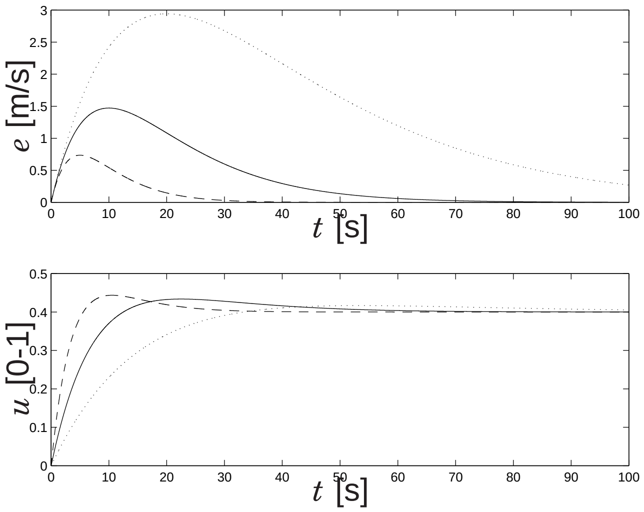
Controller parameters $\zeta = 1$, $\omega_0 = 0.1$, open loop (dashed) closed (solid) when the road has a slope of 4% (10 m/s=36 km/h=22 miles/hour)



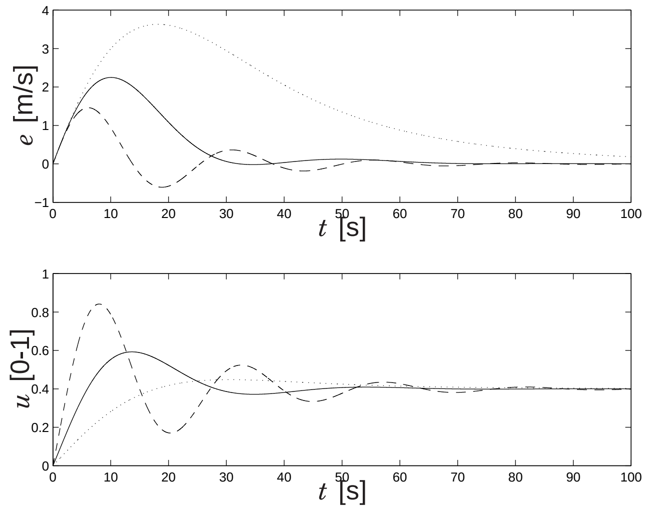
$\omega = 0.1$, $\zeta = 0.5$ (dotted), $\zeta = 1$ (solid), and $\zeta = 2$ (dashed)



$\zeta = 1$, $\omega_0 = 0.05$ (dotted), $\omega_0 = 0.1$ (solid) and $\omega_0 = 0.2$ (dashed)



$\zeta = 1$, $\omega_0 = 0.05$ (dotted), $\omega_0 = 0.1$ (solid) and $\omega_0 = 0.2$ (dashed) with extra dynamics $T = 5s$



What have we done?

- Block diagram of the system
- Derive process model in the form of ODE
- ODE = Ordinary Differential Equation
- Select controller ODE
- Process and controller described in similar ways ODE
- Eliminate variables to give relation disturbance output
- Understand how closed loop system behaves
- Select controller parameters to give desired behavior
- Fine tune parameters by simulation or experiment

Agenda

We must develop methods to

- Derive equations for the system
- Manipulate the equations
- Understand the equations (standard models)
  - Qualitative understanding concepts
  - Insight
  - Standard forms
  - Computations
- Find controller parameters
- Validate the results by simulation

3. Standard Models

- Standard models are foundations of the “language”
- Learn to deal with the standard models
- Transform problems to standard models
- Software is often based on standard models

One of the standard forms of LTI systems is the ODE

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

A high order linear time invariant differential equation that relates the output y of the system to its input u.

A First Order Equation

The homogeneous equation

$$\frac{dy}{dt} + ay = 0$$

has the solution

$$y(t) = Ce^{-at}$$

where $C = y(0)$ is an arbitrary constant. The equation

$$\frac{dy}{dt} + ay = bu$$

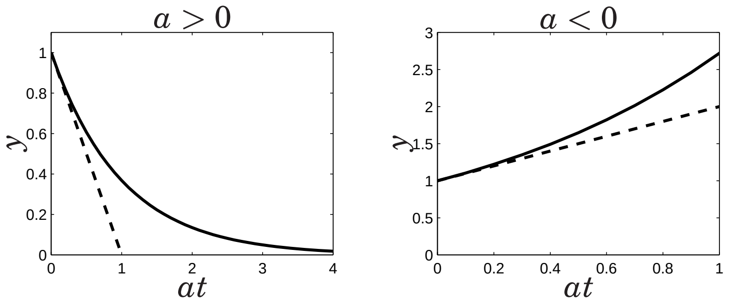
has the solution

$$y(t) = Ce^{-at} + b \int_0^t e^{-a(t-\tau)} u(\tau) dt$$

Parameter a tells a lot!

$$y(t) = y(0)e^{-at} + b \int_0^t e^{-a(t-\tau)} u(\tau) dt$$

First term depends on initial conditions, second term depends on input signal



Equations of Higher Order

Consider the equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \ldots + a_n y = 0$$

The characteristic polynomial is

$$A(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n$$

If $A(\alpha) = 0$ then $y(t) = e^{\alpha t}$ is a solution! If the characteristic equation has distinct roots α_k the solution is

$$y(t) = \sum_{k=1}^{n} C_k e^{\alpha_k t}$$

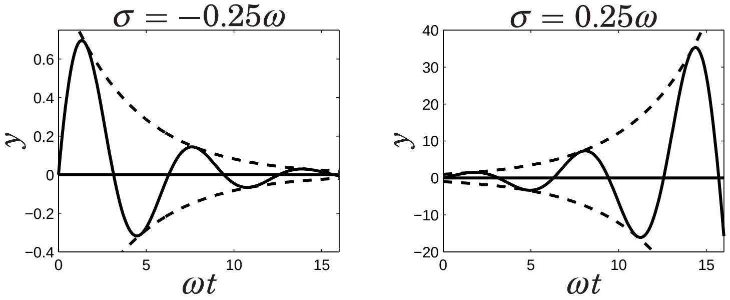
Roots of the characteristic equation gives insight!

Roots of Characteristic Equation give Insight!

A real root $s = \alpha$ to the characteristic equation corresponds to the time function $e^{\alpha t}$.

Complex roots $s = \sigma \pm i\omega$ corresponds to the time functions.

$$e^{\sigma t} \sin \omega t, \qquad e^{\sigma t} \cos \omega t$$



Multiple Roots

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \ldots + a_n y = 0$$

If α is a root to the characteristic polynomial

$$A(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n$$

of multiplicity k and $C(t)$ a polynomial of degree $k-1$ then $y = C(t)e^{\alpha t}$ is a solution to the equation.

The general solution is

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t}$$

Ordinary Differential Equations

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = u$$

has the solution

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t} + \int_0^t h(t-\tau) u(\tau) d\tau$$

Where h is the solution to the homogeneous equation

$$\frac{d^n h}{dt^n} + a_1 \frac{d^{n-1} h}{dt^{n-1}} + \ldots + a_n h = 0$$

with initial conditions

$$h(0) = 0, \quad h'(0) = 0, \quad \ldots, \; h^{(n-2)}(0) = 0, \quad h^{(n-1)}(0) = 1$$

General Linear Time Invariant System (LTI)

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

has the solution

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t} + \int_0^t g(t-\tau) u(\tau) d\tau$$

where $C_k(t)$ are polynomials and the *impulse response* g is

$$g(t) = b_1 h^{n-1}(t) + b_2 h^{n-2}(t) + \; \ldots \; + b_n h(t)$$

where h is the solution to the homogeneous differential equation with initial conditions

$$h(0) = 0, \quad h'(0) = 0, \quad \ldots, \; h^{(n-2)}(0) = 0, \quad h^{(n-1)}(0) = 1$$

Standard Forms

- Standard forms are foundations of the “language”
- Learn to deal with the standard forms
- Transform problems to standard form
- Software is often based on standard form

One of the standard forms of LTI systems is

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

A high order linear time invariant differential equation that relates the output y of the system to its input u.

Poles and Zeros

The linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

is characterized by two polynomials

$$\begin{aligned} A(s) &= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n \\ B(s) &= b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1} s + b_n \end{aligned}$$

The roots of $A(s)$ are called poles of the system. The roots of $B(s)$ are called zeros of the system.

The poles give the components of the time functions that compose the solution.

Interpretation of Poles

The poles $s = \alpha_k$ give the components of the solution

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t} + \int_0^t g(t-\tau) u(\tau) d\tau$$

where $C_k(t)$ are polynomials and

$$g(t) = \sum_{k=1}^{n} \bar{C}_{k-1}(t) e^{\alpha_k t}$$

A system is *stable* if all poles have negative real parts.

Interpretation of Zeros

Consider the linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

Introduce

$$\begin{aligned} A(s) &= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n \\ B(s) &= b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1} s + b_n \end{aligned}$$

If $s = \beta$ is a zero of $B(s)$ and $u(t) = Ce^{\beta t}$ it follows that

$$b_1 \frac{d^{n-1} u}{dt^{n-1}} + b_2 \frac{d^{n-2} u}{dt^{n-2}} \ldots + b_n u = B(\beta) C e^{\beta t} = 0$$

A zero of $B(s)$ at $s = \beta$ blocks the transmission of the signal $u(t) = Ce^{\beta t}$.

The Transfer Function

Consider the linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u$$

Introduce the polynomials

$$A(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$
$$B(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1}s + b_n$$

The rational function

$$\frac{B(s)}{A(s)}$$

is called the **transfer function** of the system

Inverse System

Consider the linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + b_2 \frac{d^{n-2}u}{dt^{n-2}} + \ldots + b_n u$$

Introduce

$$A(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n$$
$$B(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1}s + b_n$$

Notice (almost) symmetry between y and u. The **inverse system** is obtained by reversing the roles of input and output. The transfer function of the system is $\frac{B(s)}{A(s)}$ and the inverse system has the transfer function $\frac{A(s)}{B(s)}$.

Steady State Gain

Consider the linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + b_2 \frac{d^{n-2}u}{dt^{n-2}} + \ldots + b_n u$$

Assume that the and the output are constant, i.e. $y(t) = y_0$ and $u(t) = u_0$. Then

$$a_n y_0 = b_n u_0$$

The number

$$\frac{y_0}{u_0} = \frac{b_n}{a_n}$$

is called the *static gain* of the system.

Cruise Control

The relation between velocity error $e = v_r - v$ and the slope of the road θ is described by the differential equation

$$\frac{d^2 e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

where k and k_i are the controller parameters. This differential equation is characterized by the polynomials

$$A(s) = s^2 + (0.02 + k)s + k_i$$
$$B(s) = 10s$$

The poles can be given arbitrary values by choosing the controller parameters (pole placement). The choice $k = 2\omega_0 - 0.02$ and $k_i = \omega_0^2$ gives a double pole at $s = -\omega_0$. The zero at $s = 0$ blocks transmission of constant θ.

Simulation of Cruise Control

Using the Control Systems Tool box of Matlab it is very easy to simulate an LTI system. Example cruise control

```
z=1.0;
w0=0.1;
t=0:0.01:100;
th0=0.04;              %Slope of the road
k=2*z*w0-0.02;
ki=w0^2;
syseol=tf(10*th0,[1 0.02]);
sysecl=tf(10*th0*[1 0],[1 0.02+k ki]);
eol=step(syseol,t);
ecl=step(syscl,t);
plot(t,eol,'b--',t,ecl,'b-')
```

Using Matlab

It is very convenient to use Matlab to analyze dynamical systems and to plot solutions. The procedure is very simple:

- First we define the polynomials $A(s)$ and $B(s)$ that characterize the system using commands like `A=[1 2 2 1];B=[5 6];`
- The command `roots(A)` gives the roots of the polynomial A (the poles of the system) and `roots(A)` gives the roots of the polynomial B.
- To find the response to a given input signal we first create the signal. The response is then given by the command `lsim(S,u,t)`.
- There are special commands `step` and `impulse` to compute the step and impulse responses.

4. Summary

- Solution to a simple control problem:
  - Understand how the system works: use block diagram.
  - Approximate and write differential equations for the blocks.
  - Eliminate variables to obtain a differential equation for closed loop system
  - Select controller and its coefficients that give desired roots of the characteristic equation.
- The standard model and associated concepts: characteristic polynomial, poles, zeros and transfer functions, stability, inverse systems

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$