Lecture 13 - Observers

K. J. Åström

1. Introduction
2. The Observer Problem
3. Combining with State Feedback
4. Integral action
5. Response to Reference Signals
6. Summary

Theme: Estimating the state and using the estimate for feedback.

Introduction

- The white box view of dynamical systems
- *State* a number of variables that summarizes the past that is useful for prediction
- The pole placement can be solved if all states are measured
- What to do if only outputs are known?
- How to determine the state from the output?
- Combine it with state feedback
- A new view on integral action

Computing the State from the Output

System

$$\frac{dx}{dt} = Ax$$
$$y = Cx$$

Differentiate the output

$$y = Cx$$
$$\frac{dy}{dt} = CAx$$
$$\vdots$$
$$\frac{d^{n-1}y}{dt^{n-1}} = CA^{n-1}x$$

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x = W_o x = \begin{pmatrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}$$

W_o observability matrix

$$x = W_o^{-1} \begin{pmatrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}$$

An Other Attempt

The problem can be solved if the observability matrix has full rank. Can we avoid differentiations?

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Simulate a model of the system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu$$

Introduce error $\tilde{x} = x - \hat{x}$

$$\frac{d\tilde{x}}{dt} = A(x - \hat{x}) = A\tilde{x}$$

Works if A stable, but it makes no use of u.

Two Ways to Calculate the State

By differentiating the output

$$x = W_o^{-1}\begin{pmatrix} y & \frac{dy}{dt} & \dots & \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T$$

This method has the drawback that it only uses the output and that it is differentiated many times which is highly sensitive to noise.

By integrating the input

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu$$

This method has the drawback that it only uses the input and that it only works if the matrix A is stable.

Can we combine the methods?

An Alternative Solution

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

Drive the model by both u and y

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

Introduce error $\tilde{x} = x - \hat{x}$

$$\frac{d\tilde{x}}{dt} = A(x-\hat{x}) - K(y - C\hat{x}) = (A - KC)\tilde{x}$$

Notice that use of output y gives extra freedom.

Determine matrix K so that the matrix $A - KC$ has all its eigenvalues in the left half plane, then $\tilde{x}$ will go to zero!

Example - The Car

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} x + \begin{pmatrix} 0\\ 1\end{pmatrix} u\\ y &= Cx = \begin{pmatrix} 1 & 0\end{pmatrix} x\end{aligned}$$

Observability matrix

$$W_o = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$$

has full rank. The system is observable and the observer is

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \hat{x} + \begin{pmatrix} 0\\ 1\end{pmatrix} u + \begin{pmatrix} k_1\\ k_2\end{pmatrix} (y - \hat{x}_1)$$

We have $A - KC = \begin{pmatrix} -k_1 & 1\\ -k_2 & 0\end{pmatrix}$

Example - The Car ...

The matrix

$$A - KC = \begin{pmatrix} -k_1 & 1\\ -k_2 & 0\end{pmatrix}$$

has the characteristic polynomial

$$\det(sI - A + KC) = \det\begin{pmatrix} s + k_1 & -1\\ k_2 & s\end{pmatrix} = s^2 + k_1 s + k_2$$

Choosing

$$\begin{aligned} k_1 &= 2\zeta\omega_o\\ k_2 &= \omega_o^2\end{aligned}$$

gives the characteristic polynomial $s^2 + 2\zeta\omega_o s + \omega_o^2$.

Interpretation of the Observer

The observer is a dynamical system with two inputs u and y and two outputs x_1 and x_2.

$$\frac{d\hat{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (y - \hat{x}_1)$$

Input-output relations

$$\hat{X}_1(s) = \frac{1}{s^2 + 2\zeta\omega_o s + \omega_o^2} U(s) + \frac{2\zeta\omega_o s + \omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2} Y(s)$$

$$\hat{X}_2(s) = \frac{s + 2\zeta\omega_o}{s^2 + 2\zeta\omega_o s + \omega_o^2} U(s) + \frac{\omega_o^2 s}{s^2 + 2\zeta\omega_o s + \omega_o^2} Y(s)$$

Physical interpretation. Compare with $\hat{x}_2 = \dfrac{dy}{dt}$

Comparison with State Feedback

State feedback design: Find matrix L so that the matrix $A - BL$ has prescribed eigenvalues.

Observer design: Find Matrix K so that the matrix $A - KC$ has prescribed eigenvalues.

A matrix and its transpose have the same eigenvalues. We have

$$(A - KC)^T = A^T - C^T K^T$$

State feedback and observer design are the same problem. The same software can be used. Use state feedback program to obtain the observer gain by feeding it by A^T and C^T instead of A and B. Transposing L give the observer gain.

Observable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \end{pmatrix} z$$

$$\tilde{W}_o = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ a_1^2 - a_2 & -a_1 & 1 & \dots & 0 \\ \vdots & & & & \\ . & . & . & \dots & 1 \end{pmatrix}, \quad \tilde{W}_o^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -a_1 & 1 & \dots & 0 \\ -a_2 & -a_1 & \dots & 0 \\ \vdots & & & \\ -a_n & -a_{n-1} & \dots & 1 \end{pmatrix}$$

Observable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \end{pmatrix} z$$

$$\frac{d\hat{z}}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} \hat{z} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u + \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \\ k_n \end{pmatrix} (y - \hat{z}_1)$$

Summary

If the system

$$\frac{dx}{dt} = Ax$$
$$y = Cx$$

is observable its state can be determined by the observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - Cx)$$

- Combine measurements with mathematical model - indirect measurements -redundancy
- Sensor fusion in AI terminology
- Determination of filter gain K similar to state feedback design. Relation to the Kalman filter

Output Feedback

Consider a system which is observable and controllable

$$\frac{dx}{dt} = Ax$$
$$y = Cx$$

Determine the state with an observer and use state feedback from the observed state. The controller is

$$u = -L\hat{x}$$
$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

The controller is a dynamical system whose dynamics is represented by the observer

The Closed Loop System

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$
$$u = -L\hat{x}$$
$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

Introduce the state $\tilde{x} = x - \hat{x}$ instead of $\hat{x}$.

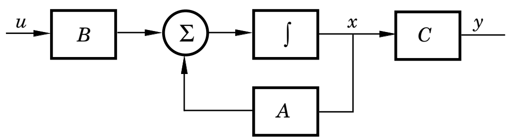
$$\frac{dx}{dt} = Ax + Bu = Ax - BL\hat{x} = Ax - BL(x - \tilde{x}) = (A - BL)x + BL\tilde{x}$$
$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x}$$

Characteristic equation

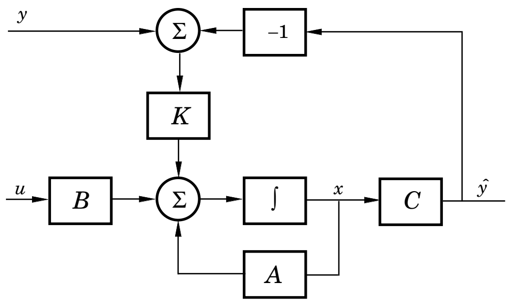
$$\det(sI - A + BL)\det(sI - A + KC) = 0$$

Block Diagram of Process and Observer

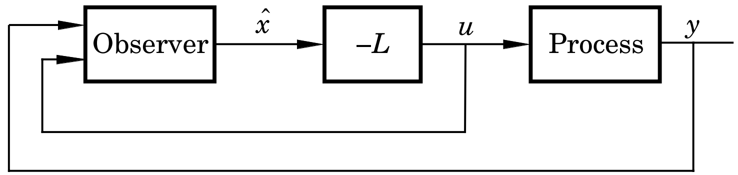
Process



Observer contains a copy of the process model



Block Diagram of Closed Loop system



A few things remains

- Integral action
- How to introduce reference signals

Observers and State Feedback - A Summary

Design of a controller for

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

can be split into two problems

- Design of a state feedback and an observer
- The closed loop system has poles corresponding to the eigenvalues of the state feedback $A_c = A - BL$ and the observer $A_o = A - KC$ matrices
- The problems are similar. The observer is obtained by the transformation $A \rightarrow A^T$, $B \rightarrow C^T$ and $L \rightarrow K^T$. The same computer program can be used.
- Interesting interpretation of controller

Integral Action by Explicit Disturbance Modeling

There are many ways to introduce integral action. Here we will give a method based on explicit modeling of disturbances.

- What is the meaning of the model

$$\frac{dx}{dt} = Ax + Bu$$

- Why do we introduce integral action?
- Disturbances!
- How to describe disturbances

Modeling Disturbances

Classical disturbance models:

- Step $\dfrac{dv}{dt} = 0, \quad v(t) = a = constant$
- Ramp $\dfrac{d^2v}{dt^2} = 0, \quad v(t) = a + bt$
- Sinusoid $\dfrac{d^2v}{dt^2} + \omega^2 v = 0, \quad v(t) = a\sin(\omega t + b)$

Constant Disturbance at Process Input

$$\frac{dx}{dt} = Ax + B(u + v)$$
$$\frac{dv}{dt} = 0$$

Feedback from estimated states

$$u = -L\hat{x} - \hat{v}$$

Observer

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ \hat{v}\end{pmatrix} = \begin{pmatrix}A & B\\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{v}\end{pmatrix} + \begin{pmatrix}B\\ 0\end{pmatrix}u + \begin{pmatrix}K\\ K_v\end{pmatrix}\varepsilon$$
$$\varepsilon = (y - C\hat{x})$$

Example - The Car

Constant disturbance $x_3 = v$, physical interpretation.

$$\frac{dx}{dt} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}x + \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}u$$
$$y = Cx = \begin{pmatrix}1 & 0 & 0\end{pmatrix}x$$

System is observable (check this!). The observer

$$\frac{d\hat{x}}{dt} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\hat{x} + \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}u + \begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix}(y - \hat{x}_1)$$

Example - The Car ...

The matrix

$$A - KC = \begin{pmatrix}-k_1 & 1 & 0\\ -k_2 & 0 & 1\\ -k_3 & 0 & 0\end{pmatrix}$$

has the characteristic polynomial

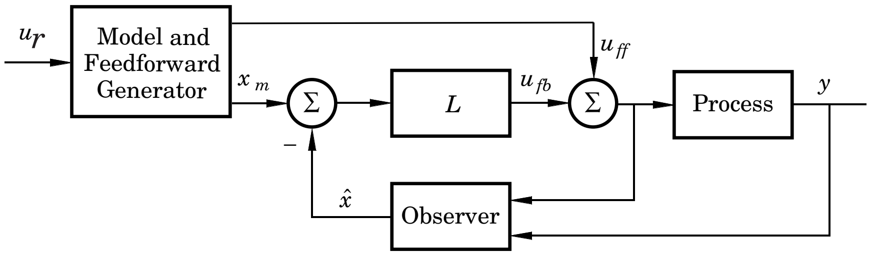
$$\det(sI - A + KC) = \begin{pmatrix}s + k_1 & -1 & 0\\ k_2 & s & -1\\ k_3 & 0 & s\end{pmatrix} = s^3 + k_1 s^2 + k_2 s + k_3$$

choose $k_1 = a + 2\zeta\omega_o$, $k_2 = 2a\zeta\omega_o + \omega_o^2$, and $k_3 = a\omega_o^2$ to give characteristic polynomial

$$(s+a)(s^2+2\zeta\omega_o s+\omega_o^2) = s^3+(a+2\zeta\omega_o)s^2+(2a\zeta\omega_o+\omega_o^2)s+a\omega_o^2$$

How to Introduce Reference Signals

To introduce the reference values we use the standard configuration with two degrees of freedom.



- A nice structure
- Decoupling of estimation of disturbances K
- Reduction of disturbances L
- Response to reference signals

Example - The Car

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = Cx = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

Desired response in Laplace transforms

$$Y_m = X_{m1} = M_y R = \frac{\omega_m^2}{s^2 + 2\zeta\omega_m s + \omega_m^2} R, \quad X_{m2} = sX_{m1}$$

Hence

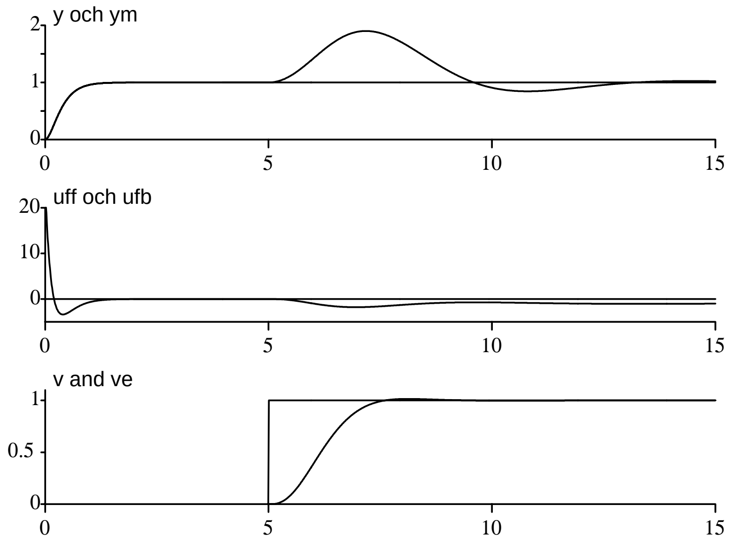
$$\begin{aligned} U_m &= P^{-1} M_y R = \frac{\omega_m^2 s^2}{s^2 + 2\zeta\omega_m s + \omega_m^2} R \\ &= \omega_m^2 (1 - \frac{2\zeta\omega_m s + \omega_m^2}{s^2 + 2\zeta\omega_m s + \omega_m^2}) R = \omega_m^2 (R - \frac{2\zeta}{\omega_m} s Y_m - Y_m) \end{aligned}$$

Example - The Car ...

The complete controller

$$\begin{aligned} u &= l_1(x_{m1} - \hat{x}_1) + l_2(x_{m2} - \hat{x}_2) - \hat{x}_3 + u_m \\ u_m &= \omega_m^2 \left(r - \frac{2\zeta\omega_m}{\omega_m^2} x_{m2} - x_{m1} \right) \\ \frac{dx_m}{dt} &= \begin{pmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta\omega_m \end{pmatrix} x_m + \begin{pmatrix} 0 \\ \omega_m^2 \end{pmatrix} r \\ \frac{d\hat{x}}{dt} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} (y - \hat{x}_1) \end{aligned}$$

A Simulation



Summary

- An elegant approach to control system design
- Interesting system structure with three components:
  - state feedback
  - observer
  - reference trajectory generator
- Strong similarity between state feedback and observers
- Sensor fusion and diagnosis $y - \hat{y}$
- A new interpretation of integral action: disturbance estimator. Can be adapted to other signals.