Lecture 14 - Controllability and Observability

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3. Structure of Linear Systems
4. Cancellation of Poles and Zeros
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Theme: A closer look at the controllability and observability and the structure of linear systems.

Introduction

- The concepts of controllability and observability introduced as conditions to solve problems of state feedback and observers
- More insight
- Kalmans decomposition
- System structures
- Cancellation of poles and zeros

The Concepts

$$\frac{dx}{dt} = Ax$$
$$y = Cx$$

Controllability: Assume that the system is at the origin initially. Can we find a control signal so that the state reaches a given position at a fixed time? Notice we do not require that it stays there!

Observability: Can the state x be determined from observations of the output y over some time interval.

Algebraic Criteria

The system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

is controllable if the matrix

$$W_c = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}$$

has full rank. The system observable if the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

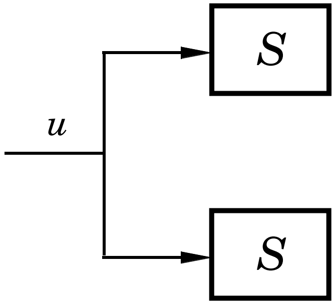
has full rank.

Prototype of Non-controllable System

Two identical systems driven by the same input. Intuitively: no way to make the systems move in opposite ways.

A simple example

$$\frac{dx_1}{dt} = -x_1 + u$$
$$\frac{dx_2}{dt} = -x_2 + u$$



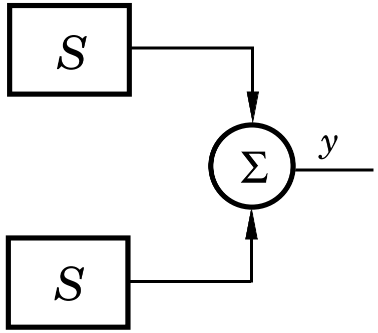
The linear combination $x_1 + x_2$ is controllable

Prototype of Non-observable System

Two identical systems whose outputs are added. Intuitively: no way to find out which system generated the output.

A simple example

$$\frac{dx_1}{dt} = -x_1 + u$$
$$\frac{dx_2}{dt} = -x_2 + u$$
$$y = x_1 + x_2$$



The linear combination $x_1 - x_2$ is not observable

Duality

Controllability of

$$\frac{dx}{dt} = Ax + Bu$$

is the same as observability for

$$\frac{dx}{dt} = A^T x$$
$$y = B^T x$$

Controllability to observability through the transformation

$$A \to A^T$$
$$B \to C^T$$
$$W_c \to W_o$$

Canonical Forms

Controllable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} z + Du$$

Observable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \end{pmatrix} z + Du$$

System Structure

The coordinates can be chosen so that a linear system has the following structure

$$\frac{d}{dt}\begin{pmatrix} x_c \\ x_{\bar c}\end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{pmatrix}\begin{pmatrix} x_c \\ x_{\bar c}\end{pmatrix} + \begin{pmatrix} B_1 \\ 0\end{pmatrix} u$$

where the states x_c are controllable and $x_{\bar c}$ are non-controllable.

$$\frac{d}{dt}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix}$$

$$y = \begin{pmatrix} C_1 & 0 \end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar o}\end{pmatrix}$$

where the states x_o are observable and $x_{\bar o}$ not observable (quiet)

Kalmans Decomposition

A linear system can be transformed to the form

$$\frac{dx}{dt} = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44}\end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0\end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix} x$$

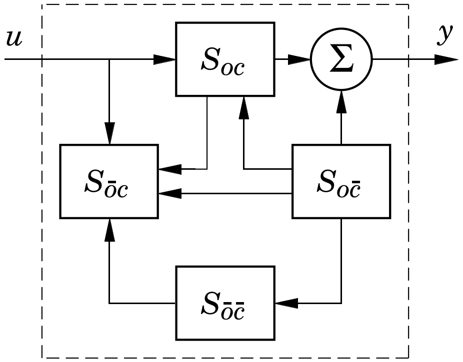
where the state vector has been partitioned as

$$x = \begin{pmatrix} x_{co} \\ x_{c\bar o} \\ x_{\bar c o} \\ x_{\bar c \bar o}\end{pmatrix}^T$$

Kalmans Decomposition

Partitioning of state space

- S_{co} controllable and observable
- $S_{c\bar o}$ controllable not observable
- $S_{\bar c o}$ not controllable observable
- $S_{\bar c\bar o}$ not controllable not observable



The transfer function is given by the subsystem S_{co}

System with State Feedback and Observers

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$
$$u = L(x_m - \hat x) + l_r r$$
$$\frac{d\hat x}{dt} = A\hat x + Bu + K(y - C\hat x)$$

Replace $\hat x$ by $\tilde x = x - \hat x$

$$\frac{dx}{dt} = (A - BL)x + B\tilde x + B(Lx_m + l_r r)$$
$$\frac{d\tilde x}{dt} = (A - KC)\tilde x$$

Observer error not controllable from r. Makes a lot of sense because we do not want reference signals to generate observer errors!

Disturbance Observer

The following system has been used to model constant load disturbances

$$\frac{dx}{dt} = Ax + B(u+v)$$
$$\frac{dv}{dt} = 0$$
$$y = Cx$$

The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$$

Notice that the state v which models the load disturbance is not controllable from u.

Cancellation of Poles and Zeros

Formal calculations with Laplace transforms sometimes leads to cancellation of poles and zeros.

Consider the system

$$\frac{dy}{dt} = \frac{du}{dt} \quad \rightarrow y(t) = u(t) + constant$$

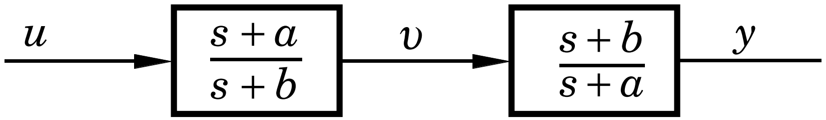
Take Laplace transforms (assuming all initial values zero)

$$sY(s) = sU(s), \quad \rightarrow Y(s) = U(s) \quad \rightarrow y(t) = u(t)$$

There are also design methods where it is deliberately attempted to cancel poles and zeros. It is important to understand what happens when this is done. The decomposition of a linear system gives good insight into what happens when working with transfer functions.

Example of Cancellation

Consider the system



The system has the transfer function $G(s) = 1$. Natural questions:

- Is the system equivalent to the system $y = u$?
- What happens with the modes that are cancelled?

Example of Cancellation ...

Introduce the state representation

$$\frac{dx_1}{dt} = -ax_1 + (b-a)(x_2+u)$$
$$\frac{dx_2}{dt} = -bx_2 + (a-b)u$$
$$y = x_1 + v = x_1 + x_2 + u$$
$$v = x_2 + u$$

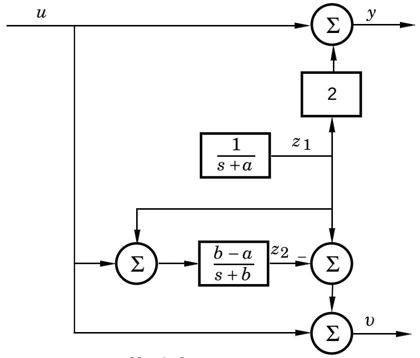
Change coordinates to

$$z_1 = \frac{1}{2}(x_1 + x_2) \qquad x_1 = z_1 + z_2$$
$$z_2 = \frac{1}{2}(x_1 - x_2) \qquad x_2 = z_1 - z_2$$

Example of Cancellation ...

Equations in new coordinates

$$
\begin{aligned}
\frac{dz_1}{dt} &= -az_1 \\
\frac{dz_2}{dt} &= -bz_2 + (b-a)z_1 + (b-a)u \\
y &= 2z_1 + u \\
v &= z_1 - z_2 + u
\end{aligned}
$$



The state z_1 is observable but not controllable

The state z_2 is controllable but not observable

Example of Cancellation ...

The system has two states, one state corresponding to the mode $s = -a$, is controllable and the state corresponding to the mode $s = -b$ is also controllable unless $a = b$. The state z_1 is observable but not the state z_2. There are four interesting cases:

- $a > 0$ and $b > 0$:
  both modes stable
- $a < 0$ and $b > 0$:
  mode $s = -a$ unstable mode $s = -b$ stable
- $a > 0$ and $b < 0$:
  mode $s = -a$ stable mode $s = -b$ unstable
- $a < 0$ och $b < 0$:
  both modes unstable

Lambda Tuning of PI Controllers

Process: $P(s) = \dfrac{b}{s+a}$

Controller: $C(s) = k + \dfrac{k_i}{s} = \dfrac{sk + k_i}{s}$

Choose controller parameters to cancel process pole at $s = -a$, hence $k_i/k = a$. The loop transfer function becomes

$$L(s) = \frac{kb}{s}$$

and the characteristic polynomial is $s + kb$. Choose controller gain k to get closed loop pole at $s = -\alpha$, hence $k = \alpha/b$.

Are there any drawbacks with the cancellation?

The Audience is Thinking ...

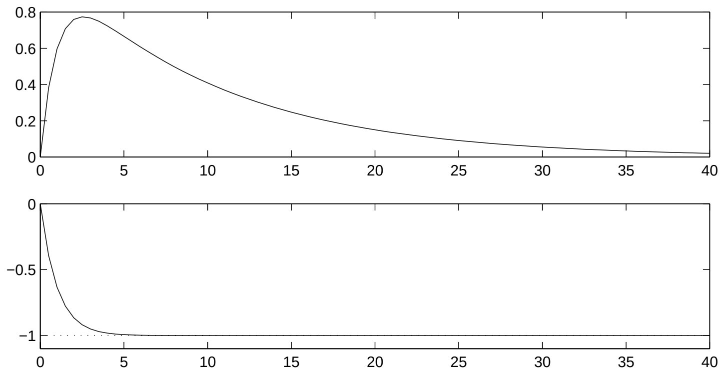
Properties of Closed Loop System

Response to reference signal R and load disturbance D

$$
\begin{aligned}
Y(s) &= \frac{b(ks+k_i)}{s^2 + (a+bk)s + bk_i} R(s) + \frac{bs}{s^2 + (a+bk)s + bk_i} D(s) \\
&= \frac{\alpha}{s+\alpha} R(s) + \frac{bs}{(s+\alpha)(s+a)} D(s) \\
U(s) &= \frac{(ks+k_i)(s+a)}{s^2 + (a+bk)s + bk_i} R(s) - \frac{b(ks+k_i)}{s^2 + (a+bk)s + bk_i} D(s) \\
&= \frac{\alpha}{b}\frac{s+a}{s+\alpha} R(s) - \frac{\alpha}{s+\alpha} D(s)
\end{aligned}
$$

Consider the situation when plant dynamics is slow (a small 0.1) and desired response fast (α large 1.0).

Properties of Closed Loop System ...



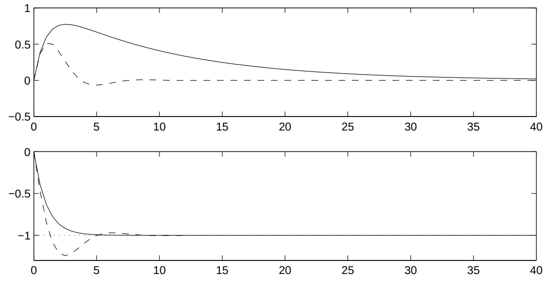
Notice that after the initial reaction the control signal does not react even if there is a large deviation. This is an effect of the cancellation. The controller zero at $s = -0.1$ blocks transmission of the signal $e^{-0.1t}$!

Easy to Fix the Problem

Characteristic polynomial

$$s^2 + (a + bk)s + bk_i = s^2 + 1.1s + k_i$$

Change k_i from 0.1 to 1!



Be careful with cancellation of unstable or slow modes!

Summary

- The notions of controllability and observability were introduced by Kalman in 1960.
- The arise from questions that are natural to ask when state models are used:
  - Can the states be moved from one position to another?
  - Can the state be computed from the measured signal.
- The conditions are required to solve the pole placement and the observer problems.
- The concepts give a natural decomposition of a system which gives insight into the cancellation problem.