Lecture 10 - Physical Modeling

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1. Introduction
2. The Concept of State
3. Linear State Models and Linearization
4. Physical Modeling
5. Homework

Theme: Physical modeling, state models, linearization

1. Introduction

- Dynamics is a major issue in control, since both the processes to be controlled and the controllers are dynamical systems.
- Dynamics has been described by differential equations relating inputs to outputs.
- How much of the past do we need to consider?
- Is there some way to minimize the information about the past that we need?
- Newton's idea!

Static and Dynamic Models

Static models

$$y = f(u)$$

The output signal $y(t)$ at time t depends only on the value of the input at time t.

The output of a dynamic system at time t depends on past values of the input, for example

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

The Concept of State

- The Great Modelers: Tycho Brahe, Kepler and Newton
- To predict the future motion of the planets it is enough to know their current positions and the velocities.
- The *state* is the least information about a system that is required for the prediction of its future development.
- Philosophical consequences, causality, predestination, (chaos).
- A detailed description of the physics of a system, balances of mass, momentum and energy.
- The state is the least number of variables required to describe storage of mass, momentum and energy.
- Synonyms: state models, internal descriptions, white boxes.

Cruise Control

A simple model of a car on a sloping road tells how its position y depends on the throttle. Let the mass be m and assume that the propelling force is proportional to the throttle. We find

$$m\frac{d^2y}{dt^2} - c\frac{dy}{dt}\left|\frac{dy}{dt}\right| = F - mg\sin\theta$$

Two states y and dy/dt.

A Simple Water Tank

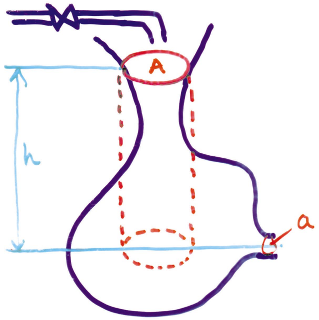
How do level h and outflow q_{out} depend on the inflow q_{in}?

Assume: Constant density

$$\frac{dV}{dt} = q_{in} - q_{out} \qquad \text{Massbalance}$$

$$V = \int_0^h A(h)dh \qquad \text{Geometry}$$

$$q_{out} = a\sqrt{2gh} \qquad \text{Energybalance}$$



Many ways to choose the state.

Analysis and Simplification

Choosing h as a state variable we find

$$\frac{dh}{dt} = \frac{1}{A(h)}(q_{in} - a\sqrt{2gh})$$

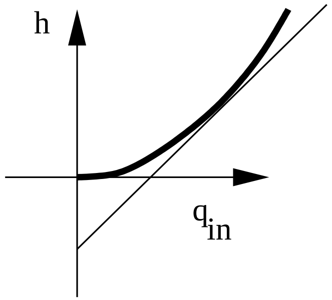
$$q_{out} = a\sqrt{2gh}$$

One function $A(h)$ and one parameter a.

Steady state relation

$$q_{out} = q_{in}$$

$$h = \frac{q_{in}^2}{2ga^2}$$



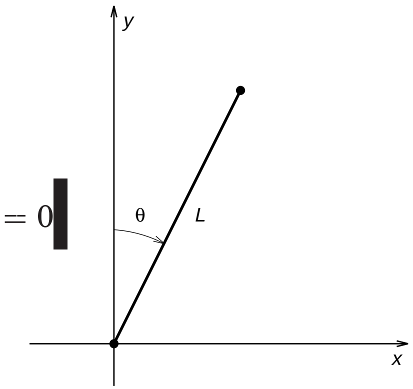
Not influenced by A!
Run ICtools or SysQuake

An Inverted Pendulum

Momentum balance
(Newton's Equation)

$$J\frac{d^2\theta}{dt^2} - mgl\sin\theta + mul\cos\theta = 0$$

Two states θ and $d\theta/dt$.



Normalize with $\omega_0 = \sqrt{mg\ell/J}$, introduce $\tau = \omega_0 t$ and $\bar{u} = u/g$ then

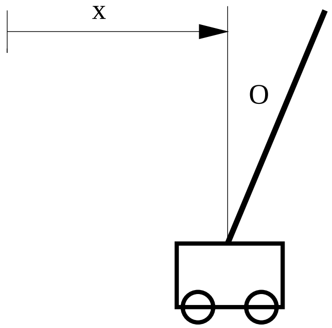
$$\frac{d^2\theta}{d\tau^2} - \sin\theta + u\cos\theta = 0$$

Pendulum on a Cart

Equations of motion Momentum balances

$$J_p\ddot{\theta} + ml \;\; x\cos\theta - mgl\sin\theta = 0$$
$$ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + M\ddot{x} = F$$

Momentum balance characterized by four variables:
θ, $\dot{\theta}$, x and $\dot{x}$
Four states are enough!



Standard Model

A system with finite number of states can be described by

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

- x state
- u input, control variable
- y output, measured variable

The model (a nonlinear ordinary differential equation (ODE)), tells that the rate of change of the state at time t is uniquely given by the state at time t, and the input at time t. If the state is known at time t, old values of x do not give any extra information.

Standard Model - Equilibrium Solutions

Given the system

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

find constant values x_0 and u_0 that satisfy the equation. Putting $dx/dt = 0$ gives

$$f(x_0,u_0) = 0$$

Difficulties with Nonlinear Equations

- Solutions may not exist for all t. Example

$$\frac{dx}{dt} = x^2, \quad x(t) = \frac{1}{1-t}$$

- There may be many solutions. Example

$$\frac{dx}{dt} = 2\sqrt{x}, \quad x(t) = \begin{cases} t^2 & \text{if } t \geq 0, \\ 0 & \text{if } t \leq a \\ (t-a)^2 & \text{if } t > a \end{cases}$$

  Compare with the water tank!
  Bad modeling!
  Not easy do discover!

- Numerical solutions require care

Inverted Pendulum

The model:

$$J_p\ddot{\theta} - mg\ell \sin\theta = mu\ell \cos\theta$$

Introduce state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, then

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= \frac{mg\ell}{J_p}\sin x_1 + \frac{m\ell}{J_p}u\cos x_1\end{aligned}$$

Find the stationary solutions!

The Audience is Thinking ...

Inverted Pendulum

The model:

$$J_p\ddot{\theta} - mg\ell \sin\theta = mu\ell \cos\theta$$

Introduce state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, then

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= \frac{mg\ell}{J_p}\sin x_1 + \frac{m\ell}{J_p}u\cos x_1\end{aligned}$$

Stationary solutions for $u = 0$ gives $\sin x_1 = 0$ and $x_2 = 0$.

Two cases:

$\theta = x_1 = 0$ and $\dot{\theta} = x_2 = 0$ (pendulum up)

$\theta = x_1 = \pi$ and $\dot{\theta} = x_2 = 0$ (pendulum down)

Expansion of Standard Model

We have used a very compact notation: $dx/dt = f(x,u)$, $y = g(x,u)$. It is important to know what this means. Writing all components of the vectors we get

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\\ &\vdots\\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\\ y_1 &= g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\\ y_2 &= g_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\\ &\vdots\\ y_r &= g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p)\end{aligned}$$

Linearization

- Nonlinear systems are difficult.
- Purpose of control is to keep variables close to desired values.
- Approximate by considering small deviations from equilibrium
- This is called *linearization*
- Major simplification
- Approximation improves with quality of control system
- Procedure:

  First determine the equilibria

  Approximate the equations around the equilibria

Linearization of Static System

Consider the system

$$y = g(u)$$

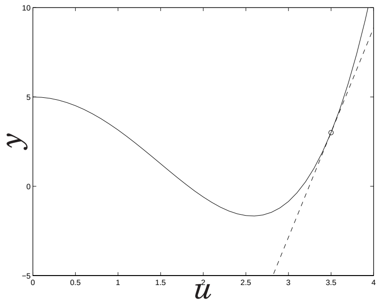
A Taylor series expansion around $u = u_0$ gives

$$y = g(u_0) + g'(u_0)(u - u_0) + \dots$$

The linearized model

$$y - y_0 = g'(u_0)(u - u_0)$$

A curve is approximated by its tangent



When is the approximation good?

Linearization of Dynamic Systems

Start with

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

Find the equilibria $u = u_0,\ x = x_0\ \ y = y_0$ by solving

$$f(x_0, u_0) = 0$$

Notice that there may be several solutions!

Decide what operating condition you want!

Linearization of Dynamic Systems ...

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

Approximate around the equilibrium!

$$x = x_0 + \delta x, \quad u = u_0 + \delta u, \quad y = y_0 + \delta y$$

Hence

$$\frac{dx}{dt} = f(x_0 + \delta x, u_0 + \delta u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x_0, u_0)\delta x + \frac{\partial f}{\partial u}(x_0, u_0)\delta u$$
$$y = g(x_0 + \delta x, u_0 + \delta u) \approx y_0 + \frac{\partial g}{\partial x}(x_0, u_0)\delta x + \frac{\partial g}{\partial u}(x_0, u_0)\delta u$$

Linearization of Dynamic Systems ...

$$\begin{aligned}
\frac{d\delta x}{dt} = \frac{dx}{dt} &= f(x_0 + \delta x, u_0 + \delta u) \\
&\approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x_0, u_0)\delta x + \frac{\partial f}{\partial u}(x_0, u_0)\delta u \\
y_0 + \delta y &= g(x_0 + \delta x, u_0 + \delta u) \\
&\approx y_0 + \frac{\partial g}{\partial x}(x_0, u_0)\delta x + \frac{\partial g}{\partial u}(x_0, u_0)\delta u
\end{aligned}$$

$$\begin{aligned}
\frac{d\delta x}{dt} &= \frac{\partial f}{\partial x}(x_0, u_0)\delta x + \frac{\partial f}{\partial u}(x_0, u_0)\delta u = A\delta x + B\delta u \\
\delta y &= \frac{\partial g}{\partial x}(x_0, u_0)\delta x + \frac{\partial g}{\partial u}(x_0, u_0)\delta u = C\delta x + D\delta u
\end{aligned}$$

Linearization of Dynamic Systems

For small deviations around an equilibrium the system

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

can be approximated by

$$\frac{d\delta x}{dt} = A\delta x + B\delta u$$
$$\delta y = C\delta x + D\delta u$$

$$A = \frac{\partial f}{\partial x}(x_0,u_0) \qquad B = \frac{\partial f}{\partial u}(x_0,u_0)$$
$$C = \frac{\partial g}{\partial x}(x_0,u_0) \qquad D = \frac{\partial g}{\partial u}(x_0,u_0)$$

A Remark on Notations

$$A = \frac{\partial f}{\partial x}(x_0,u_0)$$

Component-wise

$$a_{11} = \frac{\partial f_1(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_p)}{\partial x_1}$$
$$a_{12} = \frac{\partial f_1(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_p)}{\partial x_2}$$
$$\vdots$$
$$a_{n1} = \frac{\partial f_n(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_p)}{\partial x_1}$$
$$\vdots$$
$$a_{nn} = \frac{\partial f_n(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_p)}{\partial x_n}$$

The Water Tank

$$\frac{dh}{dt} = \frac{1}{A(h)}(q_{in} - a\sqrt{2gh}) \qquad q_{out} = q_{in} = q_0 = a\sqrt{2gh_0}$$
$$q_{out} = a\sqrt{2gh} \qquad h_0 = \frac{q_0^2}{2ga^2}$$

Assume constant cross section A, introduce $h = h_0 + \delta h$

$$\frac{d\delta h}{dt} = -\frac{a}{2A}\sqrt{\frac{2g}{h_0}}\delta h + \frac{1}{A}\delta q_{in} = -\frac{a\sqrt{2gh_0}}{2Ah_0}\delta h + \frac{1}{A}\delta q_{in}$$
$$\delta q_{out} = a\sqrt{\frac{2g}{h_0}}\delta h = \frac{a\sqrt{2gh_0}}{h_0}\delta h = \frac{q_0}{h_0}\delta h$$

Time constant

$$T = \frac{2Ah_0}{q_0} = 2 \times \frac{\text{Total water volume } [m^3]}{\text{Flow rate } [\mathrm{m}^3/\mathrm{s}]}$$

Physical interpretation!

The Inverted Pendulum

States $x_1 = \theta = y$ and $x_2 = \dot\theta$

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{mg\ell}{J}\sin x_1 + \frac{m\ell}{J}u\cos x_1$$

write in standard form

$$\frac{dx}{dt} = f(x,u) = \begin{pmatrix} x_2 \\ \frac{mg\ell}{J}\sin x_1 + \frac{m\ell}{J}u\cos x_1 \end{pmatrix}$$

Two stationary solutions

$\theta = x_1 = 0$ and $\dot\theta = x_2 = 0$ (pendulum up)

$\theta = x_1 = \pi$ and $\dot\theta = x_2 = 0$ (pendulum down)

The Inverted Pendulum ...

$$f(x,u) = \begin{pmatrix} x_2 \\ \frac{mg\ell}{J}\sin x_1 + \frac{m\ell}{J}u\cos x_1 \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ \frac{mg\ell}{J}\cos x_1 - \frac{m\ell}{J}u\sin x_1 & 0 \end{pmatrix}, \quad \frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ \frac{m\ell}{J}\cos x_1 \end{pmatrix}$$

Evaluate for $u = 0$, $x_1 = 0$ och $x_2 = 0$ (pendulum up)

$$A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ \frac{mg\ell}{J} & 0 \end{pmatrix} \qquad B = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ \frac{m\ell}{J} \end{pmatrix}$$

Evaluate for $u = 0$, $x_1 = \pi$ och $x_2 = 0$ (pendulum down)

$$A = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\frac{mg\ell}{J} & 0 \end{pmatrix} \qquad B = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 \\ -\frac{m\ell}{J} \end{pmatrix}$$

Linear Dynamical Systems - The State Model

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- Variables denote deviations from equilibrium
- Think scalar and interpret as vectors

Solution

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds \\ y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t) \end{aligned}$$

All information in the matrices A, B, C and D.

The Matrix Exponential

What is the meaning of e^{At}?

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \ldots + \frac{1}{n!}A^n t^n + \ldots$$

If A can be diagonalized $A = T\Lambda T^{-1}$, then

$$\begin{aligned} e^{At} &= T(I + \Lambda t + \frac{1}{2}\Lambda^2 t^2 + \ldots + \frac{1}{n!}\Lambda^n t^n + \ldots)T^{-1} \\ &= Te^{\Lambda t}T^{-1} = T \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix} T^{-1} \end{aligned}$$

where λ_i are the eigenvalues of the matrix A, i.e. the solutions to the equation $\det(\lambda I - A) = 0$

Calculating with the Matrix Exponential

The matrix exponential is defined as

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 \ldots + \frac{1}{n!}(At)^n + \ldots$$

Differentiate!

$$\frac{d}{dt}e^{At} = A + At + \frac{1}{2}(At)^2 \ldots + \frac{1}{(n-1)!}(At)^{n-1} + \ldots = Ae^{At}$$

Differentiation of $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

gives

$$\frac{dx}{dt} = Ax + Bu$$

Vector and Matrix Notations

- Very compact and practical notation
- Numerical calculations supported by nice software
- Learn to formulate and interpret
- Essentially the same as for scalar equations
- BUT remember that $AB \neq BA$! for matrices

Relation between Input and Output

$$
\begin{aligned}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

Input-output relation

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t)$$

Compare with first order systems! Take Laplace transforms

$$
\begin{aligned}
sX(s) - x(0) &= AX(s) + BU(s) & X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \\
Y(s) &= CX(s) + DU(s) & Y(s) &= CX(s) + DU(s)
\end{aligned}
$$

The transfer function is $G(s) = D + C(sI - A)^{-1}B$

Summary

- Obtaining dynamics from physics
- The concept of **state**
- The standard model for nonlinear finite dimensional systems

$$\frac{dx}{dt} = f(x, u), \quad y = g(x, u)$$

- Linearization and linear time invariant (LTI) systems
- The standard model for linear time invariant systems

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du$$

- Vector and matrix notations, the matrix exponential e^{At}
- Compact notation with computational tools