Lecture 8 - Specification and Limitations

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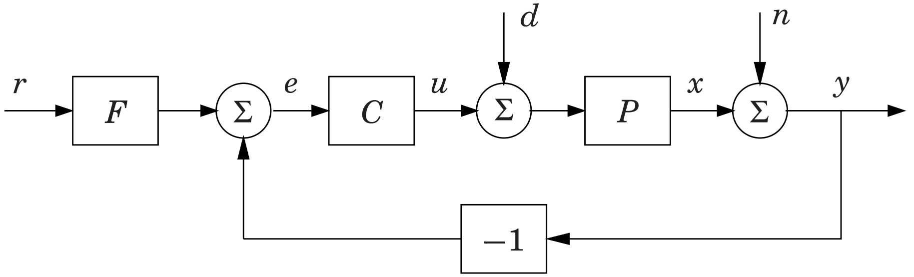
1. Introduction
2. Classical Specifications and the Gang of Six
3. Properties of Simple Transfer Functions
4. Limitations
5. Summary
6. Appendix - Examples Simple Transfer Functions

Theme: To specify and evaluate. Important particularly if you are a customer of control.

1. Introduction

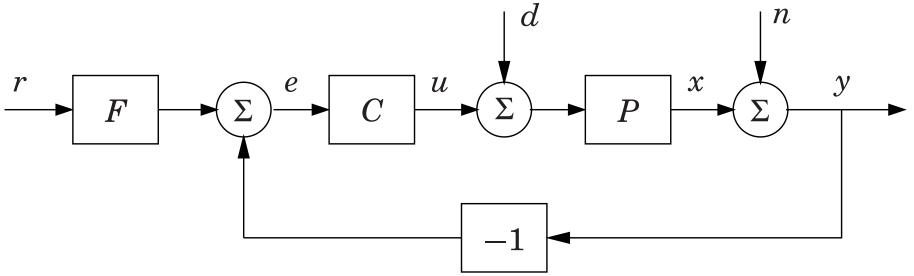
- Important to specify and evaluate systems
  - Follow reference signals
  - Reduce load disturbances
  - Do not inject too much measurement noise
  - Sensitivity to modeling errors
  - Many classical specifications were geared towards response to reference signals. Important to consider all issues.
- Important to understand fundamental limitations
- Skill in judging properties of transfer functions

Some Key Issues



- Six transfer functions should be considered
- Transient responses (steps)
- Frequency responses
  - Bode and Nyquist plots
- Poles zeros plots

Load Disturbances

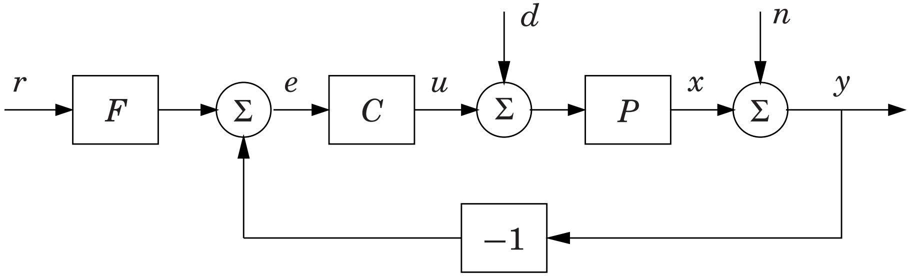


Load disturbances drive the process variable away from it desired value. The relevant transfer functions are

$$\frac{X(s)}{D(s)} = \frac{Y(s)}{D(s)} = \frac{P}{1+PC}$$

$$\frac{U(s)}{D(s)} = -\frac{PC}{1+PC}$$

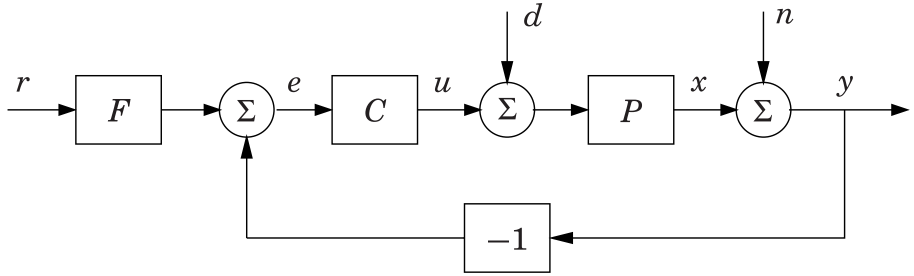
Measurement Noise



Measurement noise is fed into the system by the feedback and generates creates variations in the control signal and the process output. The relevant transfer functions are

$$\frac{U(s)}{N(s)} = -\frac{P}{1+PC}$$
$$\frac{X(s)}{N(s)} = -\frac{PC}{1+PC}$$

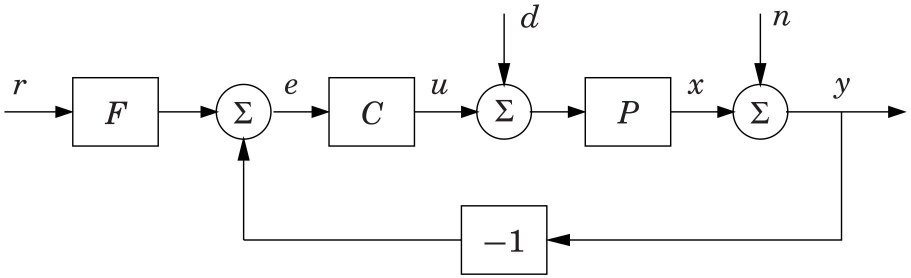
Robustness to Process Variations



It is essential that the closed loop system is not too sensitive to variations in the process dynamics. This is captured by the sensitivity functions

$$S(s) = \frac{1}{1+PC}$$
$$T(s) = \frac{PC}{1+PC}$$

Response to Reference Signal



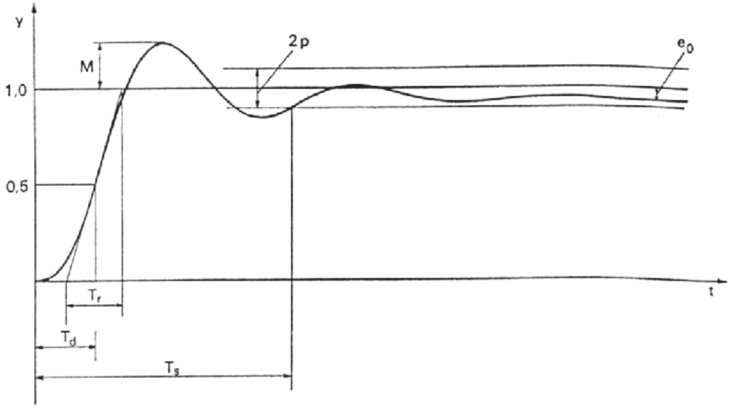
The process variable should follow the reference signal and the control signals required for this are reasonable. The relevant transfer functions are

$$\frac{X(s)}{R(s)} = \frac{Y(s)}{R(s)} = \frac{PC}{1+PC}$$
$$\frac{U(s)}{R(s)} = \frac{C}{1+PC}$$

2. Classical Specifications

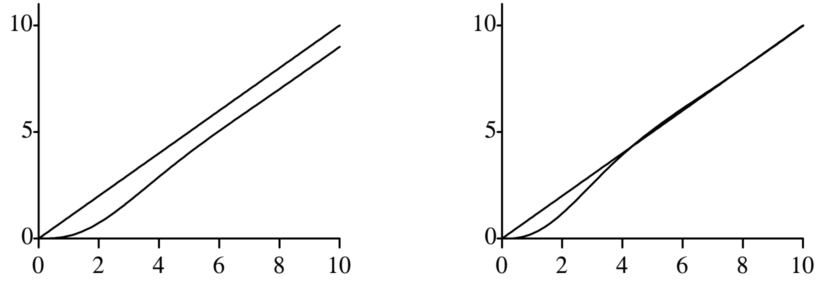
- Focused on one a few of the transfer functions, typically response to step in reference signal
- Specifications in both time and frequency domains
- Some robustness measure, e.g. gain and phase margins
- Remember that a few transfer functions only give a limited view. You must look at all six transfer functions to really know how the system behaves
- Still useful to know about this because of the tradition. This is what many persons you will encounter will talk about.

Response to Step Reference Signal



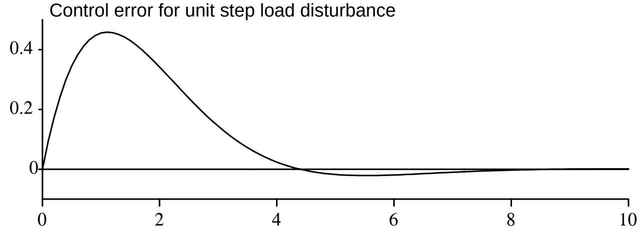
- Overshoot M
- Rise time T_r
- Solution time T_s
- Final error e_0

Response to Ramp in Reference Signal



- Useful to describe tracking of slow signals.
- Particularly important in motion control

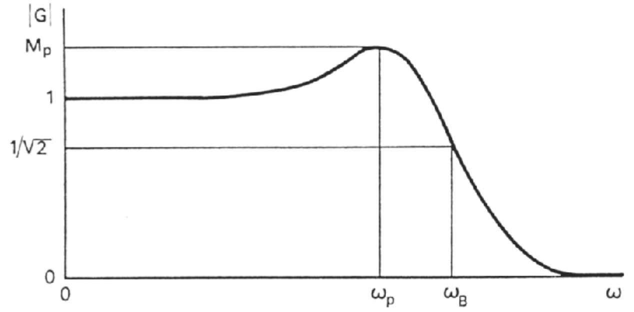
Response to Step in Load Disturbance



How to measure it?

- Max error
- Max time
- Solution time
- Integrated absolute error IE
- Integrated error IE

Frequency Response to Reference Signal



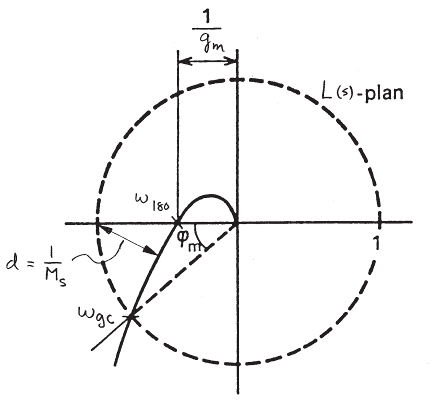
Useful parameters

- Bandwidth
- Resonance peak

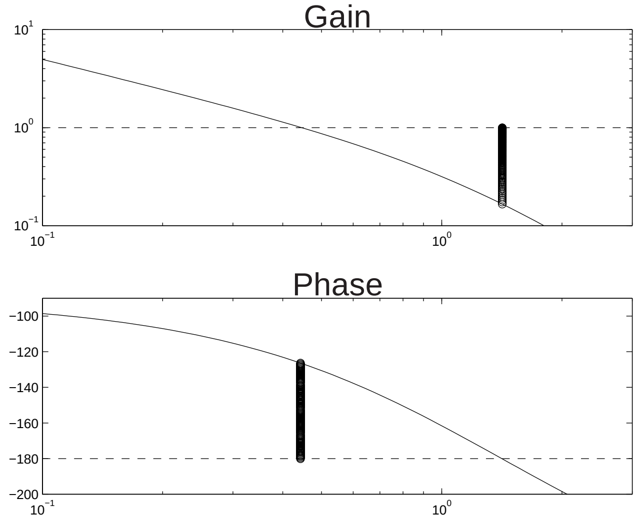
Nyquist Curve of Loop Transfer Function $L = PC$

Useful parameters

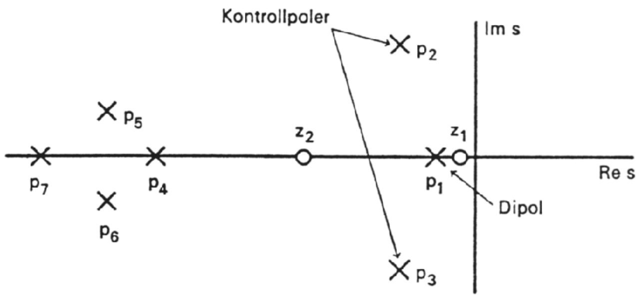
- Gain margin g_m
- Gain crossover frequency ω_{gc}
- Phase margin φ_m
- Phase crossover frequency ω_{180}
- Max sensitivities M_t, M_s
- Frequencies ω_{mt}, ω_{ms}



Bode Plot of Loop Transfer Function $L = PC$



Poles and Zeros of Closed Loop Transfer Function from r to y



More Complete Specifications

- Essential to give specifications on all **six** transfer functions
- This was not done classically and is still often neglected
- Specifications can be given in terms of time responses and frequency responses.
- Several choices, one possibility:
- Response to reference signals is given by transfer functions $FPC/(1+PC)$, $FC/(1+PC)$
- Measurement noise is captured by transfer function $-C/(1+PC)$
- Load disturbances captured by $P/(1+PC)$, $-PC/(1+PC)$
- Robustness captured by $P/(1+PC)$, $-PC/(1+PC)$

3. Properties of Simple Systems

- It is necessary for you to have a working knowledge of properties of simple systems. We will give a short discussion here.
- Practice this in the home-works.
- Use Matlab and ICtools. (Start Matlab in CAD Lab, type ICtools, team up with a friend and interact.)
- Many results are collected in an appendix to this lecture. See slides at the end.

Second Order System with a Zero

$$G(s) = \frac{\omega_0^2}{a}\frac{s+a}{s^2+2\zeta\omega_0 s+\omega_0^2} = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2} + \frac{1}{a}\frac{s\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

Notice that $G(0) = 1$. Let h_0 be the step response of

$$G_0 = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

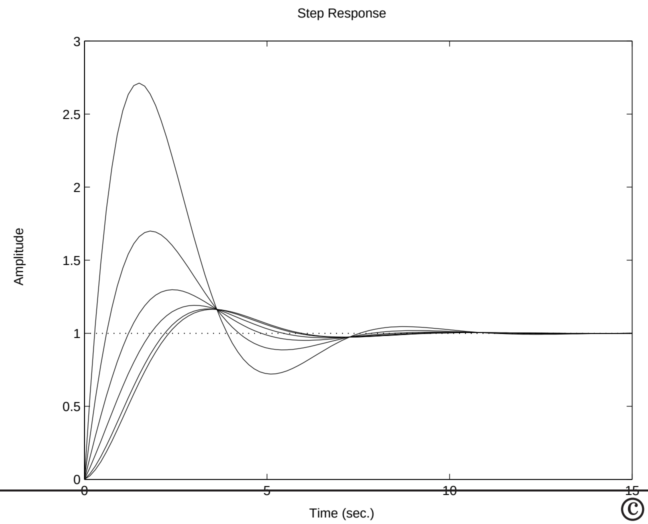
. The step response of G is then

$$h(t) = h_0(t) + \frac{1}{a}\frac{dh_0(t)}{dt}$$

- Implications for the shape of the response
- Effect of different values of a
- What happens if a is negative?

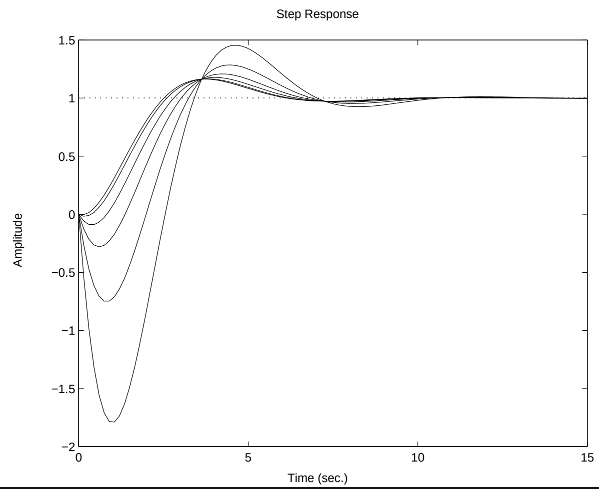
Second Order System with a LHP Zero

Step responses for $\omega_0 = 1$ and $\zeta = 0.5$ $a = 0.25$, 0.5 1, 2, 5 and 10.



Second Order System with a RHP Zero

Step responses for $\omega_0 = 1$ and $\zeta = 0.5$ $a = -0.25$, -0.5 -1, -2, -5, and -10

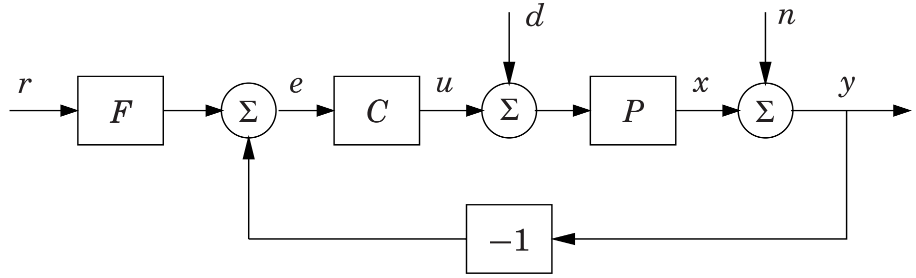


4. Limitations

There are many factors that limits the achievable performance

- Saturation and rate saturation
- Actuator friction and resolution
- Measurement noise
- Difficult dynamics, time delays, poles and zeros in the right half plane (non-minimum phase systems)
- Philosophy:
  - Understand the limitations
  - If at all possible modify the process
  - Never pose unrealistic specifications

Minimum Phase Systems



Essential limitations:

- Sensor noise and resolution
- Actuator saturation and quantization
- Actuator friction

Non-minimum Phase Systems

Factor process transfer function as $P(s) = P_{mp}(s)P_{nmp}(s)$ such that $|P_{nmp}(i\omega)| = 1$ and phase of P_{nmp} is negative, then

$$\begin{aligned}\arg L(i\omega_{gc}) &= \arg P_{nmp}(i\omega_{gc}) + \arg P_{mp}(i\omega_{gc}) + \arg C(i\omega_{gc})\\ &\geq -\pi + \varphi_m\end{aligned}$$

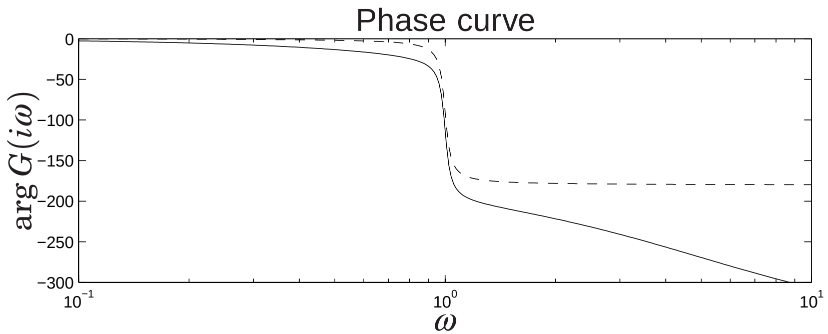
Bode's relations give $\arg P_{mp}(i\omega_{gc}) + \arg C(i\omega_{gc}) \approx n\pi/2$ where n is the slope of the gain curve of the loop transfer function at the gain crossover frequency. Hence

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n\frac{\pi}{2}$$

Simple Rule of Thumb $\varphi_m = 45^\circ$ and $n_{gc} = -0.5$ gives

$$\arg P_{nmp}(i\omega_{gc}) \geq -\frac{\pi}{2}$$

Phase Curve of Bode Plots should Look Like This



Dashed line is phase curve for non-minimum phase equivalent

System with RHP Zero

Assume

$$P_{nmp}(s) = \frac{z-s}{z+s}$$

Hence

$$\arg P_{nmp}(i\omega_{gc}) = -2\arctan\frac{\omega}{z}$$

Cross over frequency inequality

$$\arg P_{nmp}(i\omega_{gc}) = -2\arctan\frac{\omega_{gc}}{z} \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2}$$

Hence

$$\frac{\omega_{gc}}{z} \leq \tan(\frac{\pi}{2} - \frac{\varphi_m}{2} + n_{gc}\frac{\pi}{4})$$

Choosing $\varphi_m = \pi/4$ and $n_{gc} = -1/2$ gives $\omega_{gc} < z$.

A RHP zero limits the response speed of the system.

System with Time Delay

$$P_{nmp}(s) = e^{-sT}$$

Hence

$$\arg P_{nmp}(i\omega_{gc}) = -\omega_{gc}T$$

Cross over frequency inequality

$$\omega_{gc}T \leq \pi - \varphi_m + n_{gc}\frac{\pi}{2}$$

Choosing $\varphi_m = \pi/4$ and $n_{gc} = -1/2$ gives

$$\omega_{gc}T \leq \frac{\pi}{2}$$

A time delay limits the response speed of the system.

System with RHP Pole

$$P_{nmp}(s) = \frac{s+p}{s-p}$$

Hence

$$\arg P_{nmp}(i\omega_{gc}) = -2\arctan\frac{p}{\omega}$$

Cross over frequency inequality

$$-2\arctan\frac{p}{\omega_{gc}} \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2}$$

Hence

$$\omega_{gc} \geq \frac{p}{\tan(\pi/2 - \varphi_m/2 + n_{gc}\pi/4)}$$

Choosing $\varphi_m = \pi/4$ and $n_{gc} = -1/2$ gives $\omega_{gc} \geq p$

A RHP pole requires a high crossover frequency.

System with RHP Pole and Zero Pair

$$P_{nmp}(s) = \frac{(z-s)(s+p)}{(z+s)(s-p)}$$

For $z > p$ we have

$$\arg P_{nmp}(i\omega) = -2\arctan\frac{\omega_{gc}/z + p/\omega_{gc}}{1-p/z}$$

Cross over frequency inequality

$$\frac{\omega_{gc}}{z} + \frac{p}{\omega_{gc}} \leq (1-\frac{p}{z})\tan\Big(\frac{\pi}{2} - \frac{\varphi_m}{2} + n_{gc}\frac{\pi}{4}\Big)$$

Simple calculations give

$$\frac{z}{p} \geq 1 + \frac{2+2\sqrt{1+\alpha^2}}{\alpha^2}$$

RHP Pole-Zero Pair

$$\frac{z}{p} \geq 1 + \frac{2 + 2\sqrt{1+\alpha^2}}{\alpha^2}$$

Assume a phase margin $\varphi_m = \pi/4$ and a slope at the crossover frequency of $n_{gc} = -1/2$, then

$$z \geq 5.83p$$

Phase margin

$$\varphi_m < \pi + n_{gc}\frac{\pi}{2} - 4\arctan\sqrt{\frac{p}{z}}$$

z/p	2	2.24	3.86	5	5.83	8.68	10	20
φ_m	-6.0	0	30	38.6	45	60	64.8	84.6

Example - The X-29

Advanced experimental aircraft. Much design effort was done with many methods and much cost. Specifications $\varphi_m = 45^\circ$ could not be reached. Here is why!

Non-minimum phase part of the transfer function

$$P_{nmp}(s) = \frac{s-26}{s-6}$$

The zero pole ratio is $z/p = 4.33$ with $n_{gc} = -1/2$ we get

$$\varphi_m = 32.4$$

A phase margin of 45° cannot be achieved!

Example - Klein's Rear Wheel Steered Bike

Bicycle with rear wheel steering

$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ}\frac{-s+V_0/a}{s^2 - mg\ell/J}$$

Typical values:

$$m = 70 \text{ kg}$$
$$a = 0.3\text{m}$$
$$\ell = 1.2 \text{ m}$$
$$b = 0.7 \text{ m}$$
$$J = 120 \text{ kgm}^3$$
$$V_0 = 5 \text{ ms}^{-1}$$

RHP zero and pole at

$$s = z = \frac{V_0}{a}$$

$$s = p = \sqrt{\frac{mg\ell}{J}}$$

$$\frac{z}{p} = \frac{mg\ell V_0}{aJ} = \frac{mg\ell V_0}{a(J_{cm} + m\ell^2)}$$

$$\frac{z}{p} = 2.74, \quad \Rightarrow \varphi_m = -10.4^\circ$$

Summary of Dynamics Limitations

For non-minimum phase systems the limitations can be expressed by the crossover frequency inequality

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2}$$

Simple Rule of Thumb: $\arg P_{nmp}(i\omega_{gc}) \geq -\pi/2$

- RHP zeros and time delays give upper bound on ω_{gc}

  Long time delays are bad

  Slow unstable zeros are bad

- RHP poles gives a lower bound on ω_{gc}

  Fast unstable poles are bad

- RHP poles and zeros cannot be too close
- The product of a RHP pole and a time delay cannot be too large

Rules of Thumb for Limitations - Part 1

- A RHP zero z

$$\frac{\omega_{gc}}{z} \leq \begin{cases} 0.5 & \text{for } M_s,\ M_t < 2 \\ 0.2 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A time delay T

$$\omega_{gc} T \leq \begin{cases} 0.7 & \text{for } M_s,\ M_t < 2 \\ 0.37 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A RHP pole p

$$\frac{\omega_{gc}}{p} \geq \begin{cases} 2 & \text{for } M_s,\ M_t < 2 \\ 5 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

Rules of Thumb for Limitations - Part 2

- A RHP pole-zero pair with $z > p$

$$\frac{z}{p} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A RHP pole-zero pair with $z < p$

$$\frac{p}{z} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4 \end{cases}$$

- A RHP pole p and a time delay T

$$pT \leq \begin{cases} 0.16 & \text{for } M_s,\ M_t < 2 \\ 0.05 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

5. Summary

- Classical specifications are restricted
- Important to look at all six transfer functions (“Gang of Six”)
- Important to be aware of limitations
- Non-minimum phase elements give severe limitations
  - Time delays
  - RHP zeros
  - RHP poles
- Examples of assessment of a control system
- Properties of simple transfer functions

Appendix - Properties of Simple Systems

In this Appendix we have collected properties of simple systems

- First order systems
- Second order systems without zero
- Third order system without zeros
- Second order system with zero

Second Order System without Zeros

Transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Introduce

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

Poles

$$s = \begin{cases} -\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2} = -\zeta\omega_0 \pm i\omega_d & \text{if } \zeta \le 1 \\ -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1} = -\zeta\omega_0 \pm \omega_d & \text{if } \zeta > 1 \end{cases}$$

where $\omega_d = \omega_0\sqrt{1-\zeta^2}$ if $\zeta \le 1$ and $\omega_d = \omega_0\sqrt{\zeta^2-1}$ if $\zeta > 1$

Step Response

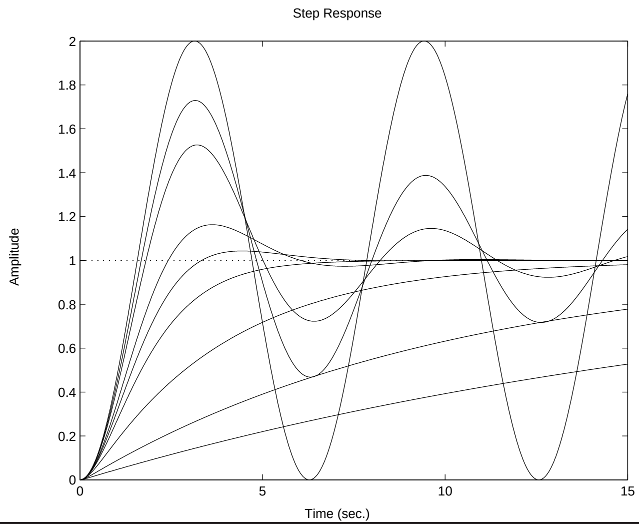
$$h(t) = \begin{cases} 1 - \dfrac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi), & \zeta < 1 \\ 1 - (1+\omega_0 t)e^{-\zeta\omega_0 t}, & \zeta = 1 \\ 1 - \left(\cosh\omega_d t + \dfrac{\zeta}{\sqrt{\zeta^2-1}}\sinh\omega_d t\right)e^{-\zeta\omega_d t} \end{cases}$$

where

$$\phi = \arccos\zeta$$

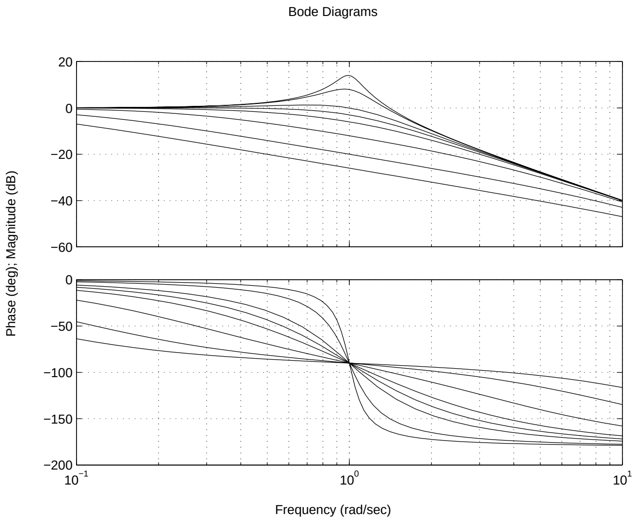
Step Response

$\zeta = 0$, 0.1, 0.2, 0.5, 0.707, 1, 2, 5 and 10.



Frequency Response

Bode plot for $\zeta = 0.1$, 0.2, 0.5, 0.707, 1, 2, 5 och 10.



Third Order System without Zeros

Transfer function

$$G(s) = \frac{a\omega_0^2}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$$

Notice $G(0) = 1$.

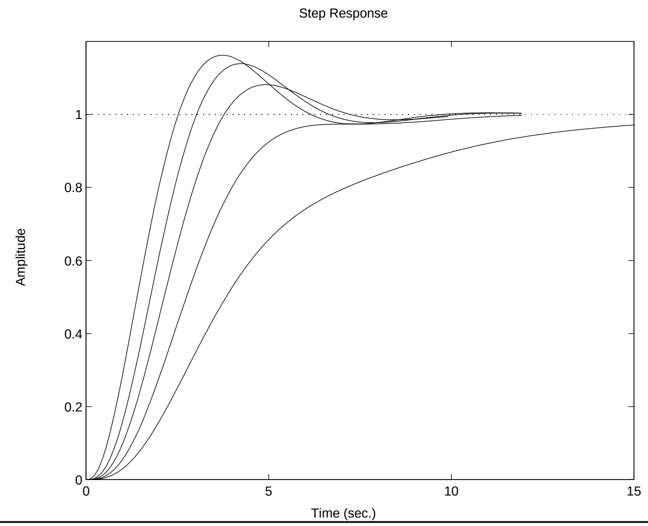
What do we mean by a solution?

```
step([a],conv([1 a],[1 2*z*w0 w0^2]))
```

ICTools and SysQuake!

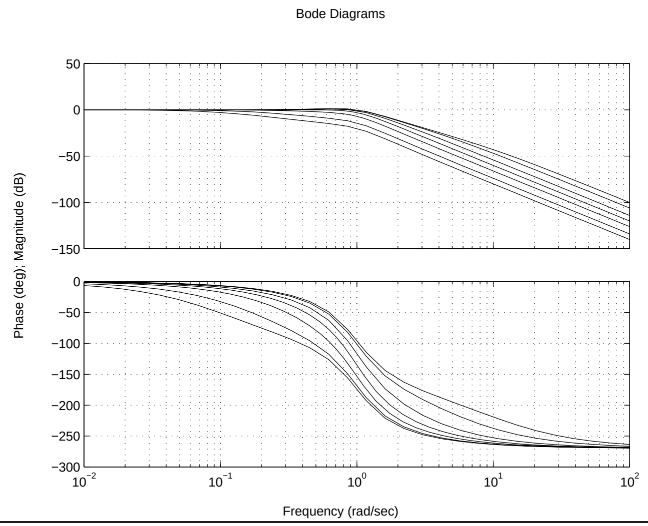
Third Order System without Zeros

Step responses for $\zeta = 0.5$ and a/ω_0 =0.25, 0.5, 1, 2 and 10.



Third Order System without Zeros

Bode plots for $\zeta = 0.5$ and a/ω_0 =0.1, 0.2, 0.5, 1, 2, 5, 10



Second Order System with a Zero

$$G(s) = \frac{\omega_0^2}{a}\frac{s+a}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} + \frac{1}{a}\frac{s\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Notice that $G(0) = 1$. Let h_0 be the step response of

$$G_0 = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

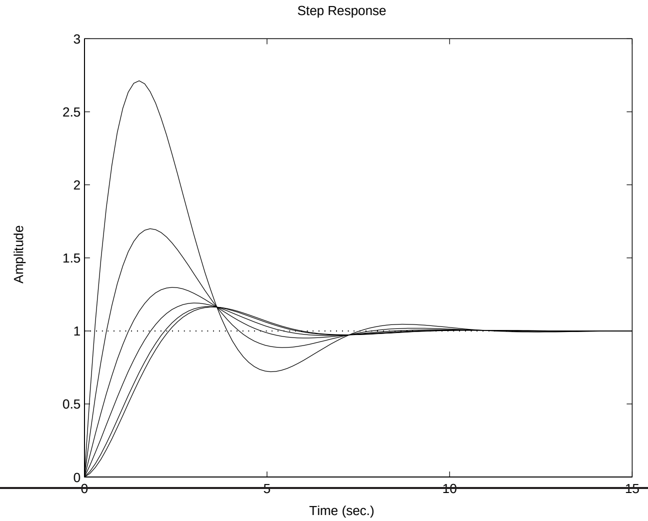
. The step response of G is then

$$h(t) = h_0(t) + \frac{1}{a}\frac{dh_0(t)}{dt}$$

- Implications for the shape of the response
- Effect of different values of a
- What happens if a is negative?

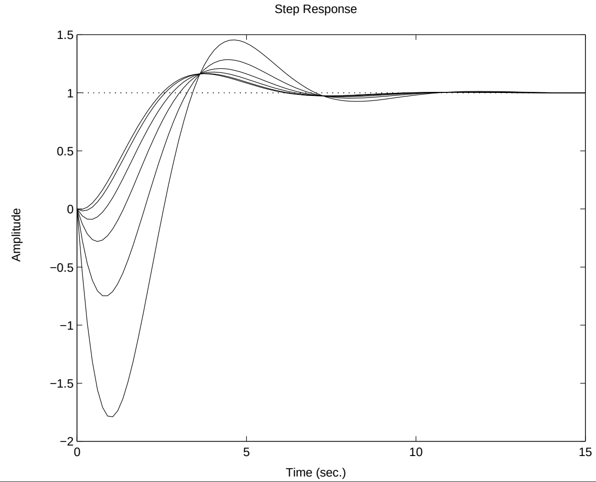
Second Order System with a LHP Zero

Step responses for $\omega_0 = 1$ and $\zeta = 0.5$ $a = 0.25$, 0.5 1, 2, 5 and 10.



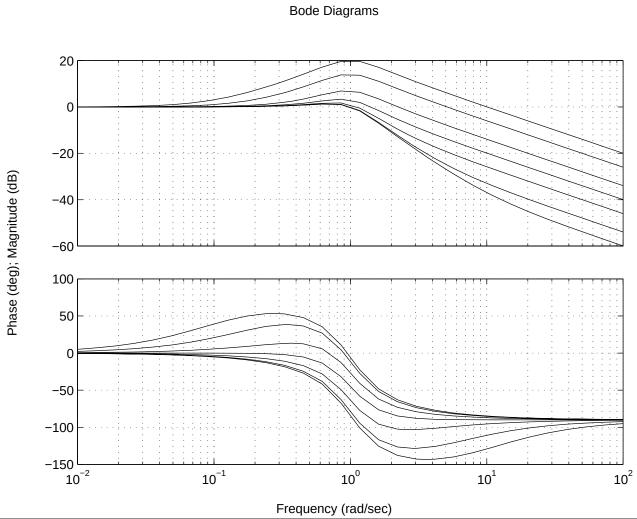
Second Order System with a RHP Zero

Step responses for $\omega_0 = 1$ and $\zeta = 0.5$ $a = -0.25$, -0.5 -1, -2, -5, and -10.



Second Order System with a LHP Zero

Bode plot for $\zeta = 0.5$ and a/ω_0 =0.1, 0.2 0.5 1 2 5 10



Second Order System with a RHP Zero

Bode plot for $\zeta = 0.5$ and a/ω_0 =-0.1, -0.2 -0.5 -1 -2 -5 -10