Lecture 15 - Review of Dynamical Systems

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1. Introduction
2. Different ways to view dynamics
3. State models
4. Input output models
5. Summary

Theme: Collecting bits and pieces.

Introduction

- Dynamics is a key foundation of control
- Linear time invariant systems has been our work horse
- A rich field with many concepts and results
- Mathematical foundations
  - Differential equations
  - Laplace transforms and complex numbers
  - Linear algebra and matrices
- The standard models
- Relations between different representations
- Computational aspects
- Intuition amplifiers: SysQuake and ICTools

Two Views on Dynamics

State Models - White Boxes

- A detailed description of the inner workings of the system
- The heritage from mechanics
- The notion of state and stability
- States describe storage of mass, energy and momentum

Input-Output Models - Black Boxes

- A description of the input output behavior
- The heritage of electrical engineering
- The notions of transfer function, poles, zeros, minimum phase
- The idea of frequency response

State Models

Standard representations:

$$\begin{aligned} \frac{dx}{dt} &= f(x,u) \\ y &= g(x,u) \end{aligned} \qquad \begin{aligned} \frac{d(x-x_0)}{dt} &= A(x-x_0) + B(u-u_0) \\ y - y_0 &= C(x-x_0) + D(u-u_0) \end{aligned}$$

where equilibrium is given by $f(x_0,u_0) = 0$ and $y_0 = g(x_0,u_0)$

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where x and u now denotes deviations from the steady state

Associated concepts

- Observability and controllability

The Concepts

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Controllability: Assume that the system is at the origin initially. Can we find a control signal so that the state reaches a given position at a fixed time? Notice we do not require that it stays there!

Observability: Can the state x be determined from observations of the output y over some time interval.

Algebraic Criteria

The system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

is controllable if the matrix

$$W_c = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}$$

has full rank. The system observable if the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

has full rank.

Input-Output Models

Standard forms for linear systems

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n s}{s^n + b_1 s^{n-1} + \dots + a_n s}$$

Associated concepts

- Impulse response $g(t)$
- Frequency response $G(i\omega)$
- Bode plots and Nyquist curves
- Poles, zeros and gain

Linear Time Invariant Systems

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Variables now denote deviations from steady state. Solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds$$
$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t)$$

First terms depend on initial condition the second on the input.

Transfer function: $G(s) = C(sI - A)^{-1} + D$

Impulse response: $h(t) = Ce^{At}B + D\delta(t)$

Coordinate Changes

Coordinate changes are often useful

$$\frac{dx}{dt} = Ax + Bu \qquad z = Tx \qquad \frac{dz}{dt} = \tilde{A}z + \tilde{B}u$$
$$y = Cx + Du \qquad x = T^{-1}z \qquad y = \tilde{C}z + \tilde{D}u$$

Transformed system has the same form but the matrices are different

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}, \qquad \tilde{D} = D$$

Transfer function and impulse response remain invariant with coordinate transformations.

$$\tilde{g}(t) = \tilde{C}e^{\tilde{A}t}\tilde{B} = CT^{-1}e^{TAT^{-1}t}TB = Ce^{At}B = g(t)$$

and

$$\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = CT^{-1}(sI - TAT^{-1})^{-1}TB = C(sI - A)^{-1}B = G(s)$$

Diagonal Form

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} z + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} u$$
$$y = \begin{pmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_n \end{pmatrix} z + Du$$

Transfer function

$$G(s) = \sum_{i=1}^{n} \frac{\beta_i \gamma_i}{s - \lambda_i} + D$$

Notice appearance of eigenvalues of matrix A

Controllable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \ldots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} b_1 & b_2 & \ldots & b_{n-1} & b_n \end{pmatrix} z + Du$$

Transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n} + D$$

The numerator of the transfer function $G(s)$ is the characteristic polynomial of the matrix A.

Observable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 0 \ldots & 0 \end{pmatrix} z + Du$$

Transfer function

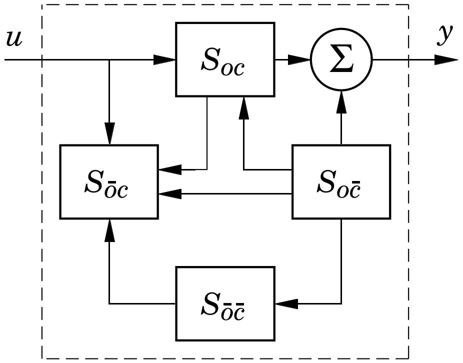
$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n} + D$$

The numerator of the transfer function $G(s)$ is the characteristic polynomial of the matrix A.

Kalmans Decomposition

Partitioning of state space

- S_{co} controllable and observable
- $S_{c\bar{o}}$ controllable not observable
- $S_{\bar{c}o}$ not controllable observable
- $S_{\bar{c}\bar{o}}$ not controllable not observable



The transfer function is given by the subsystem S_{co}

Matlab, SciLab, Octave and SysQuake

One advantage of the matrix formulation is that there is a very good computer support. This makes it easy to solve real problems. Matlab can be viewed as a matrix calculator or a matrix oriented programming environment. It was invented by Cleve Moler. SciLab and Octave are matrix oriented public domain software. SysQuake is a newer product which has been designed for a higher degree of interaction. Matlab has many tool-boxes for special domains. The *Control System Toolbox* and *Simulink* are particularly useful for control. You can find out what it contains by typing the command

help toolbox control

also look at the demos.

Examples of Matlab Functions

Creation of LTI models.

```
ss        - State-space model
zpk       - Zero/pole/gain model
tf        - Transfer function model
set       - Set/modify properties
ltiprops  - Help for LTI properties
ltimodels - Help on LTI models
```

Investigating an LTI

```
ltiview     - Response analysis GUI (LTI Viewer).
```

Transforming systems

```
ss  - Conversion to state space
zpk - Conversion to zero/pole/gain
tf  - Conversion to transfer function
```

Examples of Matlab Functions ...

Transient response

```
step     - Step response
impulse  - Impulse response
lsim     - Simulates an LTI system with given input
gensig   - Generate input signal for LSIM.
stepfun  - Generate unit-step input.
```

Frequency response

```
bode     - Bode plot
nyquist  - Nyquist plot
margin   - Gain and phase margins.
freqresp - Frequency response over a frequency grid
evalfr   - Evaluate frequency response at a frequency
```

Examples

```
SYS = SS2SS(SYS,T) performs the similarity
transformation  z = Tx on the state vector x
of the state-space model SYS.   The resulting
state-space model is described by:

                .       -1
                z = [TAT   ] z + [TB] u
                        -1
                y = [CT    ] z + D u


See also CANON, SSBAL, BALREAL.
```

Examples

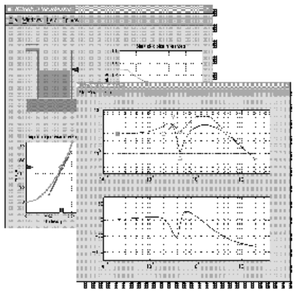
```
CSYS = CANON(SYS,TYPE) computes canonical state-space
realization CSYS of the LTI model SYS.  The string
TYPE selects the type of canonical form:
   'modal'    : Modal canonical form where the system
                eigenvalues appear on the diagonal.
                The state matrix A must be
                diagonalizable.
   'companion': Companion canonical form where the
                characteristic polynomial appears in
                the right column.

[CSYS,T] = CANON(SYS,TYPE) also returns the state
transformation matrix T relating the new state vector
z to the old state vector x by z = Tx.
```

ICTools

This is an interactive tool built in Matlab that allows you to develop an intuition for different representations of linear time invariant systems

Introduction

ICTools is a set of interactive tools for learning fundamental concepts of Automatic Control.



Currently, the tools are being used in the introductory course in Automatic Control.

See the following ICTools related links:

- ICTools in Automatic Control, Basic Course (in swedish).
- Article in the leading swedish engineering magazine Ny Teknik, 1997:48 (in swedish).

ICTools is developed by Mikael Johansson and Magnus Gäfvert, who are Ph.D. students at the Department of Automatic Control, Lund Institute of Technology, Sweden.

Commentary on Computations

- Herman Goldstine: "When things change by two orders of magnitude it is revolution not evolution."
- Important to complement computation by understanding and insight
- Hamming: "The purpose of computing is insight not numbers"
- Expect software errors! Important to check results to make sure that they are reasonable. Always look at results and **Think**
  - Can you find a special case where you know the solution
  - Can you compute an auxiliary quantity to check the results

Input-Output Models

Conceptually a huge table of input-output pairs

Strong simplification for linear time invariant systems with zero initial conditions!

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t)$$

All input output pairs are characterized by the impulse response

$$g(t) = Ce^{At}B + D\delta(t)$$

or alternatively the transfer function, which is the ratio of the Laplace transform of the input and the output when the system is initially at rest

$$G(s) = \frac{\mathcal{L}y}{\mathcal{L}u} = \frac{Y(s)}{U(s)} = \mathcal{L}g = D + C(sI - A)^{-1}B$$

Transfer Functions and Differential Equations

Consider a system with the transfer function

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}$$

It follows that

$$(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n)Y(s) = (b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n)U(s)$$

Conversion to time domain gives

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u$$

In steady state $a_n y = b_n u$, static gain $G(0) = \dfrac{b_n}{a_n}$

Poles and Zeros

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$
$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n} = \frac{B(s)}{A(s)} = \frac{B(s)}{\det(sI - A)}$$

Poles are zeros of the polynomial $A(s)$ or eigenvalues to the A-matrix, i.e. uniquely given by the A-matrix. They describe the free motion of the system. A pole $s = a$ corresponds to a motion component e^{at}.

Zeros are the zeros of the polynomial $B(s)$. Zeros depend on the matrices A, B and C, i.e. how the states are coupled to inputs and outputs. A zero $s = b$ implies that the steady state output corresponding to the input e^{bt} is zero. Zeros are blocking signal transmission.

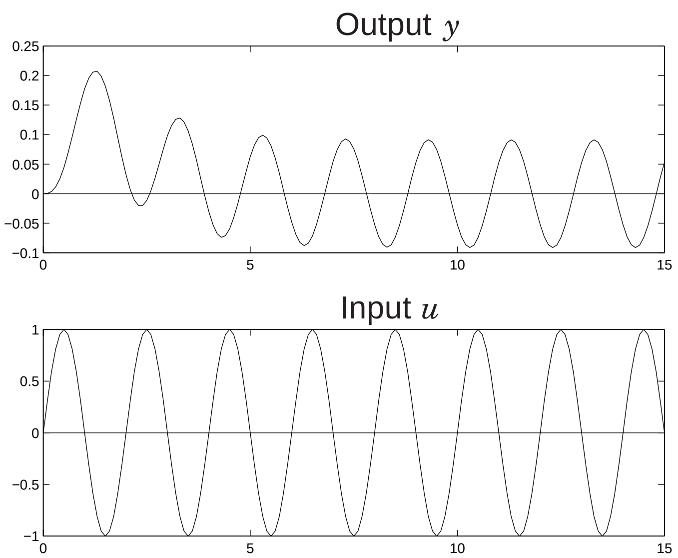
Frequency Response

The complex number $G(i\omega)$ tells how a sinusoid propagates through the system *in steady state*. If the input is $u(t) = \sin \omega t$, then the output is

$$y(t) = |G(i\omega)| \sin\big(\omega t + \arg G(i\omega)\big)$$

The number $|G(i\omega)|$ is called the gain and the number $\arg G(i\omega)$ is called the phase of the transfer function.

Notice Steady State Responses

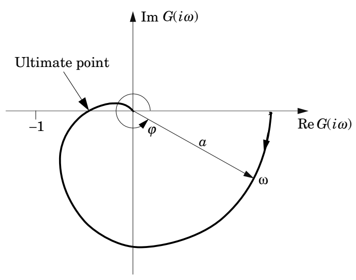


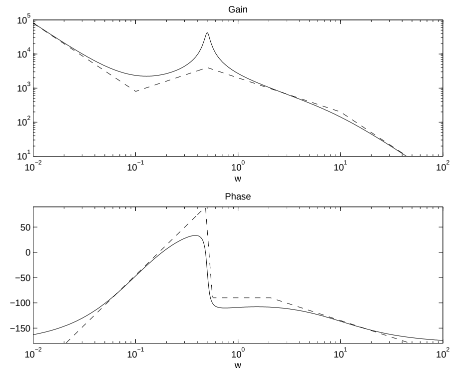
Graphical Representations

The complex function $G(i\omega)$ ($G : R \to C$) can be represented in many ways.

Bode plot

Nyquist plot





The Bode Plot

- Easy to sketch using asymptotes
- Gives a quick overview of a transfer function
- Very useful to learn how to interpret it
- The concepts of minimum phase and non-minimum phase
- Relations between gain and phase curves forminimum phase systems

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega},$$

- Minimum phase i.e. time delays and poles and zeros in the right half plane imply serious limitations! Try to redesign the system!

The Concept of Minimum Phase

A system is called a *minimum phase* system if all its poles and zeros are in the left half plane. Minimum phase systems are easy to control.

For minimum phase systems the phase curve is given by the gain curve and vice versa. An approximate relation is

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega},$$

A slope of one for the gain curve corresponds to 90° phase. The exact relations are called Bodes relations. Systems that are not minimum phase are called *non-minimum phase*. The property of non-minimum phase imposes severe limitations to what can be achieved by control.

Summary of Limitations - Part 1

- A RHP zero z

$$\frac{\omega_{gc}}{z} \leq \begin{cases} 0.5 & \text{for } M_s,\ M_t < 2 \\ 0.2 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A time delay T

$$\omega_{gc}T \leq \begin{cases} 0.7 & \text{for } M_s,\ M_t < 2 \\ 0.37 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

- A RHP pole p

$$\frac{\omega_{gc}}{p} \geq \begin{cases} 2 & \text{for } M_s,\ M_t < 2 \\ 5 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

Summary of Limitations - Part 2

- A RHP pole-zero pair with $z > p$

$$\frac{z}{p} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

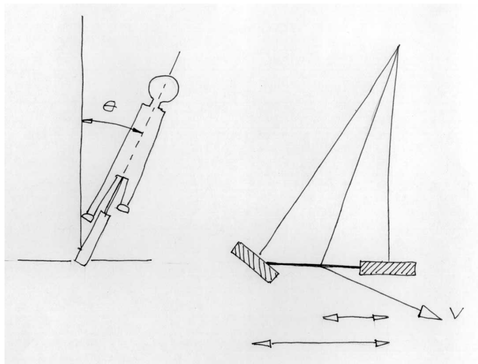
- A RHP pole-zero pair with $z < p$

$$\frac{p}{z} \geq \begin{cases} 6.5 & \text{for } M_s,\ M_t < 2 \\ 14.4 & \text{for } M_s,\ M_t < 1.4 \end{cases}$$

- A RHP pole p and a time delay T

$$pT \leq \begin{cases} 0.16 & \text{for } M_s,\ M_t < 2 \\ 0.05 & \text{for } M_s,\ M_t < 1.4. \end{cases}$$

Bicycle with Rear Wheel Steering



The tilt equation (kinematics + balance of angular momentum)

$$J\frac{d^2\theta}{dt^2} = mg\ell \sin\theta + m\ell\Big(\frac{V^2}{r}\cos\alpha - \frac{dV_y}{dt}\Big)\cos\theta$$

Compare Front and Rear Wheel Steering

Rear wheel steering:

$$J\frac{d^2\theta}{dt^2} = mg\ell \sin\theta + m\ell\Big(\frac{V^2}{r}\cos\alpha - \frac{dV_y}{dt}\Big)\cos\theta$$

Front wheel steering:

$$J\frac{d^2\theta}{dt^2} = mg\ell \sin\theta + m\ell\Big(\frac{V^2}{r}\cos\alpha + \frac{dV_y}{dt}\Big)\cos\theta$$

The Linearized Tilt Equation for Rear Wheel Steering

The linearized equation becomes

$$\frac{d^2\theta}{dt^2} = \frac{mg\ell}{J}\theta - \frac{am\ell V_0}{bJ}\frac{d\beta}{dt} + \frac{m\ell V_0^2}{bJ}\beta = \frac{mg\ell}{J}\theta + \frac{m\ell V_0^2}{bJ}\Big(-\frac{a}{V_0}\frac{d\beta}{dt} + \beta\Big)$$

The transfer function of the system is

$$P(s) = \frac{am\ell V_0}{bJ}\frac{-s + \dfrac{V_0}{a}}{s^2 - \dfrac{mg\ell}{J}}$$

One pole and one zero in the right half plane.

Transfer Function for Rear Wheel Steering

The ratio of RHP zero and RHP pole is

$$\frac{z}{p} = \frac{V_0\sqrt{J}}{a\sqrt{mg\ell}} = \frac{V_0\sqrt{J_{cm} + m\ell^2}}{a\sqrt{mg\ell}}$$

The system has an uncontrollable unstable pole if the ratio is one. The system is difficult to control robustly if the ratio is in the range of 0.25 to 4.

To make the ratio achieve large values quickly can

- Make a small by leaning forward
- Make V_0 large by biking fast (takes guts)
- Make J large by standing upright
- When the velocity is sufficiently large you can move to the seat.

Summary

- Dynamics is a very rich field that is fundamental for automatic control
- An essential part of the language of control with many useful concepts and tools
- Useful to have different views
  - State models and matrices very useful for designing control systems (state feedback and observers) and for effective computation
  - Transfer functions very useful for simple control designs and for evaluation of control performance. Also very useful to describe time delays and distributed parameter systems (PDE's) and to express model uncertainty.
- Essential to master dynamical systems and understand relations between different representations