Lecture 4 - Simple Design Problems

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1. Introduction
2. Control of First Order Systems
3. Control of Second Order Systems
4. Matlab
5. Summary

Theme: Solve simple design problems in a unified way. Dynamics of simple closed loop systems. Matlab.

1. Introduction

- The role of abstractions

  Solve an abstract problem and you have the solution to many concrete problems

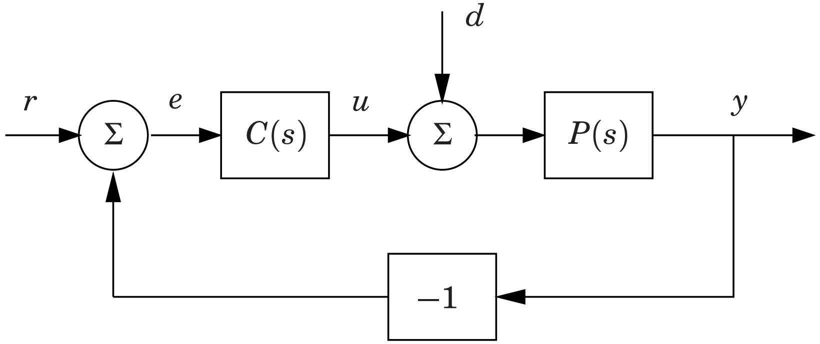
  But you must be able to translate!

  Technology transfer
- Standard process models in the form of transfer functions

$$P(s) = \frac{b}{s+a}, \qquad P(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

- Essential that storage of mass, momentum and energy can be captured by a few parameter. Description of storage more complicated if variations are rapid.
- Moderate requirements of control performance

A Simple Feedback Loop



Design controllers for the processes

$$P(s) = \frac{b}{s+a}, \qquad P(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

2. Control of First Order Systems

- Many systems are approximately of first order
- The key is that the storage of mass, momentum and energy can be captured by one parameter
- Examples
  - Velocity of car on the road
  - Control of velocity of rotating system
  - Electric systems where energy storage is essentially in one capacitor or on inductor
  - Incompressible fluid flow in a pipe
  - Level control of a tank
  - Pressure control in gas tank
  - Temperature in a body with essentially uniform temperature distribution (e.g. steam filled vessel)

A Simple Recipe

- Transform the physical problem to a standard model e.g.

$$P(s) = \frac{b}{s+a}$$

- Pick a controller e.g. PI

$$C(s) = k + \frac{k_i}{s}$$

- Closed loop system is of second order
- Design a controller where the closed loop characteristic equation has specified poles (pole placement)
- Translate results back to the physical system

PI Control of First Order Systems

Process transfer function

$$P(s) = \frac{b}{s+a}$$

Controller transfer function

$$C(s) = k + \frac{k_i}{s}$$

Closed loop transfer function from reference r to output y

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{b(ks+k_i)}{s^2+(a+bk)s+bk_i}$$

Closed loop system of second order, PI controller has two parameters.

Interpretation

Consider the equation

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{b(ks+k_i)}{s^2+(a+bk)s+bk_i}$$

Rewrite it as

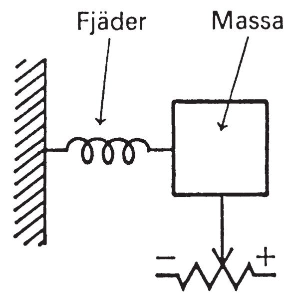
$$(s^2+(a+bk)s+bk_i)Y(s) = b(ks+k_i)R(s)$$

Interpret in time-domain

$$\frac{d^2y}{dt^2} + (a+bk)\frac{dy}{dt} + bk_iy = bk\frac{dr}{dt} + bk_ir$$

The closed loop poles can be given arbitrary values. PI control sufficient.

The Mass-Spring-Damper System



The differential equation for a mass-spring-damper system is given by

$$m\frac{d^2y}{dt^2} + d\frac{dy}{dt} + ky = 0$$

It is useful to make an analogy with this system in order to understand second order systems

The Mass-Spring-Damper Analogy

Closed loop control system

$$\frac{d^2y}{dt^2} + (a + bk)\frac{dy}{dt} + bk_i y = bk\frac{dr}{dt} + bl_i r$$

Mass-spring-damper equation

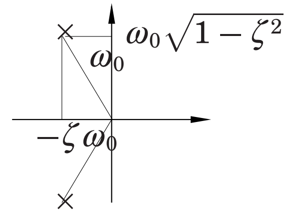
$$m\frac{d^2y}{dt^2} + d\frac{dy}{dt} + ky = 0$$

- Integral gain k_i gives spring action
- Proportional gain k gives damping action

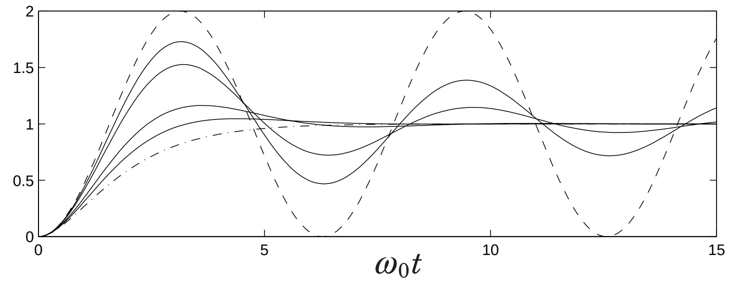
Behavior of Second Order System

The system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



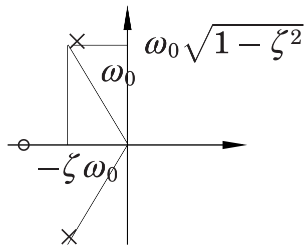
has the step response $\zeta = 0$ (dashed), 0.1, 0.2, 0.5, 0.707, 1 (dashed-dotted)



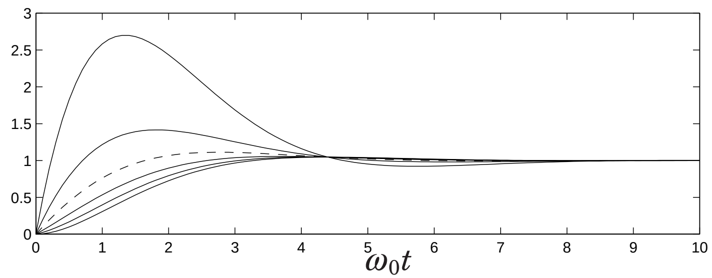
Behavior of Second Order System with a Zero

The system with $\zeta = 0.7$

$$G(s) = \frac{\omega_0}{\alpha}\frac{s + \alpha\omega_0}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



has the step response $\alpha = 0.2$, 0.5, 1 (dashed), 2, 5 and ∞.



Behavior of Second Order System with a Zero

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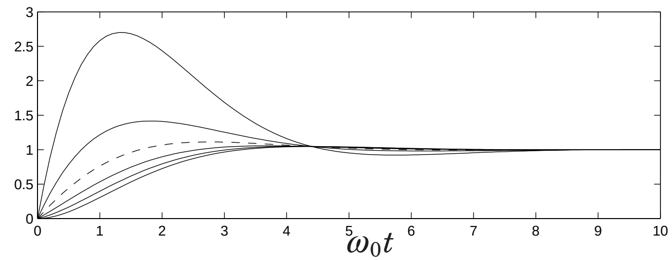


Why do all curves intersect at the same point?

The Audience is Thinking ...

3. Control of Second Order Systems

- Storage of mass, momentum and energy can be captured by two parameter
- Examples
  - Position of car on the road
  - Control of angle of rotating system
  - Stabilization of satellites
  - Electric systems where energy is stored in two elements (inductors or capacitors)
  - Levels in two connected tanks
  - Pressure in two connected vessels

PI Control of Second Order Systems

Process

$$P(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

Controller

$$C(s) = k + \frac{k_i}{s}$$

Closed loop transfer function from reference r to output y

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{(ks + k_i)(b_1 s + b_2)}{s^3 + (a_1 + b_1 k)s^2 + (a_2 + b_1 k_i + b_2 k)s + b_1 k_i}$$

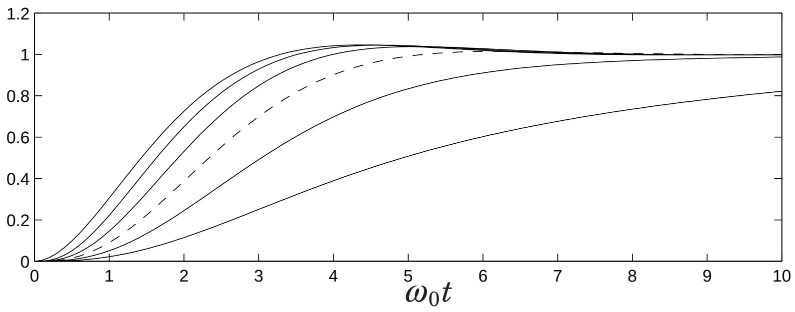
Closed loop system of third order, controller has only two parameters. Not enough degrees of freedom. A more complex controller is required to choose closed loop characteristic polynomial.

Behavior of Third Order System

The system with $\zeta = 0.7$

$$G(s) = \frac{\alpha\omega_0^3}{(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s + \alpha\omega_0)}$$

has the step response $\alpha = 0.2$, 0.5, 1 (dashed), 2, 5 and ∞.



PD Control of Second Order System

Process

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}$$

Notice that polynomial B is of zero degree. Controller

$$C(s) = k + k_d s$$

Closed loop transfer function from reference to output

$$\begin{aligned}\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} &= \frac{b(k_d s + k)}{s^2 + a_1 s + a_2 + b(k_d s + k)}\\ &= \frac{b(k_d s + k)}{s^2 + (a_1 + b k_d)s + a_2 + bk}\end{aligned}$$

Closed loop system of second order, controller has two parameters. All closed loop poles can be chosen.

PD Control of Second Order System ...

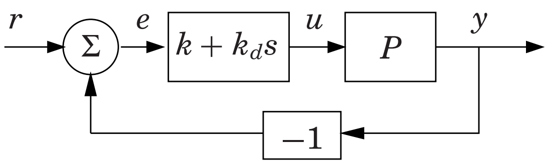
The closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{b(k_d s + k)}{s^2 + a_1 s + a_2 + b(k_d s + k)}$$

The zero $s = -k/k_d$ may cause a considerable overshoot if it is closer to the origin than the dominant poles.

This zero is caused by the controller. Can the controller be modified so that the zero is changed or removed?

The Audience is Thinking ...



Solution

The controller with error feedback

$$U(s) = kE(s) - k_d s E(s)$$

gives

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{b(k_d s + k)}{s^2 + a_1 s + a_2 + b(k_d s + k)}$$

But the controller to

$$U(s) = kE(s) - k_d s Y(s) = kR(s) - (k + k_d s)Y(s)$$

gives

$$\frac{Y(s)}{R(s)} = \frac{k}{s^2 + (a_1 + k_d)s + a_2 + k}$$

Controller with Two Degrees of Freedom

The controller

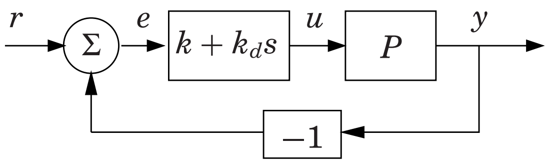
$$U(s) = kE(s) - k_d s Y(s) = kR(s) - (k + k_d s)Y(s)$$

is said to have *two degrees of freedom* because the signal path from reference R to control U is different from signal path from output Y to control U. Derivative acts only on the output.

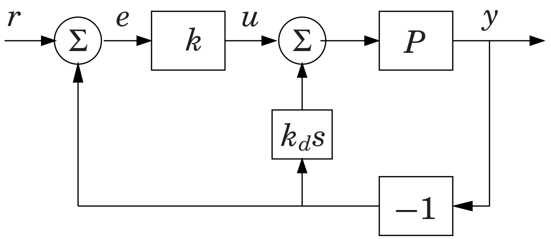
We will talk a lot about this later.

Block Diagram Representations

System with error feedback



Controller with two degrees of freedom



PID Control of Second Order System

Process: $P(s) = \dfrac{b}{s^2 + a_1 s + a_2}$

Controller: $C(s) = k + \dfrac{k_i}{s} + k_d s$

Closed loop transfer function from reference r to output y

$$\begin{aligned}\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} &= \frac{b(k_d s^2 + ks + k_i)}{s(s^2 + a_1 s + a_2) + b(k_d s^2 + ks + k_i)}\\ &= \frac{b(k_d s^2 + ks + k_i)}{s^3 + (a_1 + bk_d)s^2 + (a_2 + bk)s + bk_i}\end{aligned}$$

Closed loop system of third order, controller has three parameters. All closed loop poles can be chosen arbitrarily. Since the zeros comes from the controller they can also be chosen arbitrarily by choosing a structure with two degrees of freedom!

4. Computational Tools

Analysis and design of control system has been drastically influenced by good computational tools. A sensible approach:

- Construct a block diagram
- Derive the transfer functions for the blocks
- Calculate transfer functions between interesting signals
- Explore transfer functions qualitatively
- Use Matlab to simulate the responses of the system
- Can be done in many ways

Matlab

Matlab was designed as an interactive calculator for matrices. The matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is represented as

```
M=[1 2 3;4 5 6]
```

There are many operations for matrices. Use the command `help matrices` to find out more about them.

Polynomials

Polynomials are represented in the same way as matrices. The polynomial

$$A(s) = s^3 + 2s^2 + 3s + 4$$

is represented as

```
A=[1 2 3 4]
```

The matrix operations also apply to polynomials (overloading, polymorphism). There are also special operations for polynomials, for example `roots(A)` give the roots of the polynomial.

Transfer Functions

When polynomials are available it is also possible to represent transfer functions that are rational functions. The transfer function

$$P(s) = \frac{s+2}{s^2+5s+6}$$

is represented as

```
P=tf([1 2],[1 5 6])
```

The regular operations of addition, multiplication and division also apply to transfer functions (overloading, polymorphism).

Generating Step Responses

```
t=0:0.1:6.5*pi;                 %Generate plotting times
z=0;                            %Set damping parameter
sys=tf(1,[1 2*z 1]);            %Generate system ``sys''
y0=step(sys,t);                 %Calculate step response
z=0.1;                          %Change damping parameter
sys=tf(1,[1 2*z 1]);            %Generate system
y1=step(sys,t);                 %Generate step response
...
plot(t,y0,'r--',t,y1,'b-')      %Plot results
axis([0 15 0 2])                %Give lengths of axes
xlabel('x')                     %Add labels
ylabel('y')                     %Add labels
```

A Slick Way to Work with Systems

Transfer functions is a natural interface to Matlab. Here is one way to compute responses.

```
s=tf([1 0],1) %Define the transfer function s
P=1/(s*(s+2)) %Define process transfer function
C=2+1/s       %Define the controller transfer function
T=P*C/(1+P*C) %Compute transfer function from
              %reference to output
step(T)       %Compute and plot the step response
tzero         %Compute zeros of the transfer function
pole(T)       %Compute poles of the transfer function
              %Notice that there are cancellations
```

This is perhaps the most natural way to work with transfer functions, but may be dangerous because Matlab is not smart enough to recognize cancellations of poles and zeros.

Summary

- Draw block diagram
- Obtain simple process model
- Select a controller of sufficient flexibility (complexity)
- Derive the closed loop characteristic equation
- Pick controller parameters to give desired characteristic polynomial (pole placement). Typical prototypes:

  $$s^2 + 2\zeta\omega_0 s + \omega_0^2$$
  $$(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s + \omega_0)$$
- Behavior of second and third order systems
- The effect of poles and zeros
- Some zeros are due to the controller, they can be modified or eliminated if the controller has two degrees of freedom.