Lecture 3 - Laplace Transforms

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Review of control system analysis

1. The Basic Feedback Loop
2. Laplace Transforms
3. Analysis of Feedback Loops
4. Qualitative Understanding of Signals and Systems
5. Summary

Theme: Streamline manipulation of equations and block diagrams.

A General Method

- Get an overview of the system by drawing the block diagram.
- Make an assessment of reasonable approximations.
- Use standard model to describe the dynamics of the individual blocks.
- Pick a controller PI, PD or PID.
- Derive relations between interesting signals.
- Use simple analysis to make a preliminary assessment.
- Use computers to compute representative responses.

Construction of a Block Diagram

The block diagram gives an overview. To draw a block diagram:

- Understand how the system works.
- Identify the major components and the relevant signals.
- Key questions:

  Where is the essential dynamics?

  What are appropriate abstractions?
- Describe the dynamics of the blocks in terms of standard models.

Standard Model 1

A standard model for linear time invariant system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u$$

is characterized by two polynomials

$$A(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n$$
$$B(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_{n-1}s + b_n$$

- The roots of $A(s)$ are called poles of the system.
- The roots of $B(s)$ are called zeros of the system.
- The transfer function of the system is $G(s) = \dfrac{B(s)}{A(s)}$

Linear Time Invariant Systems (LTI)

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

has the solution

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t} + \int_0^t g(t-\tau) u(\tau) d\tau$$

where $\{\alpha_k\}$ are roots of the characteristic equation A, $C_k(t)$ polynomials and g is the *impulse response*, which has the form

$$g(t) = \sum_{k=1}^{n} \bar{C}_{k-1}(t) e^{\alpha_k t}$$

Notice appearance of α_k again!

Interpretation of the Impulse Response

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{\alpha_k t} + \int_0^t g(t-\tau) u(\tau) d\tau$$

Let the system be initially at rest, i.e. $C_k = 0$ and let the input be an impulse at time 0. The output is then

$$y(t) = g(t)$$

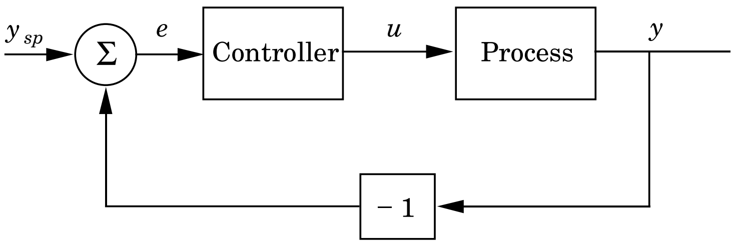
If the input is a unit step the output (the step response) is

$$y(t) = \int_0^t g(t-\tau) d\tau = \int_0^t g(\tau) d\tau$$

Experimental determination of step and impulse responses.

2. The Basic Feedback Loop

To analyze the feedback system



where the process and the controller are described as linear time invariant systems we must manipulate the equations. We will now introduce methods that drastically simplify the manipulations.

The methods will also give qualitative insight into the behavior of systems.

Recall Cruise Control

Process model

$$\frac{dv}{dt} + 0.02v = u - 10\theta$$

PI controller

$$u = k(v_r - v) + k_i \int_0^t (v_r - v(\tau)) d\tau$$

The closed loop system is described by (differentiate both equations and add them), introduce $e = v_r - v$

$$\frac{d^2 e}{dt^2} + (0.02 + k)\frac{de}{dt} + k_i e = 10\frac{d\theta}{dt}$$

The mathematical tool of Laplace transforms is ideally suited for these type of calculations. An essential part of the language of control.

3. Laplace Transforms

Consider a function f defined on $0 \le t < \infty$ and a real number $\sigma > 0$. Assume that f grows slower than $e^{\sigma t}$ for large t. The Laplace transform $F = \mathcal{L}f$ of f is defined as

$$\mathcal{L}f = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Example 1:

$$f(t) = 1, \qquad F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

Example 2:

$$f(t) = e^{-at}, \qquad F(s) = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-st}\Big|_0^\infty = \frac{1}{s+a}$$

$$\mathcal{L}f = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Transform of the derivative

$$\mathcal{L}\frac{df}{dt} = \int_0^\infty e^{-st} f'(t)dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt = -f(0) + s\mathcal{L}f$$

Initial value theorem

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\int_0^\infty se^{-st}f(t)dt = \lim_{s\to\infty}\int_0^\infty e^{-v}f(\frac{v}{s})dv = f(0)$$

Final value theorem

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\int_0^\infty se^{-st}f(t)dt = \lim_{s\to 0}\int_0^\infty e^{-v}f(\frac{v}{s})dv = f(\infty)$$

Behavior of $f(t)$ for small t is similar to behavior of $sF(s)$ for large s and vice versa!

Properties

Linearity: $\mathcal{L}(af + bg) = a\mathcal{L}f + b\mathcal{L}g$

Differentiation: $\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0)$

Integration: $\mathcal{L}\int_0^t f(\tau)d\tau = \frac{1}{s}\mathcal{L}f$

Time shift: $\mathcal{L}f(t-T) = e^{-sT}\mathcal{L}f$

Time stretch: $\mathcal{L}f(at) = \frac{1}{a}F(\frac{s}{a})$, a>0.

Convolution: $\mathcal{L}\int_0^t f(t-\tau)g(\tau)d\tau = F(s)G(s)$

Final value Theorem †: $\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$

Initial value Theorem †: $\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$

- †: *Valid only if limits exist!*

Inverse Transforms

A simple way to find time functions corresponding to a rational Laplace transform. Write $F(s)$ in a partial fraction expansion

$$F(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{(s-\alpha_1)(s-\alpha_2)\dots(s-\alpha_n)} = \frac{C_1}{s-\alpha_1} + \frac{C_2}{s-\alpha_2} + \dots + \frac{C_n}{s-\alpha_n}$$

$$C_k = \lim_{s\to\alpha_k}(s-\alpha_k)F(s) = \frac{B(\alpha_k)}{(\alpha_k-\alpha_1)\dots(\alpha_k-\alpha_{k-1})(s-\alpha_{k+1})\dots(\alpha_k-\alpha_n)}$$

The time function corresponding to the transform is

$$f(t) = C_1e^{\alpha_1 t} + C_2e^{\alpha_2 t} + \dots + C_ne^{\alpha_n t}$$

Parameters α_k give shape and numbers C_k give magnitudes.

Notice that α_k may be complex numbers. With multiple roots the constants C_k are instead polynomials.

Manipulating LTI Systems

The differentiation property $\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0)$ makes the Laplace transform very convenient for dealing with LTI systems, particularly if all initial values are zero. Differentiation of the time functions then simply corresponds to multiplication of the transform with s. We then obtain the following recipe for dealing with linear systems:

- Take Laplace transforms of the equations
- Take Laplace transforms of the signals acting on the system
- Solve linear algebraic equations to obtain the transforms of the interesting signals
- Convert the Laplace transform to a time function

Cruise Control

Process model: $\dfrac{dv}{dt} + 0.02v = u - 10\theta$

PI controller: $u = k(v_r - v) + k_i \int_0^t (v_r - v(\tau))d\tau$

Taking Laplace transforms

$$
\begin{aligned}
(s + 0.02)V(s) &= U(s) - 10\Theta(s)\\
E(s) &= V_r(s) - V(s)\\
U(s) &= kE(s) + \frac{k_i}{s}E(s)
\end{aligned}
$$

Pure algebra gives relation between Laplace transforms of slope Θ reference V_r and E by eliminating V and U

$$\big(s(s + 0.02) + ks + k_i\big)E(s) = 10s\Theta(s) + s(s + 0.02)V_r(s)$$

More Cruise Control

$$\big(s(s + 0.02) + ks + k_i\big)E(s) = 10s\Theta(s)$$

With the chosen controller $k = 2\zeta\omega_0 - 0.02$ ($\zeta = 1$) and $k_i = \omega_0^2$ and a step change of magnitude θ_0 in slope we have

$$\Theta(s) = \theta_0/s$$

and we find that the Laplace transform of the error is

$$E(s) = \frac{10s}{s^2 + 2\omega_0 s + \omega_0^2}\Theta(s) = \frac{10\theta_0}{(s + \omega_0)^2}$$

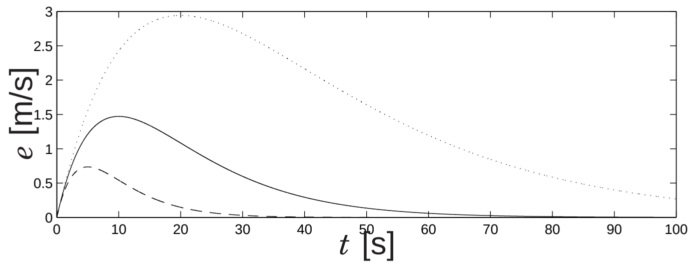
Converting this to a time function we get

$$e(t) = \frac{10\theta_0}{\omega_0} t e^{-\omega_0 t}$$

More Cruise Control ...

$$e(t) = \frac{10\theta_0}{\omega_0} t e^{-\omega_0 t}$$

The largest error $e_{max} = 10\theta_0 e^{-1}$ occurs for $t = 1/\omega_0$. Compare graph below $\theta_0 = 0.04$ $\zeta = 1$, $\omega_0 = 0.05$ (dotted), $\omega_0 = 0.1$ (solid) and $\omega_0 = 0.2$ (dashed)



What can we conclude?

Discussion

- What do we mean by a solution to a problem?
- A historical perspective

  Closed form expressions, tables, curves
- The role of computers
- The necessity of insight and understanding
- The need to check results
- What properties can we find easily using "back of an envelope" calculation.
- A perspective on use of Laplace transforms in control engineering
- A more general (*biased personal*) perspective. Technology changes fast but engineering education changes slowly.

4. Analysis of Feedback Loops

The transfer function of an LTI system was introduced in Lecture 2. Using Laplace transforms it can also be defined as follows. Consider an LTI system with input u and output y. The transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\mathcal{L}y}{\mathcal{L}u}$$

where *the Laplace transforms are calculated under the assumption that all initial values are zero.*

Transfer functions and Laplace transforms are ideal to deal with block diagram. A block is simply characterized by

$$Y(s) = G(s)U(s)$$

Signals and and systems have the same representations.

Car Model in Cruise Control

Process model

$$\frac{dv}{dt} + 0.02v = u - 10\theta$$

Transfer function from control u to velocity v

$$G_{vu}(s) = \frac{V(s)}{U(s)} = \frac{1}{s + 0.02}$$

Transfer function from slope θ to velocity v

$$G_{v\theta}(s) = \frac{\Theta(s)}{U(s)} = -\frac{10}{s + 0.02}$$

Transfer Function of PID Controller

The error e is the input and the control signal u is the output

$$u = ke + k_i \int_0^t e(\tau))d\tau + k_d \frac{de}{dt}$$

Transfer function

$$G(s) = \frac{U(s)}{E(s)} = k + \frac{k_i}{s} + k_d s$$

Transfer Function of Car

A simple model of a car on a horizontal road tells how its position y depends on the throttle. Let the mass be m and assume that the propelling force is proportional to the throttle we find

$$m\frac{d^2y}{dt^2} = F = ku$$

The transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{ms^2}$$

With suitable units we get $G(s) = s^{-2}$ a double integrator.

Transfer Function of Time Delay

Consider a system where the output y is the input u delayed T time units. The input output relation is

$$y(t) = u(t - T)$$

and the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = e^{-sT}$$

Transfer Function of Standard Model

Consider the system

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_ny = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_nu$$

Assuming that y and u and all their derivatives are zero initially. Taking Laplace transforms we get

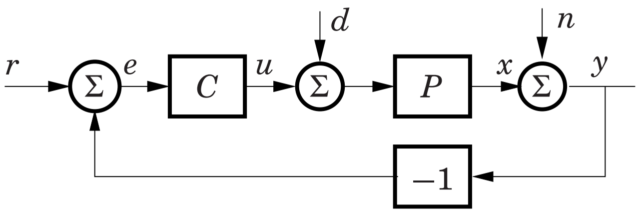
$$\begin{aligned}&(s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n)Y(s)\\ &\quad = (b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n)U(s)\end{aligned}$$

The transfer function is

$$G(s) = \frac{b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n}$$

Working with Block Diagrams

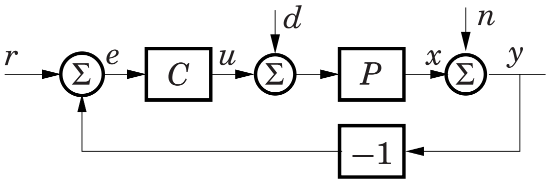
Consider the system



How is the error e related to the signals r d and n?

The Audience is Thinking ...

Solution



Introduce Laplace transforms and transfer functions. We have

$$E = R - \big(N + P(D + CE)\big)$$

Solving for E gives

$$E = \frac{1}{1+PC}R - \frac{1}{1+PC}N - \frac{P}{1+PC}D$$

Notice *form of the equations* and use of *superposition*.

5. Qualitative Understanding of Signals and Systems

Time responses can in principle be computed. Tables of Laplace transforms is a help but the work is quite tedious. Time responses are easy to compute using different types of software.

- It is a good rule to always make order of magnitude calculations to make sure that results are reasonable whenever you use software.
- Much insight can be obtained form very simple calculations (series expansions and factorization).
- Some results will be presented.
- It will be discussed more in future lectures

Insight into Signals

Consider a signal specified by a rational Laplace transform $Y(s)$. Make a partial fraction expansion

$$Y(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)} = \frac{C_1}{s-\alpha_1} + \frac{C_2}{s-\alpha_2} + \ldots + \frac{C_n}{s-\alpha_n}$$

$$C_k = \lim_{s\to\alpha_k}(s-\alpha_k)F(s) = \frac{B(\alpha_k)}{(\alpha_k-\alpha_1)\ldots(\alpha_k-\alpha_{k-1})(s-\alpha_{k+1})\ldots(\alpha_k-\alpha_n)}$$

Parameters α_k (roots of $A(s)$) are easy to compute. The signal $y(t)$ has the form

$$y(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + \ldots + C_n e^{\alpha_n t}$$

Parameters α_k give shape and C_k give magnitudes.

Example

Consider the signal

$$Y(s) = \frac{B(s)}{A(s)} = \frac{s+5}{s(s+1)(s+2)}$$

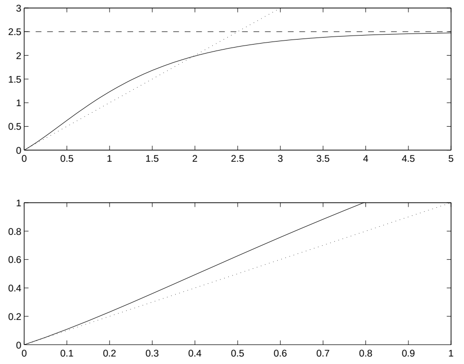
The polynomial $A(s)$ is already in factored form and its zeros are 0, -1 and -2. This means that the time function corresponding to $Y(s)$ has the components

$$\text{constant}, \quad e^{-t}, \quad \text{and} \quad e^{-2t}$$

the second and third component decay exponentially and after a short time only the constant components remains. The amplitude of the constant component is 2.5. ($\lim_{s\to 0} sY(s)$).

Example ...

For small s the Laplace transform is $Y(s) \approx 2.5/s$, which implies that for large t the time function is $y(t) \approx 2.5$. For large s we have $Y(s) \approx 1/s^2$. This means that for small t the time function is approximately $y(t) \approx t$.



Example

Consider the signal

$$Y(s) = \frac{B(s)}{C(s)} = \frac{s+5}{s^4 + 5s^3 + 7s^2 + 5s + 6}$$

The Matlab command

```
A=[1 5 7 5 6]
roots(A)
```

gives the roots of $A(s)$ to be -2, -3, and $\pm i$. This means that the time function has the components

$$e^{-2t}, \quad e^{-3t}, \quad \sin t, \quad \text{and} \quad \cos t$$

the first components decay exponentially and after a short time only the periodic components remain.

Insight from Transfer Functions

- Derive transfer function $G(s) = \dfrac{B(s)}{A(s)}$
- Compute poles α_k (roots of $A(s) = 0$)

  Free motion of system has component $Ce^{\alpha_k t}$
- Compute zeros β_k (roots of $B(s) = 0$)

  The system blocks transmission of the signal $e^{\beta_k t}$
- Compute static gain $G(0)$
- Look at behavior for small s (large t, low frequencies) and large s (small t, high frequencies)

Insight from Transfer Functions

- Make a series expansion of $G(s)$ for small s (low frequency behavior, large t)

$$G(s) = \frac{c_{-1}}{s} + c_0 + c_1 s + \ldots + c_k s^k + \ldots$$

  If $c_{-1} \neq 0$ like an integrator
  If $c_{-1} = 0$ and $c_0 \neq 0$ like a static gain.
  If $c_0 = c_2 = \ldots = c_{k-1} = 0$ and $c_k \neq 0$ like k differentiators.

- Make a series expansion of $G(s)$ for large s (high frequency behavior, small t)

$$G(s) = c_0 + \frac{c_{-1}}{s} + \frac{c_{-2}}{s^2} + \ldots + \frac{c_k}{s^k} + \ldots$$

  If $c_0 \neq 0$ like a static gain.
  If $c_{-1} = 0$ and $c_{-1} \neq 0$ like an integrator.
  If $c_{-1} = c_{-2} = \ldots = c_{-k+1} = 0$ and $c_k \neq 0$ like k integrators.

Examples

EXAMPLE 1—PID CONTROLLER
The PID controller has the transfer function

$$G(s) = k + \frac{k_i}{s} + k_d s = \frac{k_i}{s} + k + k_d s$$

This is already in series expansion form. For slow signals (small s) it behaves like an integrator and for fast signals (large s) it behaves like a differentiator.

EXAMPLE 2—PID CONTROLLER WITH DERIVATIVE FILTER

$$G(s) = k\Big(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_d/N}\Big)$$

Behaves like a static gain $k(1+N)$ for fast signals (large s).

6. Summary

- Finding relations between signals in linear time invariant systems is very simple by using Laplace transforms
- Fits very well to the block diagram description
- It is natural to think in terms of transfer functions
- The similarity of signals and systems
- Qualitative reasoning (small s corresponds to large t and large s correspond to small t) is very useful to get a quick insight into the behavior of signals and systems