Analysis Of A Pendulum Problem

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Abstract

The ball-balancer, or cart-ball system, demonstrates some basic concepts in control being nonlinear, multivariable, and non-minimum phase. It is basically an inverted pendulum problem, which is a much used benchmark problem. The objective here is to provide an analysis of the system, which can be the basis for designing different kinds of controllers.

Contents

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Tech. report no 98-E 863 (cartball), 19 Aug 1998.

1. Introduction

The ball-balancing system in this paper consists of a cart with an arc made of two parallel pipes on which a steel ball rolls. The cart moves on a pair of tracks horizontally mounted on a heavy support (Fig. 1). The control objective is to balance the ball on the top of the arc and at the same time place the cart in a desired position. The cart-ball system was built for teaching electrical engineers about automatic control, originally with a focus on state-space control theory. It is educational, because the laboratory rig is sufficiently slow for visual inspection of different control strategies and the mathematical model is sufficiently complex to be challenging.

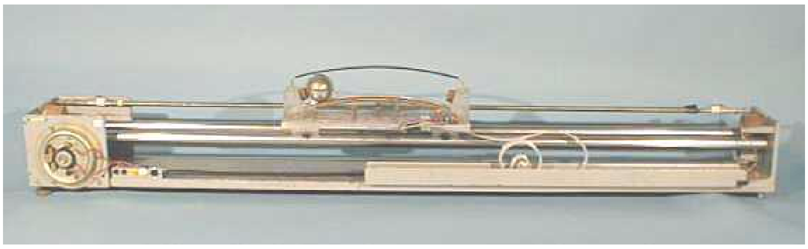

Figure 1: Laboratory rig.

The system was built during an M.Sc. project, and later the mathematical model was published in an educational journal (Jørgensen, 1974). A description, which includes equations, can be found in that paper, but it has a few errors, and much emphasis is on state-space concepts. A simulator of the same system was built in Matlab much later for a course on the Internet (Jantzen, 1996b; Jantzen & Dotoli, 1998). The simulator is based on a linearised model of the system,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{1}$$

The model is a linear state-space model, where $\mathbf{x}$ is a vector of state variables, $\dot{\mathbf{x}}$ is the vector time derivative $d\mathbf{x}/dt$, $\mathbf{u}$ is a vector of inputs to the system, and $\mathbf{y}$ is a vector of output variables. The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices of appropriate dimensions containing real numbers.

What is needed is a description and derivation of the cart-ball model, which can be downloaded from the World Wide Web, and possibly function as a benchmark problem for controller design.

Our approach is to develop the mathematical model from *first principles*, i.e., the basic laws of physics. Then to linearise the model in order to make it easier to discuss possible controller configurations.

2. The Laboratory Rig

The laboratory rig (Fig. 1), 1.5 meters long, is equipped with a power supply and equipment for both analog and digital control.

By pushing the cart left and right manually, it is possible to get the ball on top of the arc, but it is impossible to position the cart at a particular position at the same time. An automatic control system can do that, however. The cart position and ball angle from vertical are measured variables, and the manipulated variable is the horizontal force acting on the cart.

The ball rolls on curved pipes, one of which is made of aluminium while the other is a coil of resistance wire. The ball's angle from vertical is determined by measuring its position on the pipes. The ball, being made of steel, connects the pipes electrically, and acts as a voltage divider producing a voltage proportional to the position (Fig 2). The cart position is measured the same way using a carbon wheel contact, mounted on the cart, which rolls on a coil alongside the rails.

The rails are cylindric bars mounted on the support, and the cart wheels are small, low-friction ball-bearings which roll on the bars. A wire pulls the cart, passing over a pulley in one end and a wire drum in the other end, both attached to the support. The wire drum is driven by a current-driven direct current (DC) print-motor. Although the motor is current-driven, we assume the voltage is proportional to the current and in turn that the force is proportional to the current. This is an approximation, but it is a relatively fast DC motor with small electrical and mechanical time-constants. The numerical data of the rig are given in the appendix.

In the Matlab simulator (Fig. 3), a set of working control parameters are set as default and the cart and ball can be set in motion at the push of a button. When the system has come to a rest, plots show the transient response of the cart position and velocity as well as the ball position and velocity.

3. Mathematical Model

The current to the motor is essentially a function of the controlled variables $\left(y, \dot{y}, \varphi, \dot{\varphi}\right)$ in which y is the position of the cart and φ is the angular deviation from vertical of the ball position. The velocity signals are not directly measured, but obtained by differentiation in operational amplifiers.

Newton's laws and various relationships bring out a model consisting of two coupled, nonlinear differential equations (appendix). After linearisation they can be put on the state

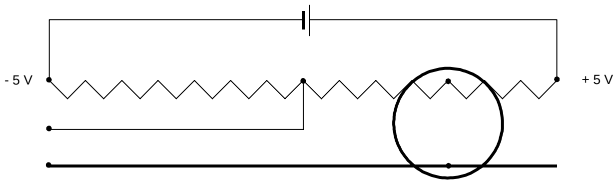

Figure 2: Ball position measurement.

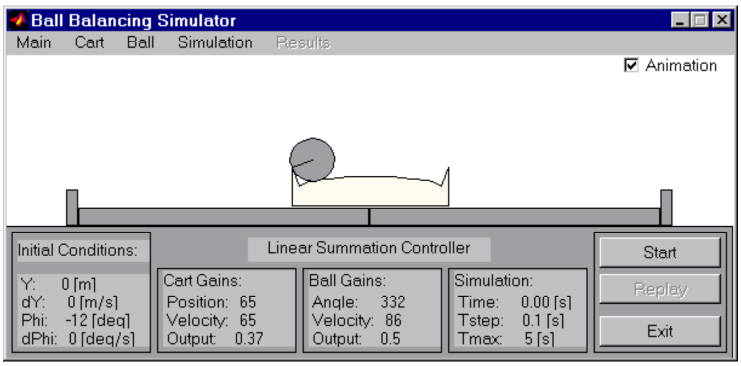

Figure 3: Matlab animation of cart-ball system.

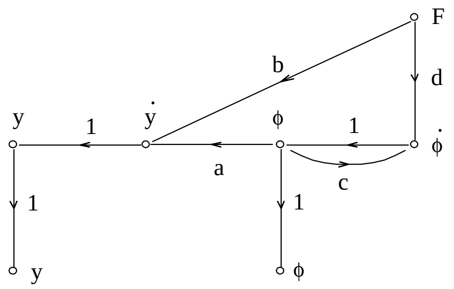

Figure 4: Signal flow in the state-space model.

space form (1), with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

The state vector **x** is the vector $\left(y, \dot{y}, \varphi, \dot{\varphi}\right)^T$ and the output vector is

$$\mathbf{y} = \begin{bmatrix} y \\ \varphi \end{bmatrix}$$

The input vector **u** consists of just one input F, the horizontal force acting on the cart, which can be substituted by the voltage U to the motor, as their relationship is 1 volt to 1 Newton; that is how the motor and the gear were designed. The constants (a, b, c, d) depend on the physical data. The matrices (2) are the basis for the Matlab simulator (Fig. 3).

An overview of the model is provided by a so-called *signal flow graph.* Given a state space model with matrices containing zeros and non-zero elements, the flow of the signals can be mapped into a directed graph, or *digraph* ; the digraph is a picture of the couplings in the model (Fig. 4).

The node set is given by an input node, four state nodes, and two output nodes; the arc set is given by the non-zero entries in the matrices. That is, if $a_{ij} \neq 0$, then there exists an arc from the jth state node to the ith state node; if $b_{ij} \neq 0$, then there exists an arc from the jth input node to ith state node; and if $c_{ij} \neq 0$, then there exists an arc from the jth state node to the ith output node. The numbers a_{ij}, b_{ij}, c_{ij} are assigned to the arcs, socalled *weights*.

If a designer decides to add a feedback connection from the output node φ to the input node F, he will create a loop $\varphi - F - \dot{\varphi} - \varphi$. If, alternatively, he adds a feedback connection from the output node y to the input node F, he will create a larger loop, $y - F - \dot{\varphi} - \varphi - \dot{y} - y$, as well as a another loop, $y - F - \dot{y} - y$. In the first case, there is no feedback from the cart to the ball, while in the second case, there is feedback through the ball into the cart. Therefore it is easier to design a ball controller than a cart controller; the ball controller can be designed and tuned independently of how the cart behaves, while a cart controller will be influenced by the ball behaviour.

Since state space models are not unique $-$ a given physical plant may be modelled by several state-space models $-$ the digraph reflects the flow of signals in the *model*, not necessarily the physical system itself. For more information about the digraph approach, see for example Jantzen (1996a).

4. Controller Configurations

Any feedback controller has to measure some or all of the state variables in the cart-ball system and derive a control signal from that. We will look at two particular configurations: a state feedback controller, and two cascaded proportional-derivative (PD) controllers. The former lends itself to mathematical analysis, while the latter is more intuitive with regard to manual tuning.

4.1 State feedback control

A state feedback controller generates a control signal

$$\mathbf{u} = \mathbf{K}\mathbf{x} \tag{3}$$

from the value of the state variables, or

$$U = k_1 y + k_2 \dot{y} + k_3 \varphi + k_4 \dot{\varphi} \tag{4}$$

Notice that the control signal is now the voltage U rather than the force F, for convenience. The block diagram in Fig. 5 shows how the four states are fed back into the controller, which combines them linearly.

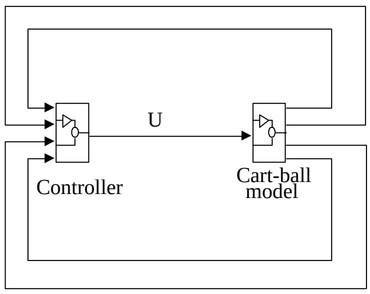

Figure 5: State feedback control.

Equation (4) can be viewed as having a cart component and a ball component, simply by placing some parentheses,

$$U = \left(k_1 y + k_2\ \dot{y}\right) + \left(k_3 \varphi + k_4\ \dot{\varphi}\right) \tag{5}$$

An equivalent block diagram will show two controllers feeding into a summation point.

The constants $k_1, \ldots, k_4$ are tuning constants, and the trick is to choose them such that the system becomes stable.

Stability can be checked mathematically. Insert (3) in (1),

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{BKx} = (\mathbf{A} + \mathbf{BK})\,\mathbf{x} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{6}$$

This is a state-space form as well, but of the closed-loop system. Stability is guaranteed if none of the eigenvalues of the closed-loop system matrix $\mathbf{A} + \mathbf{BK}$ are in the right half of the complex plane. Jørgensen (1974) went through the calculations, and his main result is that all k's must be positive. Consequently, if just one of them is zero the system will be unstable; therefore all four state variables must be available to the controller. In the lab rig, only the positions are directly measurable. The velocities are computed in hardware, however, and therefore available after all.

It is interesting to notice that the cart has positive feedback. Since k_1 must be positive, a positive (negative) position will generate a positive (negative) contribution to the control signal U. As all directions are assumed positive towards the right (appendix), the cart controller will try to push it right when it is on the right side of the centre of the track, i.e., *away* from the centre.

By trial and error, Jørgensen (1974) found the following values satisfactory,

$$\mathbf{K} = [5, 5, 120, 8]$$

With all states available, a set of gains $k_1, \ldots, k_4$ can be found using optimisation techniques, for example the `lqr` (Linear Quadratic Regulator) function in the Control Toolbox of Matlab. The optimisation requires minimising a cost function, a combined weighting of the state variables and the control signal. A fast and stable controller with little overshoot results from the following values

$$\mathbf{K} = [24, 24, 162, 44]$$

A plot of the step response is shown in Fig. 6.

4.2 Cascade control

It is quite intuitive to divide the system into two subsystems, one for the ball, another for the cart; it makes it more manageable.

The ball seems to require faster control reaction than the positioning of the cart (Jørgensen, 1974), and it is standard practice to have a fast inner loop, in this case a PD controller reacting on the ball angle, which takes commands from a slower outer loop, in this case a PD controller reacting on the cart position (Fig. 7).

The inner loop makes the ball reach its reference φ_r, preferably as quickly as possible and with as little overshoot as possible. The outer loop commands a desired angle (refer-

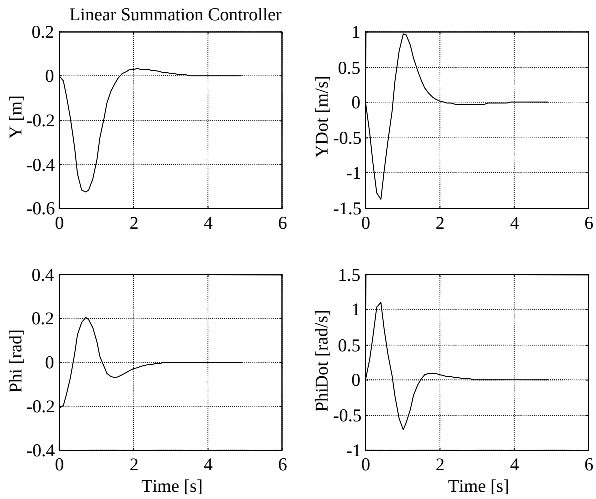

Figure 6: Step responses when $\mathbf{K} = [24, 24, 162, 44]$.

ence) φ_r of the ball controller. If the reference is, say, positive, the ball controller will try to stabilise the ball in a position on the right side of the top by accelerating the cart towards the right. In other words, the cart controller should command a positive (negative) reference, when the cart must move right (left). When the cart is near its reference, the cart controller will command $\varphi_r \approx 0$ and the ball controller will try to keep the ball near the top. In the stable point everything will be at a stand-still with $(y, \varphi) = (0, 0)$.

5. Conclusions

According to an old control engineer, a design project is 80% process knowledge and 20% controller design. If this is the case here, then this report may save the 80%, letting students concentrate on the controller design. The report will be available on the World Wide Web (reachable via `http://www.iau.dtu.dk/~jj`).

References

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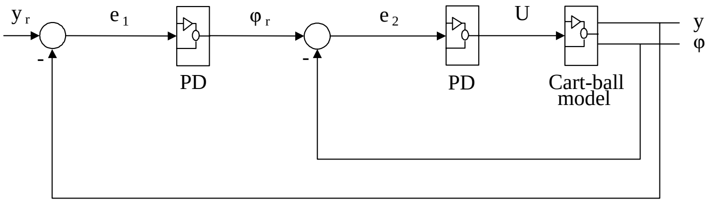

Figure 7: Cascade control.

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Appendix A. Calculations And Data

The numerical data and associated symbols for the cart-ball rig is given in Table 1 supplemented by Fig. 8. All directions are assumed positive towards the right. We will analyse the ball and cart separately and apply the basic physical equations related to the vertical reaction force V and the horizontal reaction force H. Friction forces are neglected.

- The horizontal movement of the ball

$$m\frac{d^2}{dt^2}\left[y+(R+r)\sin\varphi\right]=H \tag{A-1}$$

- The vertical movement of the ball

$$m\frac{d^2}{dt^2}\left[(R+r)\cos\varphi\right]=V-mg \tag{A-2}$$

- The rotational movement of the ball

$$I\ \ddot{\psi}=r\left(V\sin\varphi-H\cos\varphi\right) \tag{A-3}$$

- The horizontal movement of the cart

$$M\ \ddot{y}=F-H \tag{A-4}$$

- The relationship between φ and ψ

$$\psi=\frac{R+r}{r}\varphi \tag{A-5}$$

The variables (ψ, V, H) can be eliminated from (A-1)-(A-5), yielding two second order differential equations in φ and y,

$$\begin{aligned}(M+m)\,\ddot{y} &= -m\,(R+r)\left(\ddot{\varphi}\cos\varphi-\ \dot{\varphi}^2\sin\varphi\right)+F && \text{(A-6)}\\ I\frac{R+r}{r}\,\ddot{\varphi} &= mr\,(R+r)\left(-\ \ddot{\varphi}\sin^2\varphi-\ \dot{\varphi}^2\cos\varphi\sin\varphi\right) && \text{(A-7)}\\ &\quad +mgr\sin\varphi+Mr\,\ddot{y}\,\cos\varphi-Fr\cos\varphi\end{aligned}$$

They are nonlinear due to the trigonometric functions, and they are coupled such that $\ddot{y}$ occurs on the left side of (A-6) and on the right side of (A-7); the situation is the reverse in the case of $\ddot{\varphi}$. They can be solved, in principle at least, given $\left[F(t),\varphi\,(0)\,,\dot{\varphi}\,(0)\,,y\,(0)\,,\dot{y}\,(0)\right]$.

Object	Symbol	Ratings
cart length		0.35 [m]
cart width		0.12 [m]
cart radius of the arc	R	0.50 [m]
cart weight, including equivalent mass of motor and transmission	M	3.1 [kg]
cart position	y	
cart driving force	F	[N]
ball maximum angle		$\pm$0.22 [rad]
ball radius	r_1	0.0275 [m]
ball rolling radius	r	0.025 [m]
ball rolling angle [radian]	ψ	
ball angular deviation [radian]	φ	max 0.22 [radian]
ball weight	m	0.675 [kg]
ball moment of inertia, $\frac{2}{5}mr_1^2$	I	0.204*10^{-3} [kgm^2]
ball vertical reactive force [N]	V	
ball horizontal reactive force [N]	H	
bar length		1.4 [m]
bar diameter		0.025 [m]
motor power		max 21 [W]
motor voltage	U	max 13 [V]
motor transmission ratio	$U : F$	$1 : 1$
motor speed		3700 [r.p.m]
gravity	g	9.81 [ms^{-2}]

Table 1: Physical data for lab rig

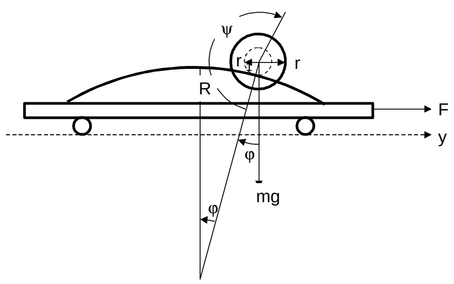

Figure 8: Symbol definitions.

Introducing the state vector **x** of state variables

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \varphi \\ x_4 &= \dot{\varphi} \end{aligned} \tag{A-8}$$

in (A-6)-(A-7), and after a lot of rearranging, the nonlinear state-space equations emerge.

$$\dot{x}_1 = x_2 \tag{A-9}$$

$$\begin{aligned} \dot{x}_2 = {} & \frac{-m\left(R+r\right)\left(-\left(r+R\right)mr\left(\sin x_3\cos^2 x_3\right)x_4^2 + mgr\sin x_3\cos x_3\right)}{\left(M+m\right)\left(\frac{I(R+r)}{r} + rm\left(\sin^2 x_3\right)\left(R+r\right) + \frac{rM(\cos^2 x_3)m(R+r)}{(M+m)}\right)} \\ & + \frac{m\left(R+r\right)\left(x_4^2\sin x_3\frac{I(R+r)}{r} + x_4^2 rm\left(\sin^3 x_3\right)\left(R+r\right)\right)}{\left(M+m\right)\left(\frac{I(R+r)}{r} + rm\left(\sin^2 x_3\right)\left(R+r\right) + \frac{rM(\cos^2 x_3)m(R+r)}{(M+m)}\right)} \\ & + \frac{\left(r+R\right)\left(mr^2+I\right)}{r\left(M+m\right)\left(\frac{I(R+r)}{r} + rm\left(\sin^2 x_3\right)\left(R+r\right) + \frac{rM(\cos^2 x_3)m(R+r)}{(M+m)}\right)}F \end{aligned} \tag{A-10}$$

$$\dot{x}_3 = x_4 \tag{A-11}$$

$$\begin{aligned} \dot{x}_4 = {} & \frac{\left(-rm^2x_4^2\frac{R+r}{M+m}\left(\cos x_3\sin x_3\right) + mgr\sin x_3\right)}{\frac{I(R+r)}{r} + rm\left(\sin^2 x_3\right)\left(R+r\right) + \frac{rM(\cos^2 x_3)m(R+r)}{(M+m)}} \\ & - \frac{r\left(\cos x_3\right)\frac{m}{M+m}}{\frac{I(R+r)}{r} + rm\left(\sin^2 x_3\right)\left(R+r\right) + \frac{rM(\cos^2 x_3)m(R+r)}{(M+m)}}F \end{aligned} \tag{A-12}$$

The model can be linearised around the origin. In order to avoid errors we will linearise (A-6)-(A-7) rather than the nonlinear state-space equations. Introduce the following approximations to the trigonometric functions,

$$\cos\varphi \simeq 1, \sin\varphi \simeq \varphi, \cos^2\varphi \simeq 1, \sin^2\varphi \simeq 0 \tag{A-13}$$

The angle φ is in the order of ± 0.22 radian, and the error introduced by the linearisation is small. Furthermore it has been shown (Jørgensen, 1974) that the influence of terms containing the factor $\dot{\varphi}^2$ is less than one percent, so those terms will be neglected. Equations (A-6)-(A-7) reduce to

$$\left(M+m\right)\ddot{y} = -m\left(R+r\right)\ddot{\varphi} + F \tag{A-14}$$

$$I\frac{R+r}{r}\ddot{\varphi} = mgr\varphi + Mr\ddot{y} - Fr \tag{A-15}$$

After some rearranging, one gets

$$\ddot{y} = -\frac{m^2r^2g}{MI+mI+mr^2M}\varphi + \frac{mr^2+I}{MI+mI+mr^2M}F \tag{A-16}$$

$$\ddot{\varphi} = \frac{mr^2 g\,(M+m)}{(R+r)\,(MI+mI+mr^2M)}\varphi - \frac{mr^2}{(R+r)\,(MI+mI+mr^2M)}F \tag{A-17}$$

Introducing the substitution variables

$$\begin{aligned} a &= -\frac{m^2r^2g}{MI+mI+mr^2M} \\ b &= \frac{mr^2+I}{MI+mI+mr^2M} \\ c &= \frac{mr^2g\,(M+m)}{(R+r)\,(MI+mI+mr^2M)} \\ d &= -\frac{mr^2}{(R+r)\,(MI+mI+mr^2M)} \end{aligned} \tag{A-18}$$

and the state vector (A-8) one obtains a linear state-space model

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{A-19}$$

The matrices are simply

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \\ d \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{A-20}$$

With the data in Table 1 the actual values of the constants are

$$(a, b, c, d) = (-1.34, 0.301, 14.3, -0.386)$$