Posicast Control of Damped Oscillatory Systems*

OTTO J. M. SMITH†, SENIOR MEMBER, IRE

Summary—**A novel method is presented for producing dead-bea response in a lightly-damped oscillatory feedback system. Complete transient response times of the order of a fraction of the natural oscillatory period can be obtained. Excellent waveshape reproduction is achieved through a linear phase lag with frequency. The method consists of exciting several transient oscillations, at closely spaced times, with magnitudes and phases so adjusted that the resultant sum of the transient oscillation phasors is zero. The steady-state output is the arithmetic sum of the excitation magnitudes.**

When a step input transient is divided into two spaced excitations, one-half cycle response is obtainable. When the input transient is divided into three excitations, one-fourth period or faster transient times are realizable, depending upon the available dynamic range or signal-to-noise ratio. The principle of design is to adjust a system to the maximum possible resonant frequency, independent of the damping factor, but stable, and then to apply the Posicast control to completely remove the oscillatory component in the output. In an electrical feedback control system, the additional hardware consists of one or two artificial transmission lines.

Introduction

THE PROBLEM in a feedback control system is to change the stored energy of an object from one value to another without subsequent oscillations or overshoot. This method can be easily demonstrated by its application to the problem of changing the at-rest position of an undamped pendulum. The system input is the point of suspension. In Fig. 1, (a) is the initial position; (b) is the condition immediately after the input step has been broken into two parts and only half of the desired change has been applied to the support; (c) is the condition after one-half cycle of the natural transient period of the pendulum. At this instant, the support is suddenly moved until it is directly over the bob and (d) shows the final position. The scheduling of the motion of the support is the transference of an equalizer, which compensates for the resonant frequency of the load by introducing attenuation at the resonant frequency. This is, however, a time-domain synthesis yielding a linear time-domain equalizer.

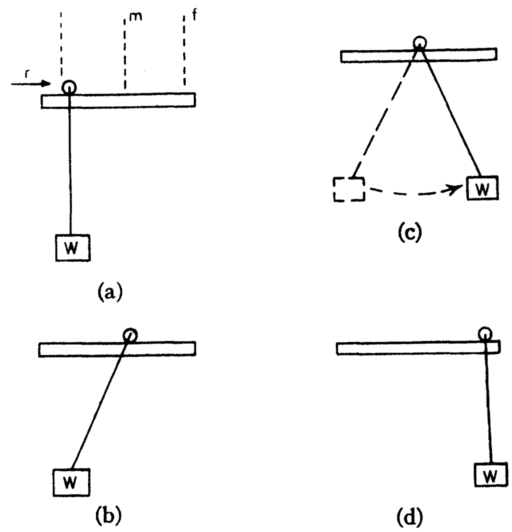

Fig. 1.

The electrical analog of the pendulum is a capacitive load with inductance and resistance in series between the supply and the load. The input is voltage across the series RLC, and the output is voltage across the capacitor. The step response for an input voltage of magnitude A is shown in Fig. 2.

If after time $T_n/2$, the input voltage has added to it a second step of magnitude B (determined from Fig. 2), so that the sum is $(A+B)$, then at this instant the current will suddenly drop to zero, since the voltage across the condenser is also $(A+B)$. The steady-state condition has been suddenly achieved, and there will be no further transient.

The final value is reached just as the velocity goes to zero. This is what happens when a fisherman drops his fly in the water at the maximum-position and zero-velocity instant. Hence the descriptive name *positive-cast* or Posicast for this kind of system.

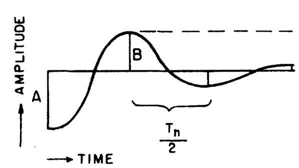

Fig. 2.

The input command was broken into two parts; the first part was applied immediately, and the second part was delayed until after one-half period of the natural transient, $T/2$. Fig. 2 shows the relative magnitudes of the two excitation functions. Amplitude A is proportional to the first excitation function, amplitude B is the magnitude of the second excitation function, and $A+B$ is the amplitude of the input driving function and of the output of the system. Mathematically, the Laplace transform of the control function is

$$\lfloor k_a + (1 - k_a)\epsilon^{-sT/2}\rfloor. \tag{1}$$

$T/2$ is the delay between the initial and the final pulse. The ratio of the first to the second pulse is a measure of the attenuation of the oscillation envelope during the transient time.

* Original manuscript received by the IRE, November 12, 1956; revised manuscript received, June 12, 1957.
† Elec. Eng. Div., Univ. of Calif., Berkeley, Calif.

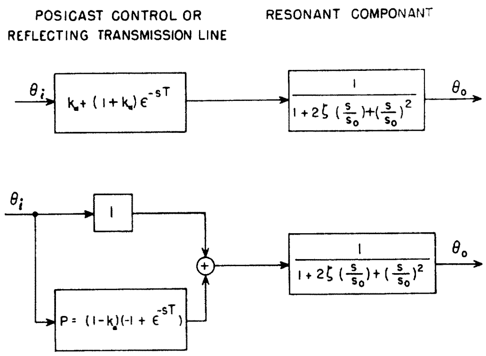

Fig. 3.

$$k = \left(\frac{k_a}{1 - k_a}\right) = \exp\left(\frac{\zeta\omega_n T_r}{\sqrt{1 - \zeta^2}}\right). \tag{2}$$

T_r is the over-all response or transient time. In this special case of half-period control, $T_r = T/2$. The natural transient radian frequency of oscillation is

$$\omega_n = 2\pi f_n = \omega_0\sqrt{1 - \zeta^2} \tag{3}$$

ζ is the per unit dimensionless damping per undamped radian of the oscillatory system. ζ is equal to α/ω_0, the sine of the angle between the $j\omega$ axis and the pole in the s plane. ζ_n is the tangent of the same angle, equal to α/ω_n or $\zeta/\sqrt{1-\zeta^2}$.

Fig. 3 shows the block diagram of the half-period control of a resonant component, in which the input is broken into an initial and delayed step. This can be represented as a unity input plus a negative pulse generator, which is shown as the block P. Fig. 4 shows the gain and phase response of the system before and after compensation. Curves A are for a lightly-damped resonant component alone. Curves B are for the lightly-damped system in Fig. 3 after the application of the Posicast control. The phase lag is approximately linear with frequency. Curves C are for a highly damped resonant component alone and curves D are for the same system after the application of Posicast control. Fig. 5 shows the s-plane plot for the undamped system. The uncompensated resonance is represented by two complex poles. The Posicast component alone has two complex zeros which exactly coincide in location in the s plane with the complex poles. The cascade combination of these two has a series of complex zeros at a very high frequency only.

This system is linear, and although it contains non-minimum phase elements, the parallel branches guarantee that the over-all system will always be minimum phase.

Even though temperature changes may affect the constants of the delay line or the oscillatory system, the basic character of the response is relatively unchanged, so long as the distance between the poles and the zeros is significantly less than the distance between the poles and the $j\omega$ axis.

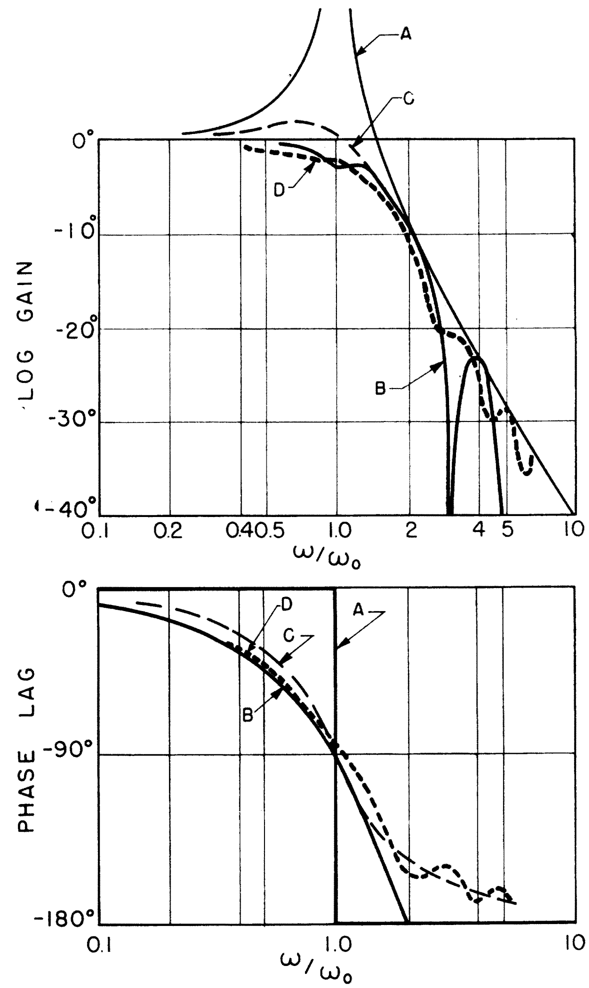

Fig. 4.

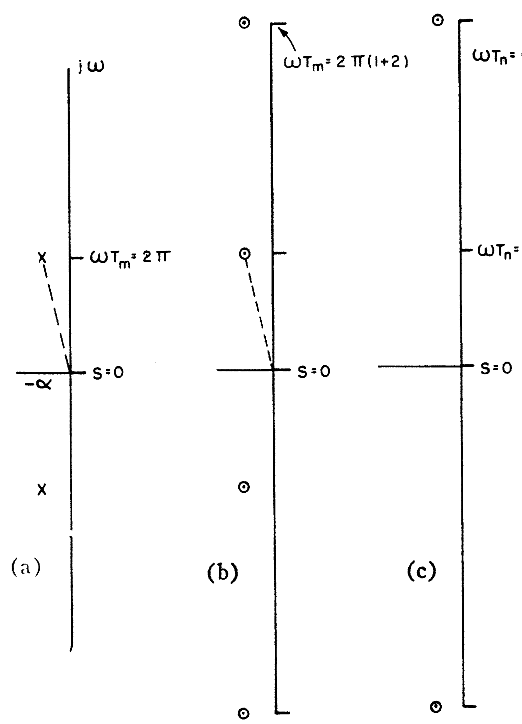

Fig. 5.—(a) $G(s)$ original lightly-damped oscillatory system, (b) $1+P$ Posicast compensating section, (c) Resultant system, one-half period response.

It can be seen that this Posicast compensator performs the function of equalization. At the frequencies for which the system gain is high, the compensator gain is low. In the Bibliography are references to other work using delay lines as equalizers. In Wiener's work [1], a line with many taps was used to approximate an impulse response whose peak was delayed in time, in order to achieve a maximum discrimination between signal and noise. In Calvert's work, multitapped delay lines were used as equalizers to generate phase lead and to adjust the attenuation on a real frequency response basis. Exponential functions of s were converted to trigonometric form and then approximated by Taylor's series.

It is not necessary to build actual LC delay lines for low frequency systems. Servomechanisms may require delays of the order of 0.05 second. These can be achieved with RC twin-tee networks, an amplifier, and negative feedback around the whole. It is beyond the scope of this paper to discuss the design of precision delay lines [5]. Process controls may require delays of 30 seconds. These can be realized pneumatically with orifices and membrane-divided capacities in a twin-tee network, with an air amplifier and negative feedback. In the megacycle range, various network configurations will yield a circular pattern of poles and zeros, each uniformly spaced in the vertical s-plane direction, and very closely approximating a delay line.

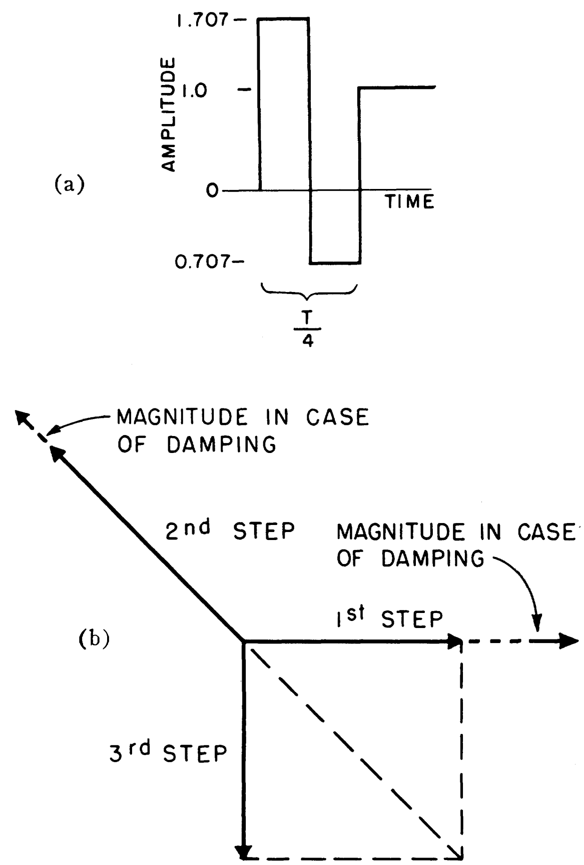

Fig. 6—(a) System input, step response of $(1+p)$. (b) Vector diagram for one-quarter-period Posicast control.

One-Quarter Cycle Response

A resonant load can be driven by an input step to produce an output step completely realized in a very small fraction of a period. The excitation function must deliver a positive step first, a negative step after a short delay, and a final positive step. These three inputs to the resonant load excite three oscillatory transients. Each can be represented by the real part of a rotating phasor which is diminishing in magnitude at the rate exp $(-\alpha t)$. These are called shrinking phasors or shrinking vectors. The vector sum of the three phasors at any time after the last input step must be zero for the transient response to have no overshoot and to remain constant at its steady-state value. The *arithmetic* sum of the magnitudes of the three steps is the steady-state output. For very short transient times (very-wide bandwidth), the first accelerating force and the second braking force must be very large compared to the final step and to the steady-state gain.

Fig. 6 shows the excitation function necessary to drive an undamped resonant load to achieve the best step response in one-quarter period. Fig. 6(b) is a vector diagram of the three phasors representing the three oscillation components excited by the three steps in Fig. 6(a). The first step is of magnitude 1.7 and starts a negative cosine oscillation of 1.7 amplitude. The second step is 45° later with magnitude of -2.4. The sum of these two is a positive sine wave of 1.7 amplitude. The third step is of magnitude 1.7 and occurs when the resultant oscillation has reached unity with zero derivative. The vector sum of the three is zero and the arithmetic sum is unity.

When the system has damping, the vectors diminish in magnitude with time, and so are called *shrinking vectors*. At the time instant of the last step, the three oscillations should be represented by the three solid vectors in Fig. 6(b). The magnitudes of the steps to produce these, however, should be larger for the earlier vectors, because of the damping. The original step magnitudes are shown with dotted vectors.

Fig. 7 shows the form of the output transient from the undamped resonant load when driven with the excitation of Fig. 6. It is the first 45° of a negative cosine wave attached to the last 45° of a positive cosine wave. The steps in Fig. 6 must be the center lines of these cosine waves. Fig. 8 is a practical circuit for obtaining the excitation function of Fig. 6. This is a doubly reflecting line, with each section having a delay of 1/16 of the resonant period. A positive input step drives the output positive and starts a transient propagating down the distortionless delay line. When this transient reaches the 0.707R it is reflected with reversed phase and this reflection reverses the output when it arrives back at the sending end of the delay line. The transient which continues to the end of the second delay line is reflected back with a doubling of amplitude and this restores the output to a positive polarity at the end of a total delay time of one-fourth period.

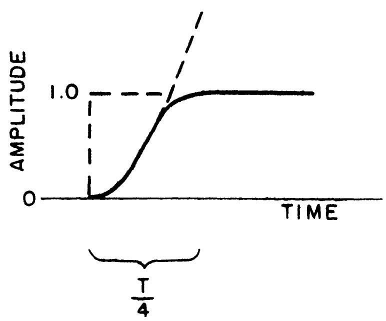

Fig. 7—System output.

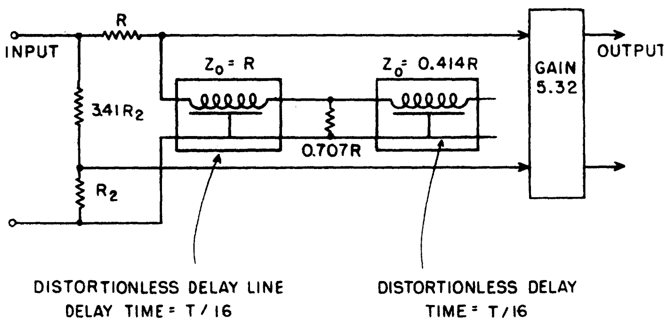

Fig. 8—Circuit for $(1+p)$.

For the complex zeros of this Posicast delay line section to cancel exactly the complex poles of a resonant component, the delay line should have the transference

$$1 + P = \frac{k - 2k^{1/2}\left(\cos\frac{\omega_n T_r}{2}\right)\epsilon^{-sT_r/2} + \epsilon^{-sT_r}}{k - 2k^{1/2}\left(\cos\frac{\omega_n T_r}{2}\right) + 1} \tag{4}$$

where k is the value previously defined. This equation is valid for all 3-step or double-pulse excitations.

The control in (4) above is derived from two restrictions on the system. For each step out of the compensator, an oscillation component is excited. After the last step, these three oscillation components can be represented by three rotating phasors. That due to the last step has a magnitude equal to the last step and a phase of zero degrees. That phasor due to the next-to-the-last step has a magnitude less than the step due to the attenuation of the oscillation, $1/\sqrt{k}$, which has occurred in the time between the middle and the last step. It has an angle of $\omega_n T_r/2$ radians, which is 45° for quarter-period control. That phasor due to the first step has a magnitude equal to the first step times $1/k$, the attenuation during the total transient period. It has an angle of $\omega_n T_r$ radians, which is 90° for quarter-period control. The sum of these three phasors must equal zero. Setting the real and imaginary parts separately equal to zero, one has two equations from which the relative magnitudes of the three steps can be calculated. These are the three numerator terms in (4).

The compensator should have unity steady-state gain, delivering a unit output for a unit input. Therefore the *arithmetic* sum of the three steps of the compensator should equal unity. This yields the calibration constant shown in the denominator of (4).

It is beyond the scope of this paper to discuss the z plane and the z transform, in which the substitution $z = \epsilon^{sT}$ is used. However, for those skilled in this method, it is apparent that (4) can be changed into a ratio of polynomials in z by making the substitution above. Eq. (4) will then have a numerator quadratic, with two complex z-plane zeros. These zeros should be designed to coincide exactly with the system s-plane poles when they are plotted in the z plane (z transform of the system). This method has great mathematical rigor and simplicity.

A compensator can be built for any multiple-pole system of any order, which will have only tangential transients, with no transient components which approach the final value asymptotically. In this case, all of the poles and zeros from the s-plane plot of the complete system are transferred to the z plane and plotted there as poles and zeros (z transform). For each z-plane pole, the compensator produces a z-plane zero. The polynomial in z for all of these zeros *is* the Posicast compensator transference. This is applicable to two or more coupled resonant frequencies.

When one attempts to achieve extremely short response times, initial pulse must be many times larger than steady-state value. This may result in saturation of amplifiers or transducers in the system. Less than one-quarter period response is feasible primarily only in low-level applications. For high-level servo systems driven near saturation, the law of diminishing returns excludes responses faster than one-fourth-period.

Feedback Systems

This method of control can be applied to any complex feedback system. With respect to the input, only a feedforward pulse generator is needed. Fig. 9 shows the block diagram. The compensator $(1+P)$ should have unity steady-state gain. When transmission lines are used, or amplifiers to simulate lines, changes in temperature will change the gain. Therefore, high steady-state stability and accuracy are achieved by dividing the function $(1+P)$ into two parts as shown in Fig. 9(b). The unity-gain input to the system is left undisturbed. In parallel with this is introduced a pulse generator, P, capacitively- or transformer-coupled with zero steady-state gain. The pulse generator P has no steady-state gain. Fig. 10(a) shows the wiring diagram for a pulse generator of this sort. Its transference is

$$P = K_0 + K_1\epsilon^{-sT_r/2} + K_2\epsilon^{-sT_r} \tag{5}$$

where

$$K_0 = \frac{2k^{1/2}\left(\cos\frac{\omega_n T_r}{2}\right) - 1}{k - 2k^{1/2}\left(\cos\frac{\omega_n T_r}{2}\right) + 1} \tag{6}$$

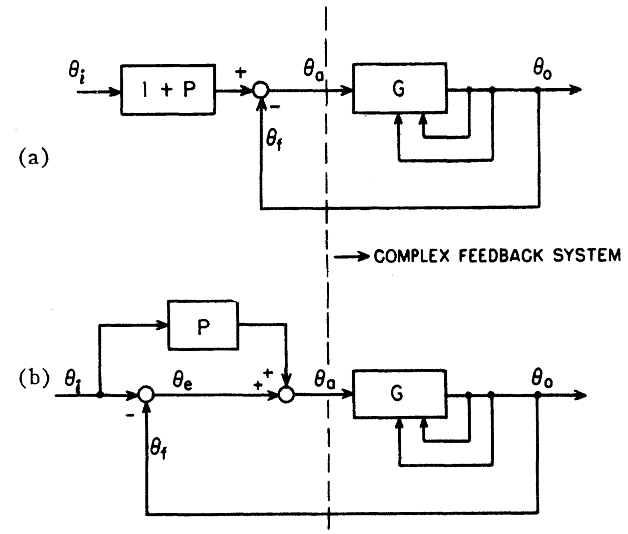

Fig. 9—(a) Statement of best control. (b) Constructional form to minimize the effects of changes in steady-state gain of p.

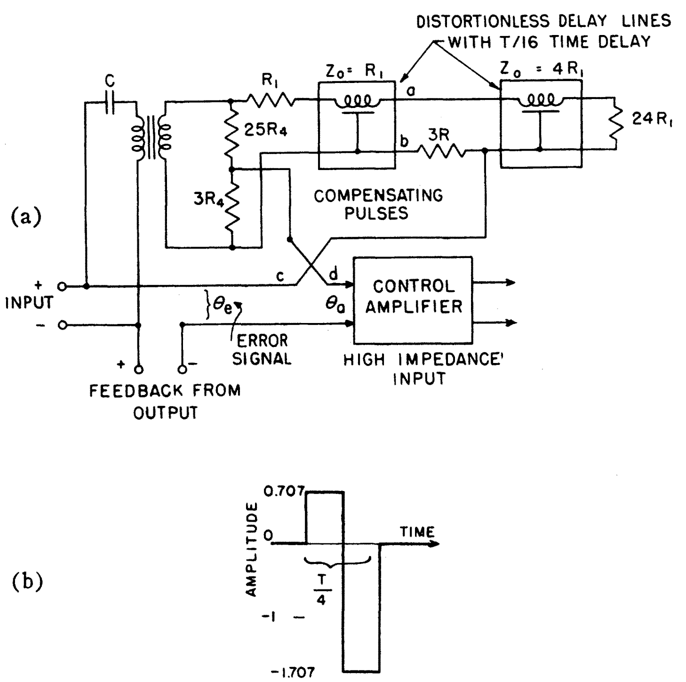

Fig. 10—(a) Circuit for p in feedforward independent of the input. (b) Step response of p alone.

$$K_1 = \frac{2k^{1/2}\left(\cos \dfrac{\omega_n T_r}{2}\right)}{k - 2k^{1/2}\left(\cos \dfrac{\omega_n T_r}{2}\right) + 1} \tag{7}$$

$$K_2 = \frac{1}{k - 2k^{1/2}\left(\cos \dfrac{\omega_n T_r}{2}\right) + 1}, \tag{8}$$

$$K_0 - K_1 + K_2 = 0. \tag{9}$$

Fig. 10(b) shows the step-response pulse output at $c-d$ in Fig. 10(a).

Compensation for Load Disturbances

Fig. 11 (next page) shows a block diagram of the original uncompensated feedback system. It is desired to make the speed of response of this system as great as possible. The functions G_1 and G_2 are adjusted to make the damped resonant frequency as high as possible, with the restriction that the system always be reliably stable for the normal variations in the parameters with signal level and with temperature. A *statement* of the optimum *realizable form* of control is the block diagram in Fig. 12. This is not the block diagram for the construction but only states that the load signal should have come through a block which divided it into two or more components so phased that the transients excited by these components would cancel out. Fig. 13 is the block diagram of the actual system. This can be derived from Fig. 12 by block diagram substitutions.

The compensation block required, P_0', is the transference P_0 with unity negative feedback. This block alone would have the transfer function

$$P_0' = \frac{P_0}{1 + P_0} = \frac{K_0 + K_1\epsilon^{-sT_r/2} + K_2\epsilon^{-sT_r}}{1 + K_0 + K_1\epsilon^{-sT_r/2} + K_2\epsilon^{-sT_r}}. \tag{10}$$

P_0' is a reentrant transmission line, or a continuously reflecting transmission line, which is terminated in values other than the characteristic impedance at each end. The transference of this block can be represented by

$$P_0' = 1 - \frac{1}{1 + K_0 + K_1\epsilon^{-sT_r/2} + K_2\epsilon^{-sT_r}} \tag{11}$$

$$P_0' = 1 - \frac{1}{(1 + K_0)}(1 - X + X^2 - X^3 + X^4 - \cdots) \tag{12}$$

where

$$X = \left(\frac{K_1}{1 + K_0}\right)\epsilon^{-sT_r/2} + \left(\frac{K_2}{1 + K_0}\right)\epsilon^{-sT_r}. \tag{13}$$

This transference has both poles and zeros. However, the use of this block within the feedback system introduces a unique mode of operation in which the poles are excited for only a short time and then are quenched. A step change or disturbance of the load produces at the input to P_0' a triple step operated on by the function $1/G_1$. These three steps have the unique relationship necessary to cancel the feedback pulses from the output of P_0, so that the input to P_0 is *not* a triple step, but is only a single step, operated on by the function $1/G_1$. The line, therefore, delivers only a triple-step output for a single-step load disturbance.

The function P_0' can be constructed like Fig. 10, except that the impedance values are so chosen as to produce continuous reflections back and forth down the line in accordance with (10)–(12). Or, the actual circuit of Fig. 10 can be used with an isolating amplifier providing negative feedback. Since this minor loop feedforward has only high-frequency response, it is not

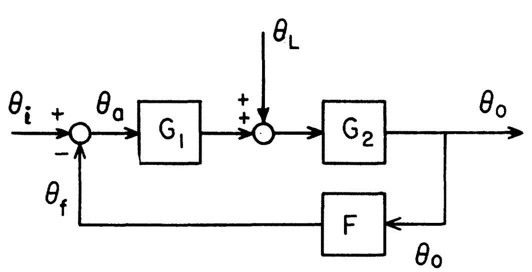

Fig. 11.

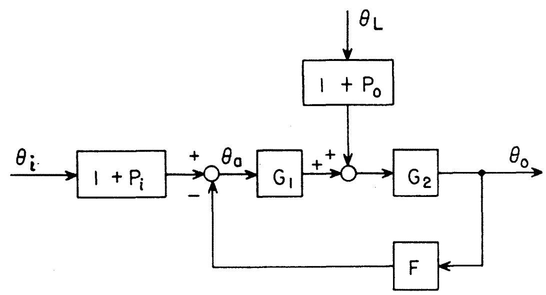

Fig. 12—Statement of best control.

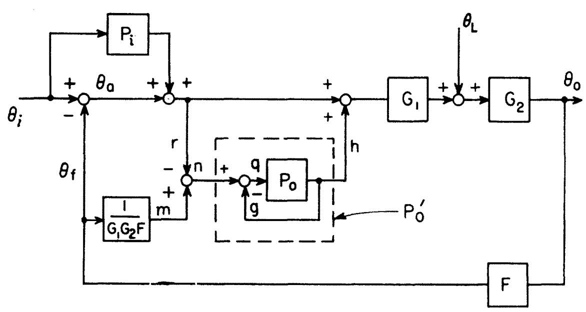

Fig. 13—Constructional arrangement for load compensator.

necessary for it to have zero frequency gain and can be transformer- or capacitively-coupled. The block $1/G_1G_1F$ is the inverse of the original system loop gain. However, this computation must be effective only for very high frequencies in the region where the loop gain is unity with a phase lag of nearly 180°.

The input of the pulse-generating line P_0 can be thought of as being located at the null position of a bridge which is driven by the output of this line. Therefore, pulses generated due to the delay within the line itself do not excite the line further, but disappear into the cancellation of the load-excited oscillation.

Fig. 14 shows the s-plane pattern of the original system, the compensator $(1+P)$ alone, and the complete feedback system with compensators for one-quarter period response.

Digital Computer Control

All of the previous systems can be adapted directly to digital computer control or periodically-sampled systems. Eq. (5) is the scheduling of the input feedforward or input computer. Eq. (10) is the scheduling of the minor-loop feedforward or load-compensating computer. The sampling period is $T_r/2$. In the special degenerative case represented by (1), half-period response can be achieved by setting T equal to both the half-period of the oscillation and to the sampling period, but this introduces problems in stability which are beyond the scope of this paper. The general control represented by (5) and (10) will yield a dead-beat response. In general, $T_r/2$ should be several sampling periods for good control without saturation problems.

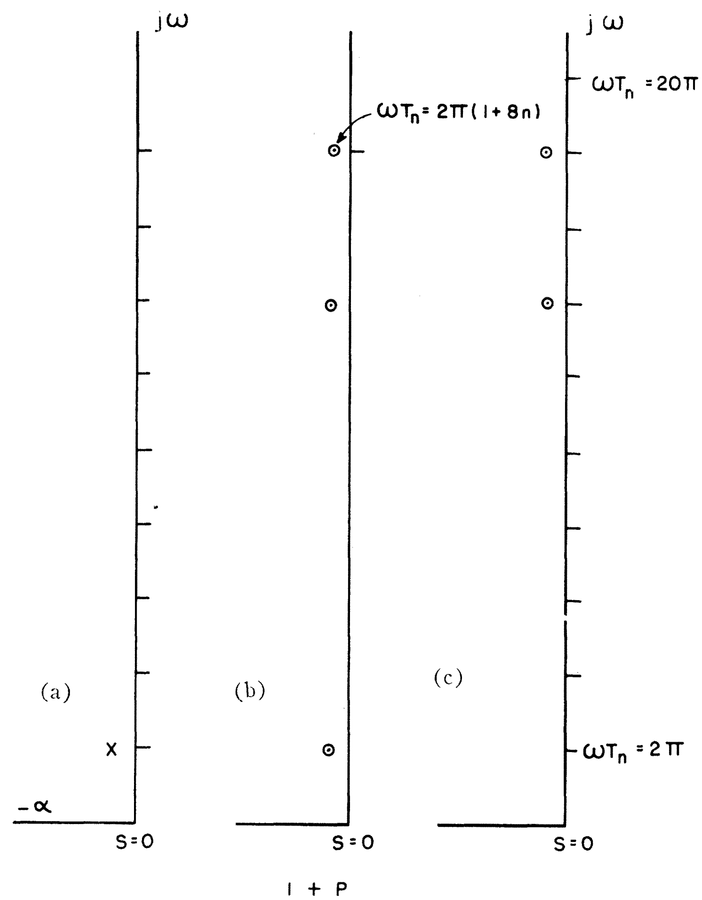

Fig. 14.—(a) Original lightly-damped system, (b) Posicast compensator for one-quarter period, (c) Final compensated system.

Error Coefficient Restrictions

This Posicast method of control can be applied to systems which have specified positional, velocity, and acceleration error coefficients. These coefficients are usually designated as k_p, k_v, and k_a, respectively. If, in addition, the system has the unalterable components of two resonant poles, we have five restrictions on the system. To fulfill these five restrictions, the control device should convert a single-step input into five step functions. The transfer function of the control device would therefore be

$$1 + P = K_0 + K_1\epsilon^{-sT_r/4} + K_2\epsilon^{-sT_r/2} + K_3\epsilon^{-sT_r3/4} + K_4\epsilon^{-sT_r}. \tag{14}$$

To solve for the coefficients of this transfer function, which is the same as the scheduling of a periodically sampled controller, one can start with the restriction that the vector sum of the five transients excited by these five steps must be zero. This yields

$$K_0\epsilon^{4\theta(-\zeta_n+j)} + K_1\epsilon^{3\theta(-\zeta_n+j)} + K_2\epsilon^{2\theta(-\zeta_n+j)} + K_3\epsilon^{\theta(-\zeta_n+j)} + K_4 = 0 \quad (15)$$

where $\zeta_n = \zeta/\sqrt{1-\zeta^2}$ and θ is the transient phase angle corresponding to time $T_r/4$.

The arithmetic sum of the coefficients of (14) above should equal one minus the reciprocal positional error coefficient. The difference between the integral of unity for time T_r and the integral of the output of the controller, that is the integral of (14) for a step input, should be the reciprocal velocity error coefficient. In a similar manner the difference between the double integral of unity and the double integral of (14) for a step input should be the reciprocal acceleration error coefficient. Setting down these integral equations and simplifying them will yield the following three restrictions:

$$\begin{aligned} K_0 + K_1 + K_2 + K_3 + K_4 &= (1 - 1/k_p) \\ 4K_0 + 3K_1 + 2K_2 + K_3 &= 4(1 - 1/k_vT_r) \\ 4^2K_0 + 3^2K_1 + 2^2K_2 + K_3 &= 16(1 - 2/k_aT_r^2). \quad (16) \end{aligned}$$

The real and the imaginary parts of (15) taken separately are two independent equations. These 5 equations are sufficient to solve for the complete schedule. This method can be extended to the control of a pair of resonant poles with any number of error coefficient restrictions.

The same technique can be applied to the control of a *complex* system with fairly simple error coefficient restrictions. For example, a system could have one real time constant and a pair of complex poles, and a zero positional error for constant velocity would be desired. The system real pole can be considered as an operator on the input signal. The resultant condition to be imposed is that there must be zero error for a delayed integral of an input step, after time T_r.

Conclusion

A method of control has been developed which eliminates the necessity for adjusting a feedback control system to have only highly damped resonant poles. It is possible to obtain significantly greater speeds of response, by adjusting the feedback system for maximum frequency of oscillation, only lightly damped, but with a reproducible and consistent closed-loop complex pole location in the s plane. The method is analogous to the race-break systems of nonlinear predictor controls, in which a large positive pulse is applied initially, a large negative pulse follows to reduce the derivative of the output, and a third steady-state step is applied permanently. These three excitations are adjusted so that the phasor sum of their phasor transients is zero after the final excitation is applied. These excitations are a mode of high-frequency control, and do not need to be included within the characteristics of the feedback system. Although this system contains nonminimum phase elements, the over-all system is minimum phase, and furthermore approaches a linear phase lag with frequency, as the original resonant poles approach zero damping. The waveshape reproduction is the best possible from any linear system with the restriction of a fixed dynamic range, or the restriction of a maximum permissible signal which will not drive the amplifiers or transducers into the nonlinear region.

Bibliography

[1] Wiener, N. *The Extrapolation, Interpolation and Smoothing of Stationary Time Series*, New York, John Wiley & Sons, Inc., 1949.

[2] Calvert, J. F. and Ford, D. J. "The Application of Short-time Memory Devices to Compensator Design," *Applications and Industry*, No. 12 (May, 1954), pp. 88–93.

[3] Calvert, J. F. and Gimpel, D. J. "Signal Component Control," *Transactions of the AIEE*, vol. 71, Part II (November, 1952), pp. 339–343.

[4] Sze, T. W., and Calvert, J. F., "Short Time Memory Devices in Closed-loop System—Steady-State Response," *Transactions of the AIEE*, Part II, vol. 74 (1955), pp. 340–344.

[5] Kuh, E. S., "Synthesis of Lumped-Parameter Precision Delay Lines," 1957 IRE National Convention Record, Part 2, pp. 160–174, 1957.