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Elementary Proofs of Some Classical Stability Criteria

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***Abstract*—Classical stability results and tests on the stability of a given polynomial are proved and derived here using a simple continuity property. The resulting new proofs given of the Hermite-Bieler theorem and the Routh and Jury tests are elementary and insightful. Most important, the proofs given here would allow the instructor to present these fundamental topics of control theory, for the first time, in an elementary, rational, and meaningful way rather than as mere sets of rules and formulae.**

I. INTRODUCTION

IN THIS PAPER, we present a unified and elementary approach to the classical problem of determining the stability of a polynomial from its coefficients. The approach consists of a systematic use of the following fact: Given a parameterized family of polynomials and any continuous path in this parameter space leading from a stable to an unstable polynomial, then, the first unstable point that is encountered in traversing this path corresponds to a polynomial whose unstable roots lie on the boundary (and not in the interior) of the instability region in the complex plane.

The above result, which we call the boundary crossing theorem, is established rigorously in the next section. The proof follows simply from the continuity of the roots of a polynomial with respect to its coefficients. The consequences of this result, however, are quite far reaching, and we demonstrate this in the subsequent sections by using it to give simple derivations of the classical Hermite–Bieler theorem, the Routh test for left half plane stability, and the Jury test for unit circle stability.

The contribution of this paper relative to the existing literature is that our simple proofs of these fundamental results make them accessible even to undergraduates, whereas the existing proofs in the literature certainly do not.

II. THE BOUNDARY CROSSING THEOREM

We first introduce two well-known results that will lead us to the main theorem.

Theorem 2.1 (Rouché's Theorem): Let $f(z)$ and $g(z)$ be two functions that are analytic inside and on a simple closed contour C. If $|g(z)| < |f(z)|$ for all z on C, $f(z)$ and $f(z) + g(z)$ have the same number of zeros (multiplicities included) inside C.

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This is just one formulation of Rouché's theorem, but it is sufficient for our purposes. Let us now state and prove a second important result [1].

Theorem 2.2: Let

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n = p_n \prod_{j=1}^{m} (s - s_j)^{t_j}, \quad p_n \neq 0$$

$$Q(s) = (p_0 + \epsilon_0) + (p_1 + \epsilon_1)s + \cdots + (p_n + \epsilon_n)s^n$$

and consider a circle C_k of radius r_k centered at s_k, which is a root of $P(s)$ of multiplicity t_k. Let r_k be fixed but satisfy

$$0 \leq r_k < \min |s_k - s_j|, \quad \text{for } j = 1, 2, \cdots, k-1, k+1, \cdots, m.$$

Then, there exists a positive number ϵ, such that if $|\epsilon_i| \leq \epsilon$, for $i = 0, 1, \cdots, n$, $Q(s)$ has precisely t_k zeros in the circle C_k.

Proof: $P(s)$ is nonzero and continuous on the compact set C_k, and therefore, it is possible to find $\delta_k > 0$ such that

$$|P(s)| \geq \delta_k > 0, \quad \forall s \in C_k.$$

On the other hand, consider the polynomial $R(s)$ defined by

$$R(s) = \epsilon_0 + \epsilon_1 s + \cdots + \epsilon_n s^n.$$

On the circle C_k, we have

$$|R(s)| \leq \sum_{j=0}^{n} |\epsilon_j| |s^j| \leq \sum_{j=0}^{n} |\epsilon_j| (|s - s_k| + |s_k|)^j \leq \epsilon \underbrace{\sum_{j=0}^{n} (r_k + |s_k|)^j}_{M_k}.$$

Thus, if ϵ is chosen so that $\epsilon < \delta_k / M_k$, we can conclude that

$$|R(s)| < |P(s)|, \quad \text{for all } s \text{ on } C_k$$

so that by Rouché's theorem, $Q(s)$ and $P(s)$ have the same number of zeros inside C_k. Since the choice of r_k ensures that $P(s)$ has just one zero of multiplicity t_k at s_k, we see that $Q(s)$ has precisely t_k zeros in C_k. ■

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Corollary 2.2: Fix m circles $C_1, \cdots, C_m$, that are pairwise disjoint and centered at $s_1, s_2, \cdots, s_m$, respectively. Then, it is always possible to find, by repeatedly applying the previous theorem, an $\epsilon > 0$ such that for any $|\epsilon_i| \leq \epsilon$, for $i = 0, 1, \cdots, n$, $Q(s)$ has precisely t_j zeros inside each of the circles C_j. Note that in this case, $Q(s)$ always has $t_1 + t_2 + \cdots + t_m = n$ zeros and must therefore remain of degree n so that necessarily $\epsilon < |p_n|$.

The above theorem and its corollary lead to the following main result.

A. Main Theorem

Let us consider the complex plane C, and let S be any given open set. We know that S, which is its boundary ∂S together with the interior U^0 of the closed set $U = C - S$, form a partition of the complex plane, that is

$$S \cup \partial S \cup U^0 = C,$$
$$S \cap U^0 = S \cap \partial S = \partial S \cap U^0 = \phi.$$

We assume, moreover, that these three sets are all nonempty. These assumptions are very general. In stability theory, we might choose for S the open left half plane C^- (for continuous time systems) or the open unit disk D^1 (for discrete time systems) or suitable subsets of these, respectively.

Now, let $P(\lambda, s)$ be a family of polynomials of fixed degree n, which is continuous with respect to λ on a fixed interval $I = [a, b]$. In other words, $P(\lambda, s)$ can be written as

$$P(\lambda, s) = p_0(\lambda) + p_1(\lambda)s + \cdots + p_n(\lambda)s^n$$

where $p_0(\lambda), p_1(\lambda), \cdots, p_n(\lambda)$ are continuous functions of λ on I and where $p_n(\lambda) \neq 0$ for all $\lambda \in I$. From the results of Theorem 2.2 and its corollary, it is immediate that in general, for any open set O, the set of polynomials of degree n that have all their roots in O is itself open. In the case that we consider above, we thus conclude that if for some $t \in I$, $P(t, s)$ has all its roots in S. It is then always possible to find a positive α such that

$$\forall t' \in (t - \alpha, t + \alpha) \cap I, P(t', s)$$

also has all its roots in S. This leads to the following fundamental result.

Theorem 2.3 (Boundary Crossing Theorem): Suppose that $P(a, s)$ has all its roots in S, where $P(b, s)$ has at least one root in U. Then, there exists at least one ρ in $(a, b]$ such that

a) $P(\rho, s)$ has all its roots in $S \cup \partial S$.
b) $P(\rho, s)$ has at least one root in ∂S.

Proof: To prove this result, let us introduce the set E of all real numbers t belonging to $(a, b]$ and satisfying the following property:

$$\mathcal{P}: \forall t' \in (a, t), P(t', s) \text{ has all its roots in } S.$$

By assumption, we know that $P(a, s)$ itself has all its roots in S, and therefore, as we saw already, it is possible to find $\alpha > 0$ such that

$$\forall t' \in [a, a + \alpha) \cap I, P(t', s)$$

also has all its roots in S. From this, we conclude that E is not empty since, for example, $a + \alpha/2$ belongs to E. Moreover, from the definition of E, it is obvious that we have the following property:

$$t_2 \in E, \text{ and } a < t_1 < t_2$$

imply that t_1 itself belongs to E. Given this, it is easy to see that E is an interval, and if we define

$$\rho = \sup_{t \in E} t \tag{1}$$

then we have that

$$E = (a, \rho].$$

A) On the one hand, it is impossible that $P(\rho, s)$ has all its roots in S. If this were the case, then necessarily, $\rho < b$, and it would be possible to find an $\alpha > 0$ such that $\rho + \alpha < b$ and

$$\forall t' \in (\rho - \alpha, \rho + \alpha) \cap I, P(t', s)$$

also has all its roots in S. As a result, $\rho + \alpha/2$ would belong to E, contradicting the definition of ρ in (1).

B) On the other hand, it is also impossible that $P(\rho, s)$ has even one root in the interior of U because a straightforward application of Theorem 2.1 would grant the possibility of finding an $\alpha > 0$ such that

$$\forall t' \in (\rho - \alpha, \rho + \alpha) \cap I, P(t', s)$$

has at least one root in the interior of U, and this would contradict the fact that $\rho - \epsilon$ belongs to E for ϵ small enough. From A) and B), we thus conclude that $P(\rho, s)$ has all its roots in $S \cup \partial S$ and at least one root in ∂S. ■

The above result is interesting but also very intuitive and just states that in going from one open set to another open set disjoint from the first, the root set of a continuous family of polynomials $P(\lambda, s)$ of fixed degree must intersect at some intermediate stage the frontier of the first open set. In the following sections, we will show the power of this simple result as we apply it to some classical stability problems.

III. The Hermite–Biehler Theorem

The first result presented below is the interlacing theorem, which is sometimes referred to as the Hermite–Biehler theorem. We first introduce some general notation and definitions that will be used in the following.

Consider a polynomial of degree n

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n.$$

$P(s)$ is said to be Hurwitz if and only if all its roots lie in the open left half of the complex plane. For a Hurwitz polynomial with real coefficients, we have the following two elementary properties:

Property 3.1: If a polynomial $P(s)$ is Hurwitz, all its coefficients are nonzero and have the same sign, either all positive or all negative.

Proof: The proof follows from the fact that $P(s)$ can be factored into a product of first- and second-degree Hurwitz polynomials for which the property obviously holds.

Property 3.2: If a polynomial $P(s)$ is Hurwitz and of degree n, the phase $\arg(P(j\omega))$ is a continuous and strictly increasing function of ω on $(-\infty, +\infty)$. Moreover, the net increase in phase from $-\infty$ to $+\infty$ is

$$\arg(P(+j\infty)) - \arg(P(-j\infty)) = n\pi. \tag{2}$$

Proof: If $P(s)$ is Hurwitz, we can write

$$P(s) = p_n \prod_{i=1}^{n} (s - s_i), \text{ with } s_i = a_i + jb_i, \text{ and } a_i < 0.$$

Then, we have

$$\begin{aligned}\arg(P(j\omega)) &= \arg(p_n) + \sum_{i=1}^{n} \arg(j\omega - a_i - jb_i)\\ &= \arg(p_n) + \sum_{i=1}^{n} \arctan\left(\frac{\omega - b_i}{-a_i}\right)\end{aligned}$$

and thus, $\arg(P(j\omega))$ is a sum of a constant plus n continuous, strictly increasing functions. Moreover, each of these n functions has a net increase of π in going from $-\infty$ to $+\infty$. ■

The even and odd parts of $P(s)$ are defined as

$$\begin{aligned}P^{\text{even}}(s) &:= p_0 + p_2 s^2 + p_4 s^4 + \cdots,\\ P^{\text{odd}}(s) &:= p_1 s + p_3 s^3 + p_5 s^5 + \cdots\end{aligned} \tag{3}$$

We also define $P^e(\omega)$ and $P^o(\omega)$ as follows:

$$\begin{aligned}P^e(\omega) &:= P^{\text{even}}(j\omega) = p_0 - p_2\omega^2 + p_4\omega^4 - \cdots,\\ P^o(\omega) &:= \frac{P^{\text{odd}}(j\omega)}{j\omega} = p_1 - p_3\omega^2 + p_5\omega^4 - \cdots\end{aligned} \tag{4}$$

$P^e(\omega)$ and $P^o(\omega)$ are both polynomials in ω^2, and as an immediate consequence, their root sets will always be symmetric with respect to the origin of the complex plane. Suppose now that the degree of the polynomial $P(s)$ is even, that is, $n = 2m$, $m > 0$. In that case, we have

$$\begin{aligned}P^e(\omega) &= p_0 - p_2\omega^2 + p_4\omega^4 - \cdots + (-1)^m p_{2m}\omega^{2m},\\ P^o(\omega) &= p_1 - p_3\omega^2 + p_5\omega^4 - \cdots\\ &\quad + (-1)^{m-1} p_{2m-1}\omega^{2m-2}.\end{aligned}$$

Definition 3.1: We say that $P(s)$ satisfies the interlacing property if and only if

a) p_{2m} and p_{2m-1} have the same sign.
b) All the roots of $P^e(\omega)$ and $P^o(\omega)$ are real, and the m positive roots of $P^e(\omega)$ together with the $m - 1$ positive roots of $P^o(\omega)$ interlace in the following manner:

$$\begin{aligned}0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \cdots\\ < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m}.\end{aligned}$$

If, on the contrary, the degree of $P(s)$ are odd, $n = 2m + 1$, $m \geq 0$, and

$$\begin{aligned}P^e(\omega) &= p_0 - p_2\omega^2 + p_4\omega^4 - \cdots + (-1)p_{2m}^m\omega^{2m}\\ P^o(\omega) &= p_1 - p_3\omega^2 + p_5\omega^4 - \cdots + (-1)^m p_{2m+1}\omega^{2m}\end{aligned}$$

and the definition of the interlacing property for this case is then naturally modified to

a) p_{2m+1} and p_{2m} have the same sign.
b) All the roots of $P^e(\omega)$ and $P^o(\omega)$ are real, and the m positive roots of $P^e(\omega)$ together with the m positive roots of $P^e(\omega)$ interlace in the following manner:

$$\begin{aligned}0 < \omega_{e,1} < \omega_{o,1} < \cdots\\ < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m} < \omega_{o,m}.\end{aligned}$$

This last definition is illustrated in Fig. 1.

We can now enunciate and prove the following theorem:

Theorem 3.1 (Interlacing Theorem for Real Polynomials): A real polynomial $P(s)$ is Hurwitz if and only if it satisfies the interlacing property.

Proof: To prove the necessity of the interlacing property, let us suppose that we start with a real Hurwitz polynomial of degree n

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n.$$

We already know from Property 3.1 that all the coefficients p_i have the same sign; thus part a) of the interlacing property is already proven, and we can assume without loss of generality that all the coefficients are positive. To prove part b), we will assume arbitrarily that $P(s)$ is of even degree so that $n = 2m$. Now, we also know from Property 3.2 that the phase of $P(j\omega)$ strictly increases from $-n\pi/2$ to $n\pi/2$ as ω runs from $-\infty$ to $+\infty$. Due to the fact that the roots of $P(s)$ are symmetric with respect to the real axis, it is also true that $\arg(P(j\omega))$ increases from 0 to $+n\pi/2 = m\pi$ as ω goes from 0 to $+\infty$. Hence, as ω goes from 0 to $+\infty$, $P(j\omega)$ starts on the positive real axis ($P(0) = p_0 > 0$), circles strictly counterclockwise around the origin $m\pi$ radians before going to infinity, and never passes through the origin since $P(j\omega) \neq 0$ for all ω. As a result, it is very easy to see that the plot of $P(j\omega)$ has to cut the imaginary axis m times so that the real part of $P(j\omega)$ becomes zero m times as ω increases, at the positive values

$$\omega_{R,1}, \omega_{R,2}, \cdots, \omega_{R,m}.$$

Similarly, the plot of $P(j\omega)$ starts on the positive real axis and cuts the real axis another $m - 1$ times as ω increases so that the imaginary part of $P(j\omega)$ also becomes zero m times (including $\omega = 0$) before growing to infinity as ω goes to infinity at

$$0, \omega_{I,1}, \omega_{I,2}, \cdots, \omega_{I,m-1}.$$

Moreover, since $P(j\omega)$ circles around the origin, we obviously have

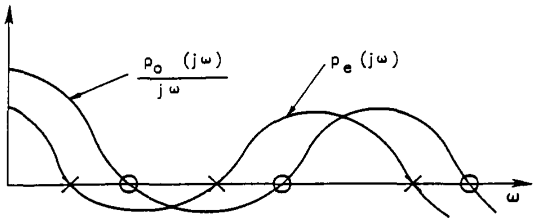

Fig. 1. The interlacing property.

$$0 < \omega_{R,1} < \omega_{I,1} < \omega_{R,2} < \omega_{I,2} < \cdots < \omega_{R,m-1} < \omega_{I,m-1} < \omega_{R,m}.$$

Now, the proof of necessity is completed by simply noticing that the real part of $P(j\omega)$ is nothing but $P^e(\omega)$, and the imaginary part of $P(j\omega)$ is $\omega P^o(j\omega)$.

For the converse, assume that $P(s)$ satisfies the interlacing property, and suppose for example that $P(s)$ is of degree $n = 2m$ and that p_{2m}, p_{2m-1} are both positive. Let us consider the roots of $P^e(\omega)$ and $P^o(\omega)$

$$0 < \omega^p_{e,1} < \omega^p_{o,1} < \cdots < \omega^p_{e,m-1} < \omega^p_{o,m-1} < \omega^p_{e,m}. \tag{5}$$

From this, we deduce that $P^e(\omega)$ and $P^o(\omega)$ can be written as

$$P^e(\omega) = p_{2m} \prod_{i=1}^{m} (\omega^2 - \omega^{p2}_{e,i})$$

$$P^o(\omega) = p_{2m-1} \prod_{i=1}^{m-1} (\omega^2 - \omega^{p2}_{o,i}).$$

Now, let us consider a polynomial $Q(s)$ that is known to be stable, of the same degree $2m$, and with all its coefficients positive. For example, we could take $Q(s) = (s + 1)^{2m}$. In any event, write

$$Q(s) = q_0 + q_1 s + q_2 s^2 + \cdots + q_{2m}s^{2m}.$$

Since $Q(s)$ is stable, we know from the first part of the theorem that $Q(s)$ satisfies the interlacing theorem so that $Q^e(\omega)$ has m positive roots $\omega^q_{e,1}, \cdots, \omega^q_{e,m}$, $Q^o(\omega)$ has $m - 1$ positive roots $\omega^q_{o,1}, \cdots, \omega^q_{o,m-1}$, and

$$0 < \omega^q_{e,1} < \omega^q_{o,1} < \cdots < \omega^q_{e,m-1} < \omega^q_{o,m-1} < \omega^q_{e,m} < \omega^q_{o,m}. \tag{6}$$

Therefore, we can also write

$$Q^e(\omega) = q_{2m} \prod_{i=1}^{m} (\omega^2 - \omega^{q2}_{e,i})$$

$$Q^o(\omega) = q_{2m-1} \prod_{i=1}^{m-1} (\omega^2 - \omega^{q2}_{o,i}).$$

Consider now the polynomial $P_\lambda(s)$ defined by

$$P^e_\lambda(\omega) := \left((1-\lambda)q_{2m} + \lambda p_{2m}\right) \cdot \prod_{i=1}^{m} \left(\omega^2 - \left[(1-\lambda)\omega^q_{e,i} + \lambda\omega^p_{e,i}\right]^2\right)$$

$$P^o_\lambda(\omega) := \left((1-\lambda)q_{2m-1} + \lambda p_{2m-1}\right) \cdot \prod_{i=1}^{m-1} \left(\omega^2 - \left[(1-\lambda)\omega^q_{o,i} + \lambda\omega^p_{o,i}\right]^2\right).$$

Obviously, the coefficients of $P_\lambda(s)$ are polynomial functions in λ, which are therefore continuous on $[0, 1]$. Moreover, the coefficient of the highest degree term of $P_\lambda(s)$ is $(1-\lambda)q_{2m} + \lambda p_{2m}$ and always remains positive as λ varies from 0 to 1. For $\lambda = 0$, we have $P_0(s) = Q(s)$ and for $\lambda = 1$, $P_1(s) = P(s)$. Suppose now that $P(s)$ is not Hurwitz. From the boundary crossing theorem, we then know that there necessarily exists some λ in $(0, 1]$ such that $P_\lambda(s)$ has a root on the imaginary axis. However, $P_\lambda(s)$ has a root on the imaginary axis if and only if $P^e_\lambda(\omega)$ and $P^o_\lambda(\omega)$ have a common root, but obviously, the roots of $P^e_\lambda(\omega)$ satisfy

$$\omega^{\lambda 2}_{e,i} = \left((1-\lambda)\omega^q_{e,i} + \lambda\omega^p_{e,i}\right)^2 \tag{7}$$

and those of $P^o_\lambda(\omega)$

$$\omega^{\lambda 2}_{o,i} = \left((1-\lambda)\omega^q_{o,i} + \lambda\omega^p_{o,i}\right)^2. \tag{8}$$

Now, take any two roots of $P^e_\lambda(\omega)$ in (7). If $i < j$, we know from (5) that $\omega^{p2}_{e,i} < \omega^{p2}_{e,j}$, and similarly, from (6), $\omega^{q2}_{e,i} < \omega^{q2}_{e,j}$ so that we also have

$$\omega^{\lambda 2}_{e,i} < \omega^{\lambda 2}_{e,j}.$$

In the same way, it can be seen that the same order as in (5) and (6) is preserved between the roots of $P^o_\lambda(\omega)$ as well as between any root of $P^e_\lambda(\omega)$ and any root of $P^o_\lambda(\omega)$. In other words, part b) of the interlacing property is invariant under such convex combinations so that we also have for every λ in $[0, 1]$:

$$0 < \omega^{\lambda 2}_{e,1} < \omega^{\lambda 2}_{o,1} < \cdots < \omega^{\lambda 2}_{e,m-1} < \omega^{\lambda 2}_{o,m-1} < \omega^{\lambda 2}_{e,m} < \omega^{\lambda 2}_{o,m}.$$

However, this shows that whatever the value of λ in $[0, 1]$, $P^e_\lambda(\omega)$ and $P^o_\lambda(\omega)$ can never have a common root, and this therefore leads to a contradiction, which completes the proof. ■

Remark 1: The same kind of theorem holds for polynomials with complex coefficients:

$$P(s) = (a_0 + jb_0) + (a_1 + jb_1)s + \cdots + (a_{n-1} + jb_{n-1})s^{n-1} + (a_n + jb_n)s^n.$$

As in the real case, one can show that the real and imaginary parts of $P(j(\omega))$ satisfy an interlacing property that is very similar to the one we defined earlier. However, these real and imaginary parts no longer correspond to the even and odd parts of $P(s)$ but rather to the two polynomials

$$P_R(s) = a_0 + jb_1 s + a_2 s^2 + jb_3 s^3 + \cdots,$$

$$P_I(s) = jb_0 + a_1 s + jb_2 s^2 + a_3 s^3 + \cdots.$$

Remark 2: In fact, it is always possible to derive results similar to the interlacing theorem with respect to any stability region S, which has the property that the phase of the polynomial evaluated along the boundary of S increases monotonically and undergoes a net change of $n\pi$. In this case, the stability of the polynomial with respect to S is equivalent to the interlacing of its real and imaginary parts evaluated along the boundary of S.

Consider, for example, the Schur or unit circle stability of a real polynomial

$$P(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0.$$

It is important to prove that the stability of $P(z)$ is equivalent to the interlacing of the real and imaginary parts of $P(z)$ evaluated along the upper half of the unit circle. More precisely, the two functions of θ

$$R(\theta) = p_n \cos(n\theta) + \cdots + p_1 \cos(\theta) + p_0$$

and

$$I(\theta) = p_n \sin(n\theta) + \cdots + p_1 \sin(\theta)$$

must interlace on $[0, \pi]$.

This condition can in fact be further refined to the interlacing on the unit circle of the two polynomials

$$P_1(z) = \tfrac{1}{2}(P(z) + z^n P(1/z))$$

and

$$P_2(z) = \tfrac{1}{2}(P(z) - z^n P(1/z)).$$

In the next two sections, we use the results of Sections II and III to give elementary proofs of Jury's (unit circle) and Routh's (left half plane) stability tests.

IV. Schur Stability

The problem of checking the stability of a discrete time system reduces to the determination of whether or not the roots of the characteristic polynomial of the system lie strictly within the unit circle or not. In this section, we develop a simple test procedure for this problem based on the boundary crossing theorem. The procedure turns out to be equivalent to Jury's test for unit circle stability.

First, we recall that a polynomial is said to be Schur if it has all its roots inside the unit circle. Now, let

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n$$

be a polynomial of degree n. We have the following necessary condition.

Property 4.1: A necessary condition for $P(z)$ to be Schur is that

$$|p_n| > |p_0|.$$

In effect, if $P(z)$ has all its roots $z_1, \cdots, z_n$ inside the unit circle, the product of these roots is given by

$$\prod_{i=1}^{n} z_i = \frac{p_0}{p_n}$$

hence

$$\left|\frac{p_0}{p_n}\right| = \prod_{i=1}^{n} |z_i| < 1.$$

Now, consider a polynomial $P(z)$ of degree n

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n$$

and let us define

$$Q(z) = z^n P(1/z) = p_0 z^n + p_1 z^{n-1} + \cdots + p_{n-1} z + p_n$$

and

$$R(z) = (1/z)(P(z) - (p_0/p_n)Q(z)),$$
$$\text{(always of degree} \leq n-1).$$

Then, we have the following key lemma, which allows the degree of the test polynomial to be reduced without losing stability information.

Lemma 4.1: If $P(z)$ satisfies $|p_n| > |p_0|$, we have the following equivalence:

$$P(z) \text{ Schur} \Leftrightarrow R(z) \text{ Schur}.$$

Proof: First notice that we obviously have

$$R(z) \text{ Schur} \Leftrightarrow zR(z) \text{ Schur}.$$

Now, consider the family of polynomials

$$P_\lambda(z) = P(z) - \lambda(p_0/p_n)\,Q(z), \quad \text{where } \lambda \in [0, 1].$$

We can see that $P_0(z) = P(z)$, and $P_1(z) = zR(z)$. Moreover, the coefficient of degree n of $P_\lambda(z)$ is $p_n - \lambda p_0^2/p_n$, and satisfies

$$|p_n - \lambda p_0^2/p_n| = |p_n - \lambda(p_0/p_n)p_0| > |p_n| - \lambda|p_0/p_n||p_0| > |p_n| - |p_0| > 0$$

so that $P_\lambda(z)$ remains of fixed degree n.

Assume now by contradiction that one of these two polynomials is stable, whereas the other one is not. Then, from the boundary crossing theorem, we can conclude that there must exist a λ in $[0, 1]$ such that $P_\lambda(z)$ has a root on the unit circle at the point $z_0 = e^{j\theta}$, $\theta \in [0, 2\pi)$, that is

$$P_\lambda(z_0) = P(z_0) - \lambda(p_0/p_n)\,z_0^n P(1/z_0) = 0. \quad (9)$$

Then, we have two cases:

1) $z_0 = \pm 1$. Then $1/z_0 = z_0$, and therefore, from (9)

$$P(z_0)\underbrace{(1 - \lambda p_0/p_n(\pm 1)^n)}_{\neq 0} = 0$$

but this implies that $P(z_0) = P(1/z_0) = 0$, and therefore, $z_0 R(z_0) = 0$, which is a contradiction since we assumed that at least one of the polynomials $P(z)$ and $zR(z)$ was stable.

2) $z_0 = e^{j\theta}$, $\theta \neq 0$, $\theta \neq \pi$. In this case, $P_\lambda(z)$ being a real polynomial, we know that $z_0^* = 1/z_0$ is also a root

of $P_\lambda(z)$. Then

$$P(z_0) - \lambda(p_0/p_n)\, z_0^n P(1/z_0) = 0$$

and

$$P(1/z_0) - \lambda(p_0/p_n)(1/z_0)^n\, P(z_0) = 0$$

so that

$$P(z_0)\underbrace{(1 - \lambda^2 p_0^2/p_n^2)}_{\neq 0} = 0.$$

Again, this implies that $P(z_0) = 0$ as well as $P(1/z_0) = 0$; thus also, $R(z_0) = 0$, which again leads to a contradiction. ■

The above lemma leads to the following procedure for successively reducing the degree and testing for stability. Starting with a polynomial $P(z)$ that one wants to check for stability, one follows the following procedure:

1) Set $P^{(0)}(z) = P(z)$.
2) Verify $|p_n^{(i)}| > |p_0^{(i)}|$.
3) Construct $P^{(i+1)}(z) = 1/z(P^{(i)}(z) - (p_0^{(i)}/p_n^{(i)})z^n P^{(i)}(1/z))$.
4) Go back to 2) until you either find that 2) is violated ($P(z)$ is not Schur) or until you reach $P^{(n-1)}(z)$ (which is of degree 1) in which case condition 2) is also sufficient, and $P(z)$ is Schur.

It can be verified by the reader that this procedure leads precisely to the Jury stability test.

Example: Consider a polynomial of degree 3 in the variable z

$$P(z) = z^3 + az^2 + bz + c.$$

According to our algorithm, we form the following polynomial

$$\begin{aligned} P^{(1)}(z) &= 1/z(P(z) - cz^3 P(1/z)) \\ &= (1 - c^2)z^2 + (a - bc)z + b - ac \end{aligned}$$

and then

$$\begin{aligned} P^{(2)}(z) &= 1/z\left(P^{(1)}(z) - \left(\frac{b - ac}{1 - c^2}\right)z^2 P^{(1)}(1/z)\right) \\ &= \frac{(1 - c^2)^2 - (b - ac)^2}{1 - c^2}\, z \\ &\quad + (a - bc)\left(1 - \frac{b - ac}{1 - c^2}\right). \end{aligned}$$

On the other hand, the Jury's table is given by

c	b	a	1
1	a	b	c
$c^2 - 1$	$cb - a$	$ca - b$	
$ca - b$	$cb - a$	$c^2 - 1$	
$(c^2 - 1)^2 - (ca - b)^2$	$(bc - a)((c^2 - 1) - (ca - b))$.		

We can see here that the first two lines of this table correspond to the coefficients of $P(z)$, the third and fourth lines to those of $P^{(1)}(z)$, and the last one to a constant times $P^{(2)}(z)$, and the tests to be carried out are the same.

V. Hurwitz Stability

We now turn to the problem of left half plane or Hurwitz stability and develop an elementary test procedure for it based on the interlacing theorem and therefore on the boundary crossing theorem. This procedure turns out to be equivalent to Routh's well-known test.

Let $P(s)$ be a polynomial of degree $n > 0$, and assume that all the coefficients of $P(s)$ are positive:

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n,$$

$$p_i > 0 \text{ for } i = 0, \cdots, n.$$

Remember that $P(s)$ can be decomposed into its odd and even parts as

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s).$$

Now, define the polynomial $Q(s)$ of degree $n - 1$ by the following:

$$\begin{aligned} \text{If } n = 2m\text{: } Q(s) &= (P^{\text{even}}(s) \\ &\quad - p_{2m}/p_{2m-1} s P^{\text{odd}}(s)) \\ &\quad + P^{\text{odd}}(s). \\ \text{If } n = 2m + 1\text{: } Q(s) &= (P^{\text{odd}}(s) \\ &\quad - p_{2m+1}/p_{2m} s P^{\text{even}}(s)) \\ &\quad + P^{\text{even}}(s). \end{aligned} \tag{10}$$

That is, in general, with $\mu = p_n/p_{n-1}$

$$\begin{aligned} Q(s) &= p_{n-1}s^{n-1} + (p_{n-2} - \mu p_{n-3})s^{n-2} + p_{n-3}s^{n-3} \\ &\quad + (p_{n-4} - \mu p_{n-3})s^{n-4} + \cdots. \end{aligned} \tag{11}$$

We then have the following key result on degree reduction:

Lemma 5.1: If $P(s)$ has all its coefficients positive

$$P(s) \text{ is stable} \Leftrightarrow Q(s) \text{ is stable.}$$

Proof: We assume, for example, that $n = 2m$ and use the interlacing theorem.

a) Assume that $P(s) = p_0 + \cdots + p_{2m}s^{2m}$ is stable and therefore satisfies the interlacing theorem. Let

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \omega_{o,2} < \cdots$$
$$< \omega_{e,2m-1} < \omega_{o,2m-1} < \omega_{e,2m}$$

be the interlacing roots of $P^e(\omega)$ and $P^o(\omega)$. One can easily check by using (10) and the definitions (4) that $Q^e(\omega)$ and $Q^o(\omega)$ are given by

$$Q^e(\omega) = P^e(\omega) + \mu\omega^2 P^o(\omega), \quad \mu = p_{2m}/p_{2m-1},$$
$$Q^o(\omega) = P^o(\omega).$$

From this, we already conclude that $Q^o(\omega)$ has the required number of positive roots, namely, the $m - 1$ roots of $P^o(\omega)$:

$$\omega_{o,1}, \omega_{o,2}, \cdots, \omega_{o,m-1}.$$

Moreover, due to the form of $Q^e(\omega)$, we can deduce that

$$Q^e(0) = P^e(0) > 0$$
$$Q^e(\omega_{o,1}) = P^e(\omega_{o,1}) < 0$$
$$\vdots$$
$$Q^e(\omega_{o,m-2}) = P^e(\omega_{o,m-2}), \text{ has the sign of } (-1)^{m-2}$$
$$Q^e(\omega_{o,m-1}) = P^e(\omega_{o,m-1}), \text{ has the sign of } (-1)^{m-1}.$$

Hence, we can already conclude that $Q^e(\omega)$ has $m - 1$ positive roots $\omega'_{e,1}, \omega'_{e,2}, \cdots, \omega'_{e,m-1}$, which interlace with the roots of $Q^o(\omega)$. Since $Q^e(\omega)$ is of degree $m - 1$ in ω^2, these are the only positive roots it can have. Moreover, we have seen that the sign of $Q^e(\omega)$ at the last root $\omega_{o,m-1}$ of $Q^o(\omega)$ is that of $(-1)^{m-1}$, but the highest coefficient of $Q^e(\omega)$ is nothing but

$$q_{2m-2}(-1)^{m-1}.$$

From this, we see that q_{2m-2} must be strictly positive, as is $q_{2m-1} = p_{2m-1}$; otherwise, $Q^e(\omega)$ would again have a change of sign between $\omega_{o,m-1}$ and $+\infty$, which would result in the contradiction of $Q^e(\omega)$ having m positive roots (whereas it is a polynomial of degree only $m - 1$ in ω^2). Therefore, $Q(s)$ satisfies the interlacing property and is stable if $P(s)$ is stable as well.

b) Conversely, assuming that $Q(s)$ is stable, we can write that

$$P(s) = \left(Q^{\text{even}}(s) + \mu s Q^{\text{odd}}(s)\right) + Q^{\text{odd}}(s).$$

By the same reasoning as in a), we can see that $P^o(\omega)$ already has the required number $m - 1$ of positive roots and that $P^e(\omega)$ already has $m - 1$ roots in the interval $(0, \omega_{o,m-1})$ that interlace with the roots of $P^o(\omega)$. Moreover, the sign of $P^e(\omega)$ at $\omega_{o,m-1}$ is the same as $(-1)^{m-1}$, whereas by adding the term $p_{2m}s^{2m}$ to $P(s)$, the sign of $P^e(\omega)$ at $+\infty$ is that of $(-1)^m$. Thus, $P^e(\omega)$ has a mth positive root:

$$\omega_{e,m} > \omega_{o,m-1}.$$

Thus, $P(s)$ satisfies the interlacing property and is therefore stable. ■

The above lemma shows how the stability of a polynomial $P(s)$ can be checked by successively reducing its degree as follows:

1) Set $P^{(0)}(s) = P(s)$.
2) Verify that all the coefficients of $P^{(i)}(s)$ are positive.
3) Construct $P^{(i+1)}(s)$ according to (11).
4) Go back to 2) until you either find that any 2) is violated ($P(s)$ is not Hurwitz) or until you reach $P^{(n-2)}(s)$ (which is of degree 2) in which case, condition 2) is also sufficient ($P(s)$ is Hurwitz).

The reader may verify that this procedure is identical to Routh's test since it generates the Routh's table. However our procedure does not allow us to count the stable and unstable zeros of the polynomial as can be done with Routh's theorem.

Example: Consider a polynomial of degree 4

$$P(s) = s^4 + as^3 + bs^2 + cs + d.$$

Following the algorithm above, we form the polynomials

$$\mu = 1/a$$

and

$$P^{(1)}(s) = as^3 + (b - c/a)s^2 + cs + d$$

and then

$$\mu = \frac{a^2}{ab - c}$$

and

$$P^{(2)} = (b - c/a)s^2 + \left(c - \frac{a^2 d}{ab - c}\right)s + d.$$

Considering that at each step, only the even or the odd part of the polynomial is modified, we have to verify the positiveness of the following set of coefficients:

$$\begin{array}{lll} 1 & b & d \\ a & c & \\ b - c/a & d & \\ c - \dfrac{a^2 d}{ab - c}. & & \end{array}$$

However, this the just the Routh table for this polynomial. Our proof also shows the following well-known property: *All* the numbers that appear in the Routh table of a Hurwitz polynomial are positive (and not only the first column).

VI. Concluding Remarks

In this paper, we have presented a unified approach to determining the Hurwitz or Schur stability of a polynomial. The unification is achieved by a systematic use of the so-called boundary crossing theorem. This results in a simple derivation of the Routh and Jury tables. We expect that many other results in stability theory can be similarly simplified by approaching them via this elementary notion.

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