Chapter 5

The inverse Nyquist array design method

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Synopsis

The general problem of the design of multivariable control systems is considered and the stability of multivariable feedback systems is examined. The concept of 'diagonal dominance' is introduced, and Rosenbrock's Inverse Nyquist Array Design Method is developed. Methods of achieving diagonal dominance are discussed and illustrated in terms of practical problems.

5.1 Introduction

The design of control systems for single-input single-output plant using the classical frequency response methods of Bode, Nyquist and Nichols is well established. However, the frequency response approach of Nyquist has been extended by Rosenbrock[1] to deal with multi-input multi-output plant where significant interaction is present.

During the last decade, interactive computing facilities have developed rapidly and it is now possible to communicate with a digital computer in a variety of ways; e.g. graphic display systems with cursors, joysticks, and light-pens. Equally, the digital computer can present information to the user in the form of graphs on a display terminal or as hard-copy on a digital plotter. The classical frequency-response methods for single-input single-output systems rely heavily on graphical representations, and Rosenbrock's 'inverse' Nyquist array' design method for multivariable systems suitably exploits the graphical output capabilities of the modern digital computer system. Also, the increased complexity of multivariable systems has made it necessary to employ interactive computer-aided design facilities, such as those developed at the Control Systems Centre, UMIST, Manchester[2], in order to establish an effective dialogue with the user.

5.2 The Multivariable Design Problem

In general, a multivariable control system will have m inputs and ℓ outputs and the system can be described by an $\ell \times m$ transfer function matrix G(s). Since we are interested in feedback control we almost always have $\ell = m$.

The uncontrolled plant is assumed to be described by an rational transfer function matrix G(s) and we wish to determine a controller matrix K(s) such that when we close the feedback loops through the feedback matrix F, as in Figure 5.1, the system is stable and has suitably fast responses.

The matrix F is assumed diagonal and independent of s, i.e.

$$F = \text{diag}\{f_i\} \tag{5.1}$$

F represents loop gains which will usually be implemented in the forward path, but which it is convenient to move into the return path. In addition, the design will have high integrity if the system remains stable as the gain in each loop is reduced.

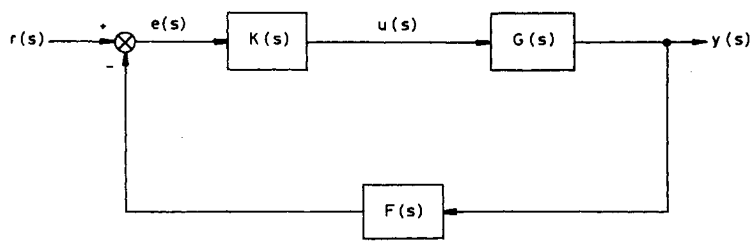

Fig. 5.1

Let us consider the controller matrix K(s) to consist of the product of two matrices $K_p(s)$ and K_d, i.e.

$$K(s) = K_p(s)\, K_d \tag{5.2}$$

where K_d is a diagonal matrix independent of s; i.e.

$$K_d = \text{diag}\ \{k_i\}, \quad i = 1,\dots,m \tag{5.3}$$

Then the system of Figure 5.1 can be re-arranged as shown in Figure 5.2 where the stability of the closed-loop system is unaffected by the matrix K_d outside the loop. For convenience, we rename the matrix K_dF as F. All combinations of gains and of open or closed-loops can now be obtained by a suitable choice of the f_i, and we shall want the system to remain stable for all values of the f_i from zero up to their design values.

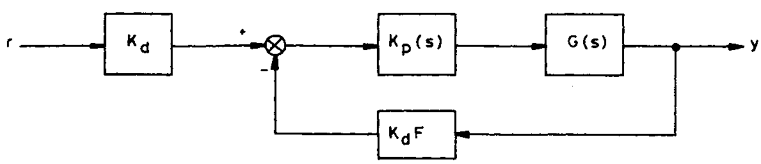

Fig. 5.2

When $f_1 = 0$ the first loop is open, and all gains in the first loop up to the design value f_{1d} can be achieved by increasing f_1.

The elements f_i of $F = \text{diag}\ \{f_i\}$ can be represented by points in an m-dimensional space which can be called the *gain space*. That part of the gain space in which $f_i > 0$, $i = 1,\dots,m$ corresponds to negative feedback in all loops, and is the region of most practical interest. The point $\{f_1, f_2, \dots, f_m\}$ belongs to the asymptotically stable region in the gain space if and only if the system is asymptotically stable with $F = \text{diag}\ \{f_i\}$.

Let

$$Q(s) = G(s)K(s) \tag{5.4}$$

then the closed-loop system transfer function matrix $H(s)$ is given by

$$H(s) = (I+Q(s)F)^{-1}Q(s) = Q(s)(I+FQ(s))^{-1} \tag{5.5}$$

Consider the open-loop system

$$Q(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} \tag{5.6}$$

Then if $f_1 = 10$ and $f_2 = 0$, $H(s)$ has all of its poles in the open left-half plane and the system is asymptotically stable. However, for $f_1 = 10$ and $f_2 = 10$, the closed-loop system $H(s)$ is unstable, as shown in Figure 5.3 .

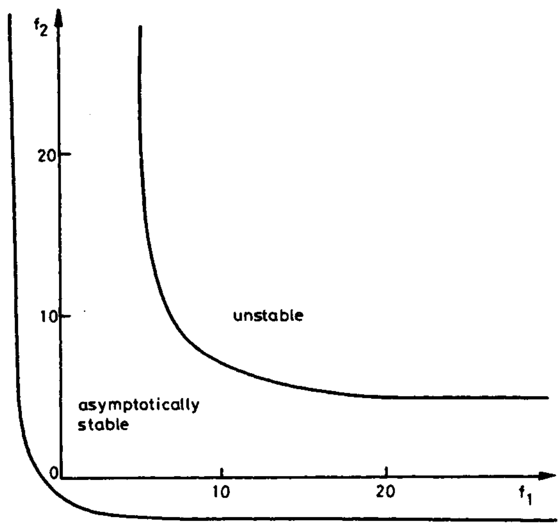

Fig. 5.3

This situation could have been predicted by examining the McMillan form of $Q(s)$ which is

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)(s+3)} & 0 \\ 0 & \frac{(s-1)}{(s+1)} \end{bmatrix} \qquad (5.7)$$

where the poles of the McMillan form are referred to as the "poles of the system" and the zeros of the McMillan form are referred to as the "zeros of the system". If any of the "zeros" of $Q(s)$ lie in the closed right-half plane, then it will not be possible to set up m high gain control loops around this system.

Consider now the gain space for the system described by

$$Q(s) = \begin{bmatrix} \frac{s-1}{(s+1)^2} & \frac{5s+1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s-1}{(s+1)^2} \end{bmatrix} \qquad (5.8)$$

which is shown in Figure 5.4 . Now, up to a certain point, increase of gain in one loop allows increase of gain in the other loop without instability. The McMillan form of this latter system $Q(s)$ is

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{s+2}{s+1} \end{bmatrix} \qquad (5.9)$$

which implies that, despite the non-minimum phase terms in the diagonal elements of $Q(s)$, no non-minimum phase behaviour will be obtained with the feedback loops closed. It is also interesting to note that this system is stable with a small amount of positive feedback. However, this is not a high integrity system, since failure of any one loop may put the system into an unstable region of operation.

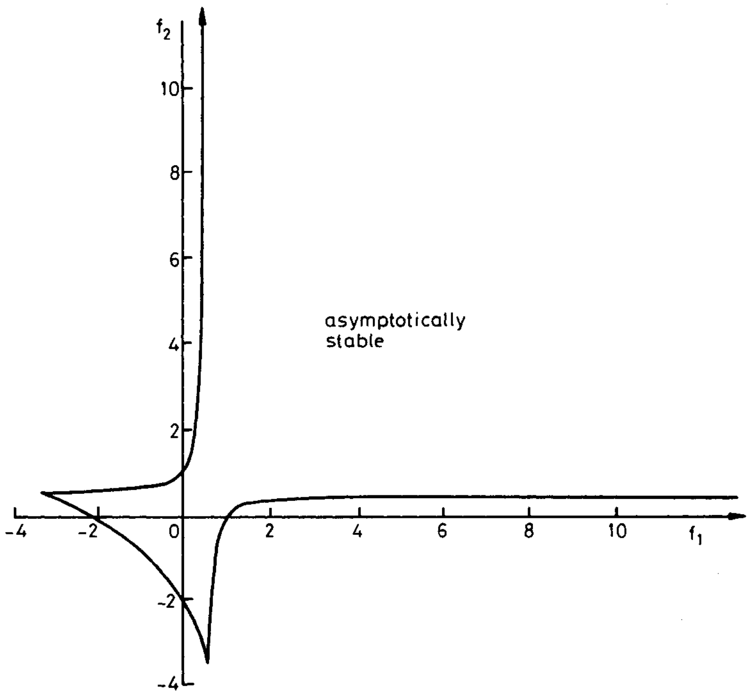

Fig. 5.4

5.3 Stability

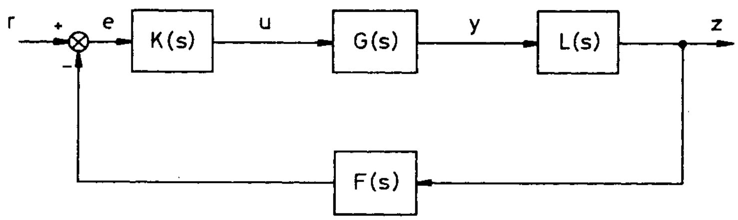

Fig. 5.5

Consider the system shown in Figure 5.5, in which all matrices are mxm; this last condition is easily relaxed (see Reference 1, p. 131 ff). Let $Q = LGK$, and suppose that G arises from the system matrix

$$P_G(s) = \begin{pmatrix} T_G(s) & U_G(s) \\ -V_G(s) & W_G(s) \end{pmatrix} \tag{5.10}$$

and similarly for $K(s)$, $L(s)$, $F(s)$. For generality, we do not require any of these system matrices to have least order. The equations of the closed-loop system can then be written as

$$\left(\begin{array}{cccccccc|c} T_K & U_K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -V_K & W_K & 0 & -I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_G & U_G & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -V_G & W_G & 0 & -I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_L & U_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -V_L & W_L & 0 & -I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_F & U_F & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & -V_F & W_F & -I_m \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{array}\right) \left(\begin{array}{c} \bar{\xi}_K \\ -\bar{e} \\ \bar{\xi}_G \\ -\bar{u} \\ \bar{\xi}_L \\ -\bar{y} \\ \bar{\xi}_F \\ -\bar{z} \\ \hline -\bar{v} \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline -\bar{z} \end{array}\right) \tag{5.11}$$

Here the matrix on the left-hand side of equation (5.11) is a system matrix for the closed-loop system shown in Figure 5.5, which we can also write as

$$P_H = \begin{pmatrix} T_H & U_H \\ -V_H & W_H \end{pmatrix} \tag{5.12}$$

The closed-loop system poles are the zeros of $|T_H(s)|$, and Rosenbrock has shown[1] that

$$|T_H(s)| = |I_m+Q(s)F(s)|\,|T_L(s)|\,|T_G(s)|\,|T_K(s)|\,|T_F(s)| \tag{5.13}$$

As the zeros of $|T_L|\,|T_G|\,|T_K|\,|T_F|$ are the open-loop poles, we need only the following information to investigate stability,

(i) The rational function $|I_m+Q(s)F(s)|$

(ii) The locations of any open-loop poles in the closed right half-plane (crhp).

Notice that this result holds whether or not the subsystems have least order. If we apply Cauchy's theorem, (ii) can be further reduced: all we need is the number p_o of open-loop poles in the crhp.

From equation (5.5)

$$|I_m+QF| = |Q(s)|/|H(s)| \qquad (5.14)$$

The Nyquist criterion depends on encirclements of a critical point by the frequency response locus of the system to indicate stability. Let D be the usual Nyquist stability contour in the s-plane consisting of the imaginary axis from -jR to +jR, together with a semi-circle of radius R in the right half plane. The contour D is supposed large enough to enclose all finite poles and zeros of $|Q(s)|$ and $|H(s)|$, lying in the closed right half plane.

Let $|Q(s)|$ map D into Γ_Q, while $|H(s)|$ maps D into Γ_H. As s goes once clockwise around D, let Γ_Q encircle the origin N_Q times clockwise, and let Γ_H encircle the origin N_H times clockwise. Then, if the open-loop system characteristic polynomial has p_o zeros in the closed right half plane, the closed-loop system is asymptotically stable if and only if

$$N_H - N_Q = p_o \qquad (5.15)$$

This form of the stability theorem is directly analogous to the form used with single-input single-output systems, but is difficult to use since $|H(s)|$ is a complicated function of $Q(s)$, namely

$$|H(s)| = |Q(s)|/|I+QF| \qquad (5.16)$$

Equation (5.5),

$$H(s) = (I+Q(s)F)^{-1}Q(s)$$

shows that the relationship between the open-loop system

$Q(s)$ and the closed-loop system $H(s)$ is not simple. However, if $Q^{-1}(s)$ exists then

$$H^{-1}(s) = F + Q^{-1}(s) \tag{5.17}$$

which is simpler to deal with. Instead of $H^{-1}(s)$ and $Q^{-1}(s)$ we shall write $\hat{H}(s) = H^{-1}(s)$ and $\hat{Q}(s) = Q^{-1}(s)$. Then, the $\hat{h}_{ii}(s)$ are the diagonal elements of $H^{-1}(s)$. In general, $\hat{h}_{ii}(s) \neq h_{ii}^{-1}(s)$, where $h_{ii}^{-1}(s)$ is the inverse of the diagonal element $h_{ii}(s)$ of $H(s)$.

We shall, in what follows, develop the required stability theorems in terms of the inverse system matrices. Also, we note that if $K(s)$ has been chosen such that $Q(s) = G(s)K(s)$ is diagonal and if F is diagonal, then we have m single loops. However, several objections to this approach can be made. In particular, it is an unnecessary extreme. Instead of diagonalising the system, we shall consider the much looser criterion of diagonal dominance.

5.3.1 Diagonal dominance[1]

A rational mxm matrix $\hat{Q}(s)$ is *row diagonal dominant* on D if

$$|\hat{q}_{ii}(s)| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ij}(s)| \tag{5.18a}$$

for $i = 1,\ldots,m$ and all s on D. *Column diagonal dominance* is defined similarly by

$$|\hat{q}_{ii}(s)| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ji}(s)| \tag{5.18b}$$

The dominance of a rational matrix $\hat{Q}(s)$ can be determined by a simple graphical construction. Let $\hat{q}_{ii}(s)$ map D into $\hat{\Gamma}_i$ as in Figure 5.6 . This will look like an inverse Nyquist plot, but does not represent anything directly measurable on the physical system. For each s on D draw a circle of radius

$$d_i(s) \quad = \quad \sum_{\substack{j=1 \\ j\neq i}}^{m} |\hat{q}_{ij}(s)| \tag{5.19}$$

centred on the appropriate point of $\hat{q}_{ii}(s)$, as in Figure 5.6 . Do the same for the other diagonal elements of $\hat{Q}(s)$. If each of the bands so produced excludes the origin, for $i = 1,\ldots,m$, then $\hat{Q}(s)$ is row dominant on D. A similar test for column dominance can be defined by using circles of radius

$$d_i'(s) \quad = \quad \sum_{\substack{j=1 \\ j\neq i}}^{m} |\hat{q}_{ji}(s)| \tag{5.20}$$

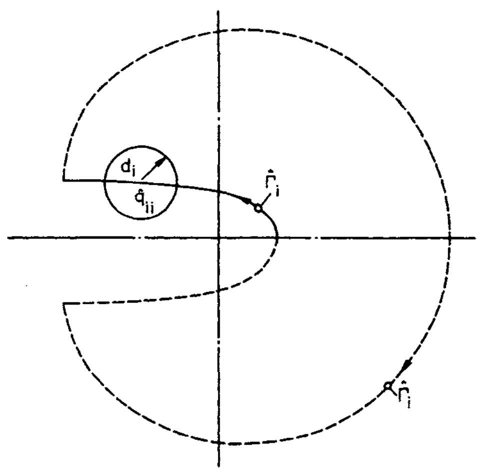

Fig. 5.6

5.3.2 Further stability theorems

If $\hat{Q}(s)$ is row (or column) dominant on D, having on it no zero of $|\hat{Q}(s)|$ and no pole of $\hat{q}_{ii}(s)$, for $i = 1,\ldots,m$, then let $\hat{q}_{ii}(s)$ map D into $\hat{\Gamma}_i$ and $|\hat{Q}(s)|$ map D into $\hat{\Gamma}_Q$. If $\hat{\Gamma}_i$ encircles the origin $\hat{N}_i$ times and $\hat{\Gamma}_Q$ encircles the origin $\hat{N}_Q$ times, all encirclements being clockwise, then Rosenbrock[1] has shown that

$$\hat{N}_Q = \sum_{i=1}^{m} \hat{N}_i \tag{5.21}$$

A proof of this result is given in the Appendix.

Let $\hat{Q}(s)$ and $\hat{H}(s)$ be dominant on D, let $\hat{q}_{ii}(s)$ map D into $\hat{\Gamma}_{qi}$ and let $\hat{h}_{ii}(s)$ map D into $\hat{\Gamma}_{hi}$. Let these encircle the origin $\hat{N}_{qi}$ and $\hat{N}_{hi}$ times respectively. Then with p_o defined as in (5.15), the closed-loop system is asymptotically stable if, and only if,

$$\sum_{i=1}^{m} \hat{N}_{qi} - \sum_{i=1}^{m} \hat{N}_{hi} = p_o \tag{5.22}$$

This expression represents a generalised form of Nyquist's stability criterion, applicable to multivariable systems which are diagonal dominant.

$$\text{For } |Q|, \quad \hat{N}_Q = - \sum_{i=1}^{m} \hat{N}_{qi} \tag{5.23}$$

and since we are considering inverse polar plots stability is determined using

$$\hat{N}_H - \hat{N}_Q = - p_o + p_c \tag{5.24}$$

where $p_c = 0$ for closed-loop stability.

$$\text{Hence,} \quad \hat{N}_Q - \hat{N}_H = p_o \tag{5.25}$$

replaces the relationship used with direct polar plots, given by equation (5.15).

5.3.3 Graphical criteria for stability

If for each diagonal element $\hat{q}_{ii}(s)$, the band swept out by its circles does not include the origin or the critical point $(-f_i,0)$, and if this is true for $i = 1,2,\ldots,m$, then the generalised form of the inverse Nyquist stability criterion, defined by (5.25), is satisfied. In general, the $\hat{q}_{ii}(s)$ do not represent anything directly measurable on the system. However, using a theorem due to Ostrowski[3]

$$h_i^{-1}(s) = h_{ii}^{-1}(s) - f_i \tag{5.26}$$

is contained within the band swept out by the circles centred on $\hat{q}_{ii}(s)$, and this remains true for all values of gain f_i in each other loop j between zero and f_{jd}. Note that $h_{ii}^{-1}(s)$ is the inverse transfer function seen between input i and output i with all loops closed. The transfer function $h_i(s)$ is that seen in the ith loop when this is open, but the other loops are closed. It is this transfer function for which we must design a single-loop controller for the ith loop.

The theorems above tell us that as the gain in each other loop changes as long as dominance is maintained in the other loops, $h_{ii}^{-1}(s)$ will also change but always remains inside the ith Gershgorin band. The band within which $h_i^{-1}(s)$ lies can be further narrowed. If $\hat{Q}$ and $\hat{H}$ are dominant, and if

$$\phi_i(s) = \max_{\substack{j \\ j \neq i}} \frac{d_j(s)}{|f_j + \hat{q}_{jj}(s)|} \tag{5.27}$$

then $h_i^{-1}(s)$ lies within a band based on $\hat{q}_{ii}(s)$ and defined by circles of radius

$$r_i(s) = \phi_i(s) d_i(s) \tag{5.28}$$

Thus, once the closed-loop system gains have been chosen such that stability is achieved in terms of the larger bands, then a measure of the gain margin for each loop can be determined by drawing the smaller bands, using the 'shrinking factors' $\phi_i(s)$ defined by (5.27), with circles of radius r_i. These narrower bands, known as Ostrowski bands, also reduce the region of uncertainty as to the actual location of the inverse transfer function $h_{ii}^{-1}(s)$ for each loop.

5.4 Design Technique

Using the ideas developed in the previous section, the frequency domain design method proposed by Rosenbrock consists essentially of determing a matrix $K_p(s)$ such that the product $[G(s)K_p(s)]^{-1}$ is diagonal dominant. When this condition has been achieved then the diagonal matrix $K_d(s)$ can be used to implement single-loop compensators as required to meet the overall design specification. Since the design is carried out using the inverse transfer function matrix then we are essentially trying to determine an inverse pre-compensator $\hat{K}_p(s)$ such that $\hat{Q}(s) = \hat{K}_p(s)\hat{G}(s)$ is diagonal dominant. The method is well suited to interactive graphical use of a computer.

One method of determining $\hat{K}_p(s)$ is to build up the required matrix out of elementary row operations using a graphical display of all of the elements of $\hat{Q}(s)$ as a guide. This approach has proven successful in practice and has, in most cases considered to-date, resulted in $K_p(s)$ being a simple matrix of real constants which can be readily realized.

Another approach which has proved useful is to choose $\hat{K}_p = G(o)$, if $|G(o)|$ is nonsingular. Here again $\hat{K}_p(s)$ is a matrix of real constants which simply diagonalizes the plant at zero frequency.

For example, Figure 5.7 shows the inverse Nyquist array (INA) of an uncompensated system with 2 inputs and 2 outputs.

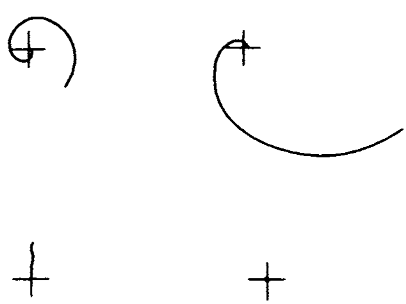

Fig. 5.7

An inspection of this diagram shows that element 1,2 is larger than element 1,1 at the maximum frequency considered, and similarly element 2,1 is very much larger than element 2,2 over a wide range of frequencies . Thus, the open-loop system is not row diagonal dominant, nor is it column diagonal dominant. However, by choosing $\hat{K}_p = G(o)$ the resulting INA of $\hat{Q}(j\omega)$ is shown in Figure 5.8 with the Gershgorin circles for row dominance superimposed. Both $\hat{q}_{11}$ and $\hat{q}_{22}$ are made diagonal dominant with this simple operation. The dominance of element $\hat{q}_{22}$ of this array can be further improved by another simple row operation of the form

$$R_2' = R_2 + \alpha R_1 \tag{5.29}$$

where $\alpha \approx 0.5$, in this case.

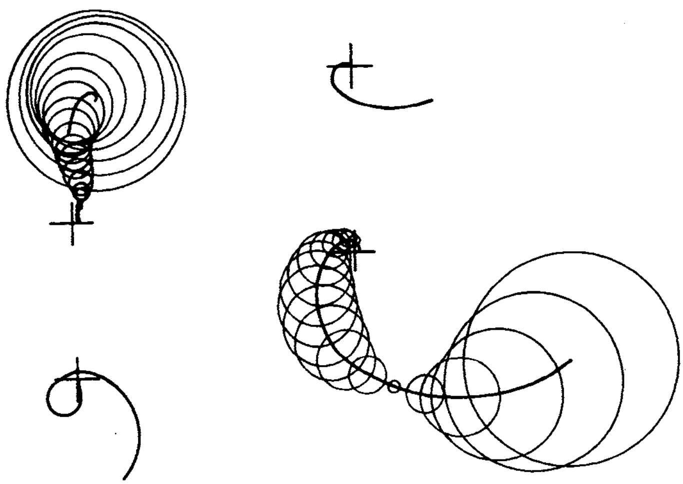

Fig. 5.8

A further approach, which is perhaps more systematic than those mentioned above, is to determine $\hat{K}_p$ as the 'best', in a least-mean-squares sense, wholly real matrix which most nearly diagonalizes the system $\hat{Q}$ at some frequency $s = j\omega$, (Rosenbrock[1] and Hawkins[4]). This choice of $\hat{K}_p$ can be considered as the best matrix of real constants which makes the sum of the moduli of the off-diagonal elements in each row of $\hat{Q}$ as small as possible compared with the modulus of the diagonal element at some frequency $s = j\omega$. The development of the two forms of this latter approach is given in the following paragraphs.

Consider the elements $\hat{q}_{jk}$ in some row j of $\hat{Q}(j\omega) = \hat{K}\hat{G}(j\omega)$, i.e.

$$\hat{q}_{jk}(j\omega) = \sum_{i=1}^{m} \hat{k}_{ji}\, \hat{g}_{ik}(j\omega) \tag{5.30}$$

$$= \sum_{i=1}^{m} \hat{k}_{ji}(\alpha_{ik}+j\beta_{ik}) \tag{5.31}$$

Now choose $\hat{k}_{j1}, \hat{k}_{j2}, \ldots, \hat{k}_{jm}$ so that

$$\sum_{\substack{k=1\\k\neq j}}^{m} |\hat{q}_{jk}(j\omega)|^2 \tag{5.32}$$

is made as small as possible subject to the constraint that

$$\sum_{i=1}^{m} \hat{k}_{ji}^2 = 1 \tag{5.33}$$

Using a Lagrange multiplier, we minimize

$$\phi_j = \sum_{\substack{k=1\\k\neq j}}^{m} \left| \sum_{i=1}^{m} \hat{k}_{ji}(\alpha_{ik}+j\beta_{ik}) \right|^2 + \lambda\left(- \sum_{i=1}^{m} \hat{k}_{ji}^2 \right) \tag{5.34}$$

$$= \sum_{\substack{k=1\\k\neq j}}^{m} \left(\left(\sum_{i=1}^{m} \hat{k}_{ji}\alpha_{ik} \right)^2 + \left(\sum_{i=1}^{m} \hat{k}_{ji}\beta_{ik} \right)^2 + \lambda\left(1 - \sum_{i=1}^{m} \hat{k}_{ji}^2 \right) \right) \tag{5.35}$$

and taking partial derivatives of ϕ_j with respect to $\hat{k}_{j\ell}$ (i.e. the elements of the row vector $\hat{k}_j$), we get, on setting these equal to zero,

$$\frac{\partial \phi_j}{\partial \hat{k}_{j\ell}} = \sum_{\substack{k=1\\k\neq j}}^{m} \left(2\left(\sum_{i=1}^{m} \hat{k}_{ji}\alpha_{ik} \right)\alpha_{\ell k} + 2\left(\sum_{i=1}^{m} \hat{k}_{ji}\beta_{ik} \right)\beta_{\ell k} \right) - 2\lambda \hat{k}_{j\ell} \tag{5.36}$$

$$= 0 \quad \text{for } \ell = 1,2,\ldots,m$$

Now writing

$$A_j = \{a_{i\ell}^{(j)}\}$$

$$= \left\{ \sum_{\substack{k=1 \\ k\neq j}}^{m} (\alpha_{ik}\alpha_{\ell k} + \beta_{ik}\beta_{\ell k}) \right\} \tag{5.37}$$

and $\hat{k}_j = (\hat{k}_{j\ell})$ (5.38)

the minimization becomes

$$A_j\hat{k}_j^T - \lambda\hat{k}_j^T = 0 \tag{5.39}$$

since

$$\sum_{\substack{k=1 \\ k\neq j}}^{m} |\hat{q}_{jk}(j\omega)|^2 = \hat{k}_j A_j \hat{k}_j^T = \lambda\hat{k}_j\hat{k}_j^T = \lambda \tag{5.40}$$

Thus, the design problem becomes an eigenvalue/eigenvector problem where the row vector $\hat{k}_j$, which pseudodiagonalizes row j of $\hat{Q}$ at some frequency $j\omega$, is the eigenvector of the symmetric positive semi-definite (or definite) matrix A_j corresponding to the smallest eigenvalue of A_j .

Figure 5.9 shows the INA of a 4-input 4-output system over the frequency range 0 → 1 rad/sec. Although the Gershgorin circles superimposed on the diagonal elements show that the basic system is diagonal dominant, the size of these circles at the 1 rad/sec end of the frequency range indicate that the interaction in the system may be unacceptable during transient changes. Using the pseudodiagonalisation algorithm described above, at a frequency of 0.9 rad/sec in each row, a simple wholly real compensator $\hat{K}_p$ can be determined which yields the INA shown in Figure 5.10 . Here, we can see that the size of the Gershgorin discs has in fact been considerably reduced over all of the bandwidth of interest.

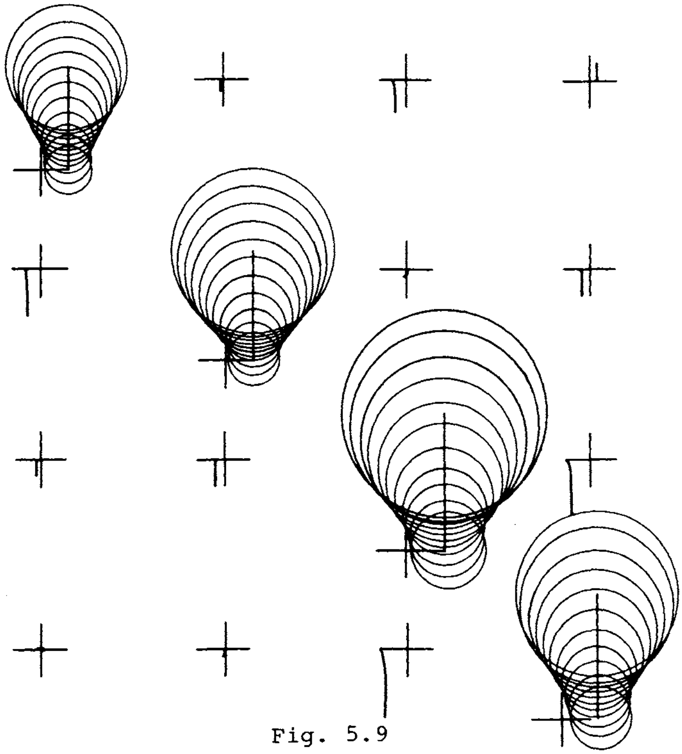

Fig. 5.9

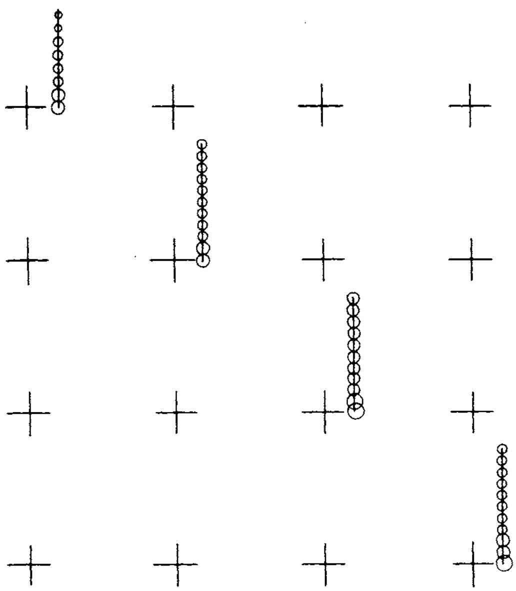

Fig. 5.10

However, in general, we may choose a different frequency ω for each row of $\hat{K}$, and it is also possible to pseudo-diagonalize each row of $\hat{Q}$ at a weighted sum of frequencies. The formulation of this latter problem again results in an eigenvalue/eigenvector problem (see Rosenbrock[1]).

Although this form of pseudodiagonalization frequently produces useful results, the constraint that the control vector $\hat{k}_j$ should have unit norm does not prevent the diagonal term $\hat{q}_{jj}$ from becoming very small, although the row is diagonal dominant, or vanishing altogether.

So, if instead of the constraint given by (5.33), we substitute the alternative constraint that

$$|\hat{q}_{jj}(j\omega)| = 1 \tag{5.41}$$

then a similar analysis leads to

$$A_j\hat{k}_j^T - \lambda E_j\hat{k}_j^T = 0 \tag{5.42}$$

where A_j is as defined by equation (5.37) and E_j is the symmetric positive semidefinite matrix

$$\begin{aligned} E_j &= \{e_{i\ell}^{(j)}\} \\ &= [\alpha_{ij}\alpha_{\ell j}+\beta_{ij}\beta_{\ell j}] \end{aligned} \tag{5.43}$$

Equation (5.42) now represents a generalized eigenvalue problem, since E_j can be a singular matrix, and must be solved using the appropriate numerical method.

5.5 Conclusions

The inverse Nyquist array method offers a systematic way of achieving a number of simultaneous objectives, while still leaving considerable freedom to the designer. It is easy to learn and use, and has the virtues of all frequency response methods, namely insensitivity to modelling errors including nonlinearity, insensitivity to the order of the system, and visual insight. All the results for inverse plots apply with suitable changes to direct Nyquist plots (see Reference 1, pp. 174-179). We can also obtain multivariable generalisations of the circle theorem for systems with nonlinear, time-dependent, sector-bounded gains[1,5]. Not only do the Gershgorin bands give a stability criterion; they also set bounds on the transfer function $h_{ii}(s)$ seen in the ith loop as the gains in the other loops are varied.

Several further ways of determining $\hat{K}_p(s)$ such that $\hat{Q}(s)$ is diagonal dominant are the subject of current research (see Leininger[6]). However, several industrial multivariable control problems have already been solved using this design method[7,8,9,10].

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APPENDIX

To prove the result given by equation (5.21), we note that by (5.18a and b) and the conditions of the theorem there is no pole of $\hat{q}_{ii}$ on D, $i,j = 1,2,\ldots,m$, nor is there any zero of $\hat{q}_{ii}$ on D, $i = 1,2,\ldots,m$, because D is a compact set and $\hat{q}_{ii} > 0$ on D. Moreover, by Gershgorin's theorem, there is no zero of $|\hat{Q}|$ on D. Let $\hat{Q}(\alpha,s)$ be the matrix having

$$\begin{aligned} \hat{q}_{ii}(\alpha,s) &= \hat{q}_{ii}(s) \\ \hat{q}_{ij}(\alpha,s) &= \alpha\hat{q}_{ij}(s), \quad j \neq i \end{aligned} \tag{5.44}$$

where $\hat{q}_{ii}(s)$, $\hat{q}_{ij}(s)$ are the elements of $\hat{Q}(s)$ and $0 \leq \alpha \leq 1$. Then every element of $\hat{Q}(\alpha,s)$ is finite on D, and so therefore is $|\hat{Q}(\alpha,s)|$. Consider the function

$$\beta(\alpha,s) = \frac{|\hat{Q}(\alpha,s)|}{\prod_{i=1}^{m} \hat{q}_{ii}(s)} \tag{5.45}$$

which is finite for $0 \leq \alpha \leq 1$ and all s on D, and which satisfies $\beta(0,s) = 1$. Let the image of D under $\beta(1,\cdot)$ be Γ. Let the image of (0,1) under $\beta(\cdot,s)$ be γ_s. Then γ_s is a continuous curve joining the point $\beta(0,s) = 1$ to the point $\beta(1,s)$ on Γ. As s goes once round D, γ_s sweeps out a region in the complex plane and returns at last to its original position.

Suppose, contrary to what is to be proved, that Γ encircles the origin. Then the region swept out by γ_s as s goes once round D must include the origin. That is, there is some $\alpha(0,1)$ and some s on D for which $\beta(\alpha,s) = 0$. But the $\hat{q}_{ii}$ are all finite on D, so by (5.45) $|\hat{Q}(\alpha,s)|=0$.

By Gershgorin's theorem this is impossible. Hence the number of encirclements of the origin by Γ is, from (5.45),

$$0 = \hat{N}_Q - \sum_{i=1}^{m} \hat{N}_i \tag{5.46}$$

which is (5.21).

The Inverse Nyquist Array Design Method - Problems

P.1 A plant described by

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

has arisen from a system with

$$|T_G| = (s-1)(s+1)^2$$

If feedback is to be applied to this system through a feedback matrix $F = \text{diag}\{1,2\}$, determine whether or not the resulting closed-loop system is stable.

Comment on cancellations occurring in the formulation of $G(s)$, and on the encirclements obtained in the resulting INAs.

{Acknowledgement for this problem and its solution are hereby made to Professor H.H. Rosenbrock of the Control Systems Centre, UMIST.}

P.2 Given a system described by the transfer-function matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

(i) State one reason why the inverse systems $G^{-1}(s)$ and $H^{-1}(s)$ are used in Rosenbrock's inverse Nyquist array design method.

(ii) Sketch the inverse Nyquist array for $G^{-1}(s)$, and comment on the diagonal dominance of the uncompensated system.

(iii) Determine a wholly real forward path compensator K such that the closed-loop system $H(s)$, with unity feedback, is decoupled.

(iv) Introduce integral action into both control loops and sketch the resulting root-locus diagrams for the final loop-tuning of the compensated system.