Determination of stability regions with the inverse Nyquist array

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ABSTRACT

A new criterion for closed-loop stability and the design of linear multivariable control systems is developed using the inverse-Nyquist-array method and an Ostrowski theorem. Estimates of the gain margins in each loop and bounds on the stable-gain space are obtained using the bands of Ostrowski circles superimposed on the diagonal elements of the inverse Nyquist array. The manner of application of this new approach is similar to the way in which the bands of Gershgorin circles are used. The new criterion allows the diagonal dominance requirements to be relaxed in one row or in one column of the inverse Nyquist array at each point on the D contour, while still permitting the origin encirclements of the determinant of the inverse-transfer-function matrix to be determined from those of its diagonal elements. As a consequence the estimated stability region is larger than that obtained when strict dominance requirements are imposed.

LIST OF PRINCIPAL SYMBOLS

s = Laplace-transform variable
ω = frequency variable
$\mathbf{Q}(s)$ = open-loop transfer-function matrix (t.f.m.)
$\mathbf{G}(s)$ = plant t.f.m.
$\mathbf{K}(s)$ = controller t.f.m.
$\mathbf{H}(s)$ = closed-loop t.f.m.
$\mathbf{F}(s)$ = transducer t.f.m.
m = number of plant inputs and outputs
$\hat{\mathbf{Q}}(s), \hat{\mathbf{G}}(s), \hat{\mathbf{K}}(s), \hat{\mathbf{H}}(s)$ = inverses of $\mathbf{Q}(s), \mathbf{G}(s), \mathbf{K}(s)$, and $\mathbf{H}(s)$
$\hat{h}_{ij}(s)$ = ij th element of $\hat{\mathbf{H}}(s)$
k_i = controller gain in ith control loop
$\det|\hat{\mathbf{H}}(s)|$ = determinant of $\hat{\mathbf{H}}(s)$
$\Delta(s)$ = characteristic polynomial

1 INTRODUCTION

The design of linear multivariable control systems can often be simplified if the plant interaction is reduced to a level whereby the important closed-loop system properties can be deduced from the properties of the individual control loops. Usually the most important requisite of control-system design is closed-loop stability. In theory, the designer could determine the region defined by all combinations of individual loop gains for which the closed-loop system is asymptotically stable. Although it is often impracticable to determine it precisely, let us call this region the stable-gain space.

When significant interaction is present, it is often difficult, and may sometimes be impossible, to choose control loops and single-loop gains so that the closed-loop system operates satisfactorily with all loops closed. In these circumstances, the inverse-Nyquist-array (i.n.a.) method[1] and the concept of diagonal dominance[1] provide a criterion for designing matrix compensation that may reduce interaction to a level where the control-system design can be completed with single-loop design concepts.[1-4]

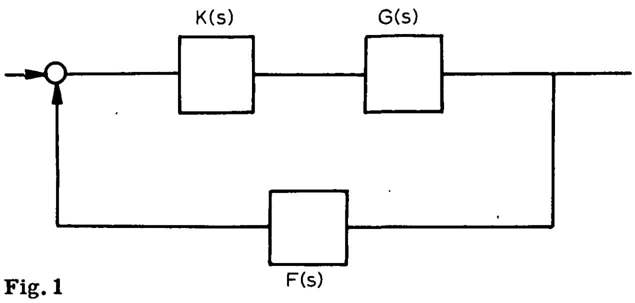

Fig. 1
Multivariable feedback control system

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In this paper, a less stringent requirement than dominance is developed using a theorem of Ostrowski.[5] When the pertinent conditions are satisfied, bands of Ostowski circles, which have been used elsewhere[4, 6] for another purpose, are used to assess stability in a way similar to that used with the bands of Gershgorin circles[1] (i.e. diagonal dominance). The estimated-gain margins exceed those obtained with diagonal dominance, and hence they enlarge the estimate of the stable-gain space.

2 MULTIVARIABLE FEEDBACK CONTROL SYSTEMS

Consider the feedback control system of Fig. 1. A convenient notation is to denote the open-loop transfer-function matrix (t.f.m.) by

$$\mathbf{Q}(s) = \mathbf{G}(s)\mathbf{K}(s) \tag{1}$$

where s is the Laplace-transform variable and K(s) is a matrix controller. Similarly the closed-loop t.f.m. can be denoted by

$$\mathbf{H}(s) = \{\mathbf{I}_m + \mathbf{Q}(s)\mathbf{F}(s)\}^{-1}\mathbf{Q}(s) \tag{2}$$

where $\mathbf{I}_m$ is the m × m identity matrix and $\mathbf{F}(s)$ is a matrix of transducer transfer functions and is usually a diagonal matrix whose diagonal elements are equal to 1 or to 0. When $\mathbf{F}(s) = \mathbf{I}_m$, this indicates that all feedback loops are closed.

The inverse t.f.m. of the closed-loop system is given by

$$\mathbf{H}^{-1}(s) = \hat{\mathbf{H}}(s) = \mathbf{F}(s) + \hat{\mathbf{K}}(s)\hat{\mathbf{G}}(s) = \{\hat{h}_{ij}(s)\} \tag{3}$$

The circumflex over a matrix signifies the inverse of that matrix and the elements of the inverse matrix are denoted in the same way.

In the i.n.a. method, the relationship

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} |\hat{h}_{ij}(s)| = \theta_i(s)|\hat{h}_{ii}(s)| \qquad i = 1, 2, \ldots, m \tag{4}$$

is of particular interest. When $\theta_i(s) < 1$ for all s on the familiar Nyquist D contour[1] and for $i = 1, 2, \ldots, m$, then $\hat{\mathbf{H}}(s)$ is said to be row diagonally dominant.

3 DETERMINATION OF ORIGIN ENCIRCLEMENTS OF DET$|\hat{\mathbf{H}}(s)|$ USING OSTROWKI'S THEOREM

Consider the inverse Nyquist diagram of $\hat{h}_{ii}(j\omega)$ and its band of Gershgorin circles given in Fig. 2. If $\hat{\mathbf{H}}(s)$ is row diagonally dominant, then these diagrams can be used to assess closed-loop stability.[1-4] If $\mathbf{Q}(s)$ is open-loop stable, an estimate of the stable-gain space can be deduced in the form[1, 4]

$$0 \leqslant k_i \leqslant k_{i\max} \qquad i = 1, 2, \ldots, m \tag{5}$$

where k_i represents the controller gain in the ith control loop and $k_{i\max}$ is determined as outlined elsewhere.[4]

An alternative approach can be developed from a result of Ostrowski which is discussed by Marcus and Minc.[5] Now let s be a point on the Nyquist D contour. In our terminology, if

$$|\hat{h}_{ii}(s)|\,|\hat{h}_{jj}(s)| > \left(\sum_{\substack{k=1\\k\neq i}}^{m} |\hat{h}_{ik}(s)|\right)\left(\sum_{\substack{p=1\\p\neq j}}^{m} |\hat{h}_{jp}(s)|\right)$$

$$i, j = 1, 2, \ldots, m \qquad i \neq j \tag{6}$$

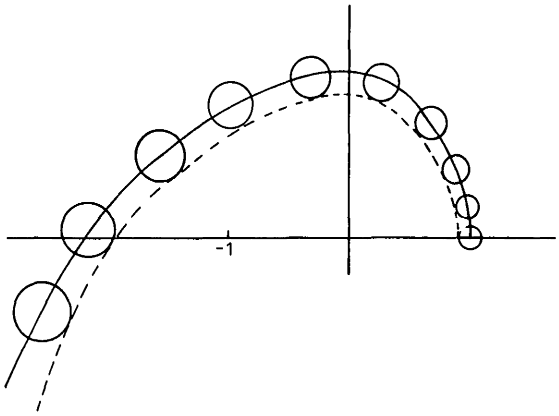

Fig. 2

Inverse Nyquist diagram of $\hat{h}_{ii}(j\omega)$ with its band of Gershgorin circles

then $\det|\hat{H}(s)| \neq 0$. Now let $\theta_i(s)$ be as defined in eqn. 4, but no longer restrict its value to be less than 1, which means that dominance no longer must be satisfied. Further define

$$\phi_i(s) = \max_{\substack{j=1,2,\ldots,m\\ j\neq i}} \theta_j(s) \qquad i = 1, 2, \ldots, m$$

Assuming that neither $|\hat{h}_{ii}(s)|$ nor $|\hat{h}_{jj}(s)|$ are zero, then expr. 6 can be restated as

$$\phi_i(s)\theta_i(s) < 1 \qquad i = 1, 2, \ldots, m \tag{7}$$

which is a sufficient condition for the nonsingularity of $\det|\hat{H}(s)|$ at that particular point on the Nyquist D contour.

If expr. 7 is satisfied for each row of $\hat{H}(s)$, at all points on the Nyquist D contour, then we assert that the encirclements[1] of $\det|\hat{H}(s)|$ are equal to the algebraic sum of the encirclements of all the diagonal elements of $\hat{H}(s)$. The proof for this assertion is based on the same reasoning used by Rosenbrock in proving a similar result for the bands of Gershgorin circles.[3] Of course, the same result can be applied to the columns of $\hat{H}(s)$. Once the encirclements of $\det|\hat{H}(s)|$ are known, closed-loop stability is determined by the encirclement criterion.[2]

4 DETERMINATION OF STABILITY BOUNDARIES USING THE OSTROWSKI BANDS

To illustrate the engineering significance of this result, suppose that $\hat{H}(s)$ is row diagonally dominant and that the closed-loop system is asymptotically stable by the conditions of the multivariable encirclement theorem.[1-3] Let Fig. 2 represent the plot of $\hat{h}_{ii}(j\omega)$ and its band of Gershgorin circles. Further suppose that Fig. 3 represents the plot of $\hat{h}_{ii}(j\omega)$ upon which is centred a band of circles of radius $\phi_i(j\omega)\theta_i(j\omega)|\hat{h}_{ii}(j\omega)|$. By convention these circles are known as the band of Ostrowski circles. If the gain in the ith loop is increased while satisfying $1/\phi_i > \theta_i > 1$ for all s on the D contour, it follows that

$$1 > \phi_i\theta_i = \phi_i\phi_j \geqslant \theta_j\phi_j \qquad j = 1, 2, \ldots, m \tag{8}$$

is satisfied for all s on the D contour and hence $\phi_j\theta_j$ is less than 1 the remaining $m - 1$ rows of $\hat{\mathbf{H}}(j\omega)$. This argument means that an estimate of the maximum permissible gain in the ith loop can be determined, when the gains in the remaining loops are fixed at set values, by calculating the gain margin associated with the band of Ostrowski circles superimposed on the inverse Nyquist plot of $\hat{h}_{ii}(j\omega)$.

If $\hat{H}(s)$ is row diagonally dominant, then expr. 7 is automatically satisfied; however the converse is not generally true. As a consequence, the region containing all the values of single-loop gains satisfying expr. 7 is larger than that obtained with diagonal dominance as given in expr. 5. The

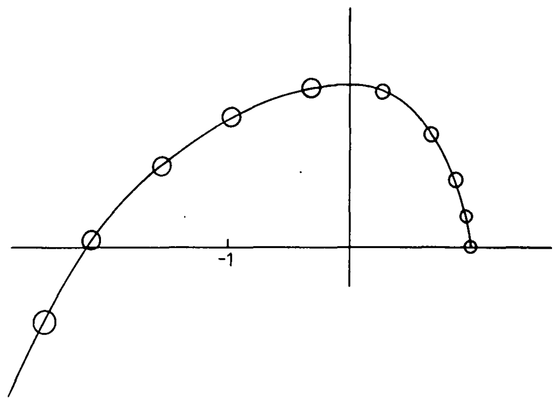

Fig. 3

Inverse Nyquist diagram of $\hat{h}_{ii}(j\omega)$ with its band of Ostrowski circles

estimated gain margin associated with a particular loop will also be larger than that determined by $k_{i\,max}$ in expr. 5. The manner in which both bands of circles are used is very similar; however there is an important difference between the two. First consider the case where the Gershgorin circles are used and a stable-gain space such as that defined in expr. 5 is obtained. When all loop gains lie within the region defined by expr. 5, estimated-gain margins for each loop can be calculated using the bands of Gershgorin circles, and hence these margins are independent of the gains in the remaining loops. Now suppose that the Ostrowski condition (expr. 7) is satisfied for all s on the D contour. In this case, the value of ϕ_i which helps increase the estimated gain margin in the ith loop depends on the remaining values of θ. This means that the estimated gain margin for each loop depends on the gains in all the remaining loops.

The substance of these results is that an estimate of the stable gain space can be determined with the bands of Ostrowski circles. Here the requirement is that the plot of each diagonal element of $\hat{H}(j\omega)$ with its band of Ostrowski circles should not enclose the critical point. This provides a better estimate of the stable (and unstable) gain space than obtained previously.[4] Of course there is still a region of uncertainty; however, its size is smaller.

When the bands of Ostrowski circles are used, expr. 7 permits one θ_i to be greater than 1 at any one frequency. As the phase crossover frequencies of each $\hat{h}_{ii}(j\omega)$ will usually be different, this may permit the θ_i of two or more rows (or columns) of $\hat{H}(j\omega)$ to exceed 1 at their respective phase crossover frequencies.

4.1 Application of Ostrowski bands

4.1.1 Example 1

As an example consider an open-loop stable, 2×2 multivariable system for which the inverse Nyquist diagrams of $\hat{h}_{11}(j\omega)$ and $\hat{h}_{22}(j\omega)$ and the bands of Gershgorin circles are as shown in Fig. 4. The phase crossover frequencies for $\hat{h}_{11}(j\omega)$ and $\hat{h}_{22}(j\omega)$ are ω_6 and ω_7, respectively. Now suppose that the gains in each loop are such that the critical points for the inverse Nyquist diagrams of $\hat{h}_{11}(j\omega)$ and $\hat{h}_{22}(j\omega)$ both lie on the bands of Gershgorin circles as indicated in Fig. 4. As the bands of Ostrowski circles represented by the dashed lines in Fig. 4 do not enclose the critical points, the system remains stable and a conservative estimate of the gain margin in each loop can be calculated with the gain in the other loop fixed. Furthermore stability may still be assured when both critical points lie within the bands of Gershgorin circles, provided neither critical point lies within the corresponding band of Ostrowski circles. In the general case, whenever a gain in one loop is changed, the radii of the circles in all the remaining bands of Ostrowski circles must be recalculated. If both loop gains were increased in the example given in Fig. 4, then both bands of Ostrowski circles must be reconstructed before checking for the enclosure of the critical point.

4.1.2 Example 2

The Ostrowski result can also provide useful information when neither $\hat{Q}(s)$ nor $\hat{H}(s)$ can be made diagonally dominant. For example, consider the t.f.m.

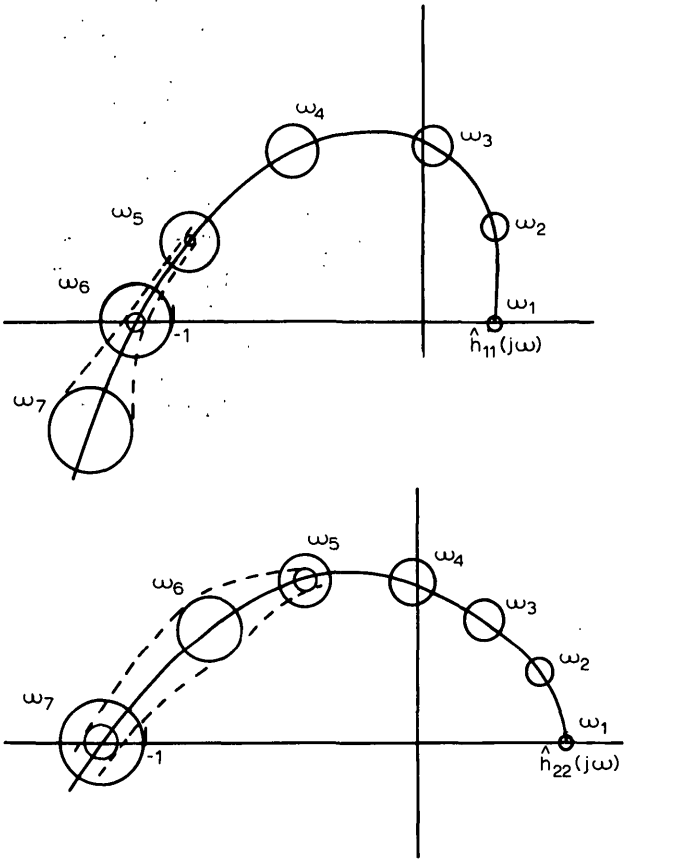

Fig. 4

Inverse Nyquist diagrams of diagonal elements of $\hat{H}(j\omega)$ for a 2 × 2 system with Gershgorin circles drawn at the indicated frequencies and Ostrowski circles shown near phase cross-over frequencies

$$G(s) = \frac{1000}{\Delta(s)} \begin{Bmatrix} 0{\cdot}01496(s+1{\cdot}7)(s+100) & 95{\cdot}15(s+1{\cdot}898)(s+10) \\ 0{\cdot}0852(s+1{\cdot}44)(s+100) & 124{\cdot}0(s+2{\cdot}037)(s+10) \end{Bmatrix} \quad (9)$$

$$\Delta(s) = (s^2 + 3{\cdot}225s + 2{\cdot}525)(s+10)(s+100)$$

which is the model of a 2-shaft gas-turbine jet engine.[7,8] The i.n.a. of $\hat{G}(j\omega)$ is not diagonally dominant nor is the i.n.a. of $\hat{Q}(j\omega) = \hat{K}\hat{G}(j\omega)$ with the matrix compensation

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (10)$$

In fact, it has been suggested elsewhere[9] that it is impossible to make both rows of $\hat{Q}(j\omega) = \hat{K}\hat{G}(j\omega)$ satisfy the dominance requirements when **K** is restricted to be a constant matrix. This means that the origin encirclements[1] of $\det|\hat{Q}(s)|$ cannot be deduced by just using the diagonal elements of the array $\hat{Q}(j\omega)$ and the bands of Gershgorin circles; however, the origin encirclements of $\det|\hat{Q}(s)|$ can be deduced using the array $\hat{Q}(j\omega)$ and the bands of Ostrowski circles. The control-system design can be completed on this basis with the matrix compensation given in eqn. 10, and this agrees with Mueller's suggestion that the control loops should be interchanged.[7]

5 DISCUSSION

(i) When first introduced by Rosenbrock,[4] the plots of $h_{ii}(j\omega)$ with their bands of Ostrowski circles were used to determine bounds on the location of the diagonal elements of the Nyquist array. As shown above, the bands of Ostrowski circles can also be used for the assessment of stability in a similar manner to the way that the bands of Gershgorin circles are used.[4] The stability information so obtained enlarges the estimates of the stable (and unstable) gain space. Another way of approaching the stability boundary determination is suggested in a forthcoming book.[10] When dominance is satisfied in all the rows of $\hat{H}(s)$ and if the gain in one loop is varied, the designer has essentially a single-loop situation and he may treat the Ostrowski band for that loop as though it were a single-loop inverse Nyquist plot if the band is narrow enough.[10]

(ii) The extent to which the Ostrowski bands improve the estimated stable-gain space obtained with the bands of Gershgorin circles depends on the particular system. In Fig. 4, the gain in the first loop could be increased roughly 9%, while holding the gain in the second loop constant, before the critical point touches the band of Ostrowski circles associated with $\hat{h}_{11}(j\omega)$. In other systems, the improvements could be significantly larger or smaller. In some cases, the Ostrowski bands can be used when diagonal dominance cannot be satisfied.

(iii) Experience has shown[11] that these result are best used in conjunction with interactive computer-aided-design and graphic display facilities.[12] In these circumstances the designer can display and redisplay the i.n.a. diagrams and the appropriate bands of circles until a satisfactory controller has been designed.

As the bands of Ostrowski circles depend on the values of θ in the other rows of $\hat{H}(j\omega)$, altering the gain in one loop necessitates the recalculation of the Ostrowski bands before looking at another loop. These calculations can be performed automatically by a computer or they can also be performed manually. In the latter case, the designer would have to replot the Ostrowski bands for each change in the gain settings.

(iv) The estimate of the stable-gain space determined from the Gershgorin bands assumes the very simple form in expr. 5. Except in trivial cases, the estimate of the stable-gain space using the Ostrowski result will require the calculation of several points on its boundary. This is a straightforward task for a 2 × 2 system and the graphical result is easily presented. For larger systems, i.e. $m \geqslant 3$, the calculations could be performed; however, graphical presentation would involve the representation of multidimensional surfaces. When $m \geqslant 3$, it may be more practical to think in terms of the grain margin for one or several sets of gains as required.

6 CONCLUSIONS

The Ostrowski bands of circles can be used in a similar way as the Gershgorin bands of circles to provide information about closed-loop-system properties of multivariable control systems in terms of the gains in the individual control loops. The estimate of the stable-gain space obtained with the Ostrowski bands is usually larger than that obtained with Gershgorin bands of circles. As the estimates of gain margins are still conservative, these results are very useful for design purposes.

7 ACKNOWLEDGMENTS

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EFFECTS OF FLEXIBILITY ON A MOMENTUM-STABILISED COMMUNICATION-SATELLITE ATTITUDE-CONTROL SYSTEM

In their paper [Proc. IEE, **120**, (5), pp. 613-619], Gething et al. studied the effects of flexibility on momentum-stabilised satellite attitude-control systems. Their method of analysis, Likin's hybrid co-ordinates method, is suitable for structually complex appendages; however, many of the current and proposed solar arrays have a simplicity of construction which suggests that, under rather bland assumptions, the continuous equations for simple beams and sheets may prove tractable. For example, flexible arrays have recently been modelled[A] as a spring-dashpot system mounted on a rigid massless rod attached to the main body of the satellite via a coil spring to simulate torsional effects. Contrary to the authors' claim (Section 1), other investigators have studied the effects of flexibility on the attitude-control systems of spin-stabilised satellites with flexible and comparatively rigid appendages, and also on flywheel-stabilised satellites[B] and vehicles having long flexible appendages stabilised by gravity gradient.[C] Some general appraisals of the effects of structural flexibility on control-system performance have been made[D] using existing data, from which possible interaction problems are more readily identified and dealt with. This avoids deficiences in structural-dynamic analysis and in the knowledge of the flight environment and structural data.

Gething et al. (Section 3) consider only the lowest-modal frequency of the flexible appendage. However, it has been shown that, when using a linear feedback law, if the flexure modal frequencies are well separated they are little affected by the control gains associated with the rigid mode, and are uncoupled from each other owing to a resonance phenomenon of each mode at its own loop frequency. If, in addition to rate and proportional feedback, there are lags and leads, associated with sensors and actuators in the control loop, and if their time constants are small compared with the period of the rigid mode loop, then they will have little effect upon it. However, for higher modes a critical frequency is reached where positive feedback is obtained and the control increases structural deflections rather than retarding them, leading to instability of an otherwise stable mode.

Section 1 considers the effects of sensor dynamics and signal noise as a secondary effect in the flexibility problem. However, I have shown[F] that, for the pseudorate pulse-frequency controller used, random external disturbing torques and sensor noise cause considerable variations in the limit-cycle behaviour of the attitude-control system; for large-variance noise, either a complete removal of limit cycling with a strong probability of instability occurs, or a condition of limit-cycle annuli is reached which prevents the controller from reaching a condition of stable equilibrium. This method,[F] called the stochastic describing function (a generalisation of the ordinary describing function), is demonstrated for the pseudorate satellite-attitude controller shown in Fig. A. An illustration of the effects of Gaussian-distributed noise on the self-oscillatory modes of this impulse attitude controller is shown in the Nyquist plot of Fig. B, where

$$G(p)H(p) = \frac{2\{1 - \exp(-p)\}}{p^2(p+1)}$$

represents the linear elements of Fig. 1, and N is the stochastic describing function of the on-off switch with $D = 1$, $\Delta_1 = 0{\cdot}25$ and $\Delta_2 = 0{\cdot}125$. From Fig. B it can be seen that, as the sensor noise etc. variances increase, a limit-cycle annuli condition is reached, with a deterioration in stability. Thus noise is comparable with flexibility in the stability of pulse-controlled attitude controllers. This is not an entirely surprising result, since, at best, flexibility coefficients are only reasonable approximations (as in the Apollo control service module) and are greatly affected by random environmental factors such as solar radiation, and the two problems of flexure and noise can be considered as one. Thus the conclusions of Section 5 of utilising onboard

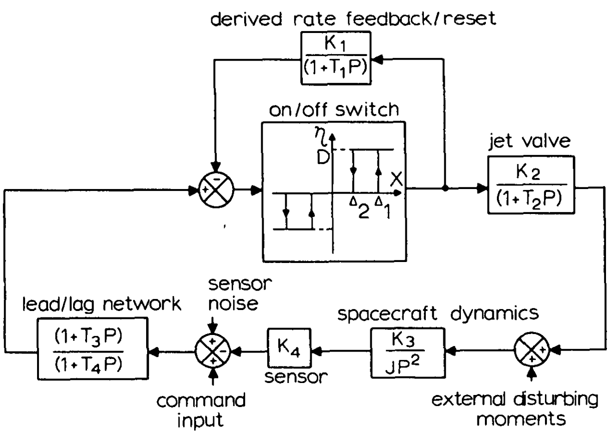

Fig. A

Generalised single-axis gyroless attitude controller

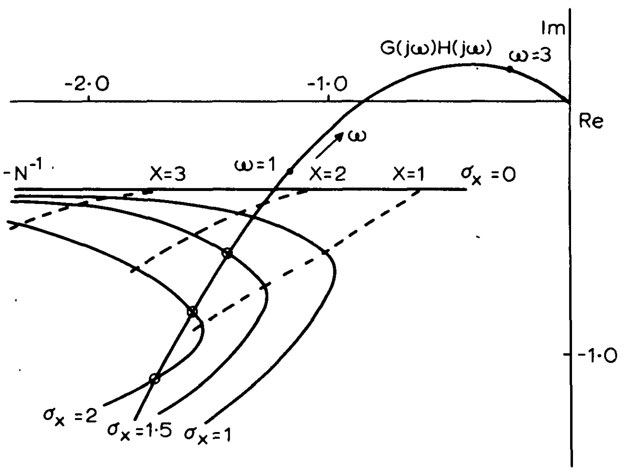

Fig. B

Nyquist plot for controller of Fig. A