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Design of multivariable control systems using the inverse Nyquist array

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Abstract

The inverse Nyquist array is a set of diagrams corresponding to the elements of the inverse of the open-loop transfer function of a control system. A number of theorems are proved which show how this array can be used to investigate the stability of multivariable control systems. The application of the array to the design of such systems is illustrated.

1 Introduction

Though increasing attention is being given to the design of multivariable systems, this has been a relatively neglected subject. Many methods have been suggested, and a survey is being compiled by MacFarlane.[1] Most of them, however, suffer from three defects.

First, if they are specialised to single-loop systems, they are generally less useful than the traditional methods. Secondly, they usually make it difficult to include engineering constraints such as a restriction on the phase advance produced by the controller. Thirdly, they tend to produce complicated control schemes where simpler schemes would be equally satisfactory.

For these reasons, there is room for alternative methods such as the one presented here. It allows engineering constraints to be imposed, and in the single-loop case reduces to the traditional methods. Its scope has yet to be determined, but in examples it gives rise to simple and satisfactory control systems. Work is actively proceeding to develop the method further.

2 Inverse Nyquist array

The system which will be considered first is shown in Fig. 1: a more general system is suggested in Section 4. The

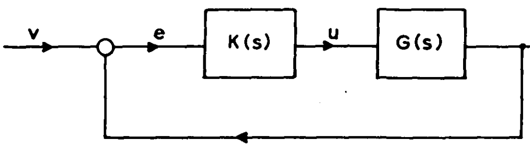

Fig. 1
Multivariable control system
Vectors v, e, u and y all have m components

plant has the $m \times m$ transfer function matrix $G(s)$, and the controller is represented by an $m \times m$ matrix $K(s)$. The object is to find a suitable matrix $K(s)$ which will ensure that the closed-loop system meets certain performance specifications.

It will be assumed that the elements of $G(s)$ and $K(s)$ are rational polynomial functions of s, and that neither $|G(s)|$ nor $|K(s)|$ is identically zero. It will also be assumed that all the zeros of $|K(s)|$ are in the open left halfplane; it has previously been shown[2] that right halfplane zeros in $|G(s)K(s)|$ give rise to control difficulties, so that there will be no incentive to introduce them in $|K(s)|$. Finally, it is assumed that the plant from which $G(s)$ arises is asymptotically stable before control is applied, and that $K(s)$ has all its poles in the open left halfplane. These assumptions are valid in many situations of practical interest.

The open-loop transfer function is

$$Q(s) = G(s)K(s) \qquad (1)$$

A notation for the elements of the inverse matrices G^{-1}, K^{-1}, Q^{-1} etc. will be required; these matrices will be written

$$G^{-1} = \hat{G} \quad K^{-1} = \hat{K} \quad Q^{-1} = \hat{Q} \text{ etc.} \qquad (2)$$

The elements of the matrices will be represented, in the usual way, by q_{ij}, $\hat{q}_{ij}$ etc. The cofactor of q_{ij} will be denoted by Q_{ij}, and similarly for the other matrices. Notice that

$$\hat{q}_{ij} = Q_{ji}/|Q| \qquad (3)$$

$$\neq q_{ij}^{-1} \qquad (4)$$

in general. From Fig. 1,

$$y = GKe = Q(v - y) \qquad (5)$$

so that the closed-loop transfer function which relates $y(s)$ to $v(s)$ is

$$H = (I_m + Q)^{-1}Q \qquad (6)$$

and $$H^{-1} = \hat{H} = I_m + \hat{Q} \qquad (7)$$

For the purposes of design a more general result than eqn. 7 is desirable. Let only p of the feedback loops from y be closed, and let F be a matrix having all its entries zero except for unit entries on the principal diagonal corresponding to those loops which are closed. For example if $m = 3$, $p = 2$ and the first and third loops are closed,

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

Then from Fig. 1,

$$y = GKe = Q(v - Fy) \qquad (9)$$

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so that the transfer function relating $y(s)$ to $v(s)$ becomes

$$R = (I_m + QF)^{-1}Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and $\quad R^{-1} = \hat{R} = F + \hat{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

When $F = 0$, $\hat{R} = \hat{Q}$, while $F = I_m$, $\hat{R} = \hat{H}$.

The inverse Nyquist array (I.N.A.) is the set of m^2 diagrams representing the elements $\hat{q}_{ij}(j\omega)$ of $\hat{Q}(j\omega)$. The I.N.A. allows the elements of $\hat{R}(j\omega)$ to be obtained in an elementary way, because $\hat{r}_{ij} = \hat{q}_{ij}$ when $i \neq j$, $\hat{r}_{ii} = \hat{q}_{ii}$ if the ith loop is open and $\hat{r}_{jj} = 1 + \hat{q}_{jj}$ if the jth loop is closed. In the last case, all that is needed is to shift the origin in the diagram for $\hat{q}_{jj}$ to the point $(-1, 0)$, when the new diagram represents $\hat{r}_{jj}$.

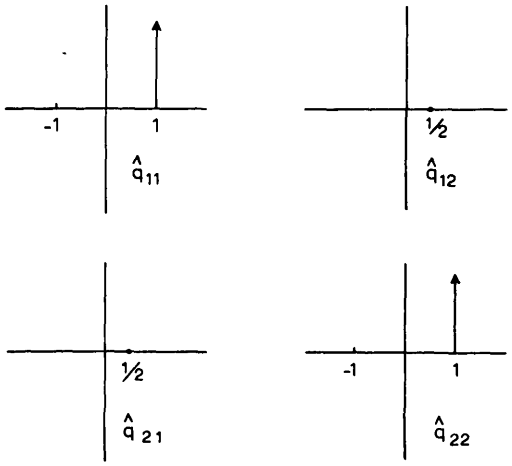

Fig. 2
Inverse Nyquist array corresponding to eqn. 13

As an example, let

$$G(s) = \begin{bmatrix} \dfrac{4(1+s)}{(1+2s)(3+2s)} & \dfrac{-2}{(1+2s)(3+2s)} \\ \dfrac{-2}{(1+2s)(3+2s)} & \dfrac{4(1+s)}{(1+2s)(3+2s)} \end{bmatrix} \quad . \quad (12)$$

which can arise from a second-order system, and let $K = I_2$. Then

$$\hat{Q}(s) = \begin{bmatrix} 1+s & \frac{1}{2} \\ \frac{1}{2} & 1+s \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the I.N.A. is shown in Fig. 2.

2.1 Structure of controller

Since the objective is to design a suitable controller $K(s)$, it is desirable to know what structure is adequate to describe a general $K(s)$. It will be assumed that $K(s)$ is a rational polynomial matrix, with all its poles in the open left halfplane, and that $|K(s)| \not\equiv 0$ and has all its zeros in the open left halfplane. It is shown, in theorem 1 of Appendix 7, that any such $K(s)$ can be written as a product

$$K(s) = K_a K_b(s) K_c(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where the three matrices K_a, $K_b(s)$ and $K_c(s)$ have the following properties:

(*a*) The matrix K_a is a permutation matrix. It therefore represents a preliminary renumbering of the inputs to G, which usually will be done so that the new input i affects chiefly the output i. The inverse $\hat{K}_a$ of K_a is another permutation matrix.

(*b*) The matrix $K_b(s)$ has determinant $|K_b(s)| = 1$ and represents a sequence of elementary column operations. Each such operation consists of adding, to column j of the matrix Q operated on, a multiple of $\alpha_{ij}(s)$ by column i. Here α_{ij} is a rational polynomial function having as its denominator either 1 or a polynomial with all its zeros in the open left halfplane. The inverse $\hat{K}_b$ of K_b can be expressed as a corresponding sequence of row operations. For example when $m = 3$ the matrix

$$K_{b1} = \begin{bmatrix} 1 & 0 & 1/(1+s) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

represents the addition of $1/(1+s)$ times column 1 of Q to column 3 of Q. Its inverse is

$$\hat{K}_{b1} = \begin{bmatrix} 1 & 0 & -1/(1+s) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which represents the subtraction, from row 1 of $\hat{Q}$, of $1/(1+s)$ times row 3 of $\hat{Q}$.

(*c*) The matrix $K_c(s)$ is diagonal, and its nonzero entries have all their poles and zeros in the open left halfplane. If K_c is written

$$K_c(s) = \text{diag}\,\{k_i(s)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the inverse $\hat{K}_c$ is

$$\hat{K}_c(s) = \text{diag}\,\{\hat{k}_i(s)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where $\hat{k}_i(s) = k_i^{-1}(s)$ and has all its poles and zeros in the open left halfplane. Note that if $|k_i(s_0)|$ is large for some s_0, then $|\hat{k}_i(s_0)|$ is small.

The structure which corresponds to eqn. 14 is illustrated for $m = 3$ in Fig. 3. The matrix $K_b(s)$ accomplishes a modification of the interaction in the plant, while $K_c(s)$ represents m independent controllers. The m loops which contain the $k_i(s)$ will be called the m principal loops. The importance of the decomposition of K into K_a, K_b and K_c is that the successive application of K_a, K_b and K_c is sufficient to generate the most general K satisfying the conditions on K stated above. This is not immediately obvious; e.g., if

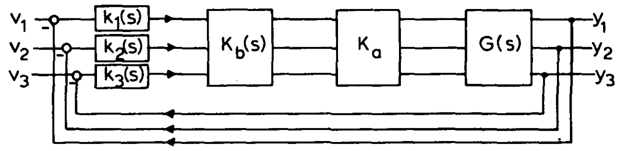

Fig. 3
Structure of multivariable control system resulting from eqn. 14

$$K(s) = \begin{bmatrix} \dfrac{(1-s)}{(1+s)^2} & \dfrac{-s}{1+s} \\ \dfrac{1}{1+s} & 1 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

the ordinary process of Gauss reduction[3] leads to

$$K(s) = \left\{ \begin{bmatrix} 1 & 0 \\ \dfrac{1+s}{1-s} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{-s(1-s)}{1+s} \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} \dfrac{1-s}{(1+s)^2} & 0 \\ 0 & \dfrac{1}{1-s} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (20)$$

$$= \{K_b'(s)\}K_c'(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

but here both K_b' and K_c' have right halfplane poles. By the procedure used in the proof of theorem 1, K can be put in the alternative form

$$K(s) = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1+s}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{2}{1+s} \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} \dfrac{2}{(1+s)^2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (22)$$

There is, of course, no need to generate K in the form shown in eqn. 14 if some other form commends itself. For example it may be profitable to make the real part of Q diagonal at some particular frequency ω_0, which can be done by a matrix

$$K = \{\text{Re}\,Q(j\omega_0)\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

if this exists. No matter how K is generated, however, it can be put in the form of eqn. 14 provided that K satisfies the initial assumptions. Subject to these assumptions, therefore, there is no loss of generality in obtaining K by successive choice of K_a, K_b and K_c.

The aim of the method to be presented is to find such K_a and $K_b(s)$ that the final stage, of finding $K_c(s)$, can be done by the single-loop theory. If $|G(s)|$ has all its zeros in the open left halfplane, theorem 2 shows that K_a and $K_b(s)$ can be found so that $G(s)K_aK_b(s)$ is diagonal. This has been proposed[4] as a possible design method, but it has two defects.

First, it often leads to relatively complicated controllers. Secondly, when $|G(s)|$ has a zero in the right halfplane, it is shown in theorem 3 that generally no K_a and $K_b(s)$ exist (subject to the conditions imposed earlier) which will make $G(s)K_aK_b(s)$ diagonal; if $K_e(s)$ is such that $G(s)K_e(s)$ is diagonal, theorem 4 shows that in general $|K_e(s)|$ has a zero in the right halfplane. This is known[2] to be undesirable. This second difficulty can be avoided by asking only that $G(s)K_aK_b(s)$ should be triangular, when the single-loop theory can again be used. The first difficulty, that of unnecessary complication, persists. We therefore wish to find a less severe preliminary transformation [represented by K_a and $K_b(s)$] which still allows the single-loop theory to be applied.

The procedure for generating K_a and $K_b(s)$ will be developed in terms of the inverse matrices $\hat{K}_a$ and $\hat{K}_b(s)$. These have a simple interpretation in terms of their effect on the I.N.A. The final stage is then to design controllers $k_i(s)$ for each of the principal loops. If $\hat{Q}$ (and therefore Q) had previously been made diagonal or triangular, this final stage would also have a simple representation in the I.N.A., as the diagonal elements $\hat{q}_{ii}$ of $\hat{Q}$ would then give conventional inverse Nyquist diagrams for the m loops, and the corresponding design methods are well known. If $\hat{Q}$ is not first brought to diagonal (or triangular) form, this design procedure is not immediately available. This problem is therefore considered in Sections 2.2 and 2.3.

2.2 Loop transfer functions

When some of the principal loops are open and some are closed, the transfer function between input v_i and output y_i is

$$r_{ii}(s) = \hat{R}_{ii}(s)/|\hat{R}(s)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The inverse Nyquist diagram for this path is obtained from

$$r_{\bar{i}\bar{i}}^{-1}(s) = |\hat{R}(s)|/\hat{R}_{ii}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which can be expanded to give

$$r_{\bar{i}\bar{i}}^{-1}(s) = \hat{r}_{ii}(s) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \hat{r}_{ij}(s)\hat{R}_{ij}(s)/\hat{R}_{ii}(s) \quad . \quad . \quad . \quad (26)$$

or alternatively

$$r_{\bar{i}\bar{i}}^{-1}(s) = \hat{r}_{ii}(s) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \hat{r}_{ji}(s)\hat{R}_{ji}(s)/\hat{R}_{ii}(s) \quad . \quad . \quad . \quad (27)$$

Eqns. 26 and 27 are valid whether the ith principal loop is open or closed.

If $\hat{R}$ is diagonal, eqns. 26 and 27 show, as expected, that $r_{\bar{i}\bar{i}}^{-1} = \hat{r}_{ii}$. The same is true when $\hat{R}$ is triangular. In both cases, as is otherwise evident, the diagrams on the principal diagonal of the I.N.A. can be used as conventional inverse Nyquist diagrams to design the controllers in the principal loops. In the general case, eqns. 26 and 27 can be used to compute the correction to $\hat{r}_{ii}$ which gives $r_{\bar{i}\bar{i}}^{-1}$. This will entail the use of a computer, and will usually not permit much insight into the nature of the $\hat{K}_a$ and $\hat{K}_b$ which will bring the system to the desired form.

On the other hand, if the sum in eqns. 26 or 27 is sufficiently small, we shall expect to be able to work with $\hat{r}_{ii}$ alone as an approximation to $r_{\bar{i}\bar{i}}^{-1}$. Section 2.3 gives a result of this type. With its aid, the system can first be modified until the sum in eqns. 26 or 27 is small enough to be safely neglected. Then the design can be carried on using the $\hat{r}_{ii}$ as though they were inverse Nyquist diagrams for separate loops. The remaining effect of interaction can be found if desired from eqns. 26 or 27, or we may revert to simulation to finalise the design. It should be noted[2] that, at any frequency ω_0, a high enough gain in all other loops except the ith will make the sum in eqns. 26 or 27 negligible.

2.3 A multivariable stability theorem

Let S_0 be the open-loop system consisting of a controller with transfer-function matrix $K(s)$ cascaded with the given plant. Because the plant is asymptotically stable and $K(s)$ has all its poles in the open left halfplane we may assume that S_0 is asymptotically stable. Let S_c be the closed-loop system which results from feedback according to eqn. 9. Define

$$d_i(s) = \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ij}(s)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and let D be a contour in the complex plane consisting of the imaginary axis from $-j\alpha$ to $+j\alpha$ and a semicircle of radius α in the right halfplane. Here α is sufficiently large to ensure that all finite poles and zeros of the functions of interest ($|Q|$, $|R|$, q_{ij}, $\hat{q}_{ij}$, r_{ij}, $\hat{r}_{ij}$) lying in the open right halfplane are inside D, and all such poles and zeros on the imaginary axis lie on D.

From theorem 6 of Appendix 7, the following sufficient condition for the asymptotic stability of S_c is immediately deduced. Suppose that $\hat{q}_{ii}(s)$ maps D into $\hat{\Gamma}_{0i}$; i.e. $\hat{\Gamma}_{0i}$ is the inverse Nyquist diagram (completed in the usual way) obtained from $\hat{q}_{ii}(s)$. S_c is asymptotically stable if the three following conditions are fulfilled:

(*a*) For each loop j which is closed in S_c, $\hat{\Gamma}_{0j}$ encircles the point $(-1, 0)$ the same number of times (in the same direction) as it encircles the origin.

(*b*) For $i = 1, 2, \ldots, m$ and for all s on D,

$$|\hat{q}_{ii}(a)| - d_i(s) \geqslant \epsilon > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

(*c*) For each loop j which is closed in S_c and for all s on D,

$$|\hat{r}_{jj}(s)| - d_j(s) \geqslant \epsilon > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The first of these conditions resembles the usual Nyquist criterion for a single loop around a plant with transfer function $q_{\bar{j}\bar{j}}^{-1}$. The remaining conditions ensure that the interactions are sufficiently small to allow stability to be deduced from the diagonal elements $\hat{q}_{jj}$ alone. As the criterion is sufficient but not necessary, instability cannot be inferred from the failure of the conditions. In conditions (*b*) and (*c*) d_i can be replaced throughout by

$$\delta_i = \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ji}(s)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

3 Examples

The design procedures to which the I.N.A. lends itself will now be illustrated by two examples. The first is the system having a transfer-function matrix G given by eqn. 12. The I.N.A. is shown for $k_1 = k_2 = 1$ in Fig. 2, and it is easily verified that conditions (*a*), (*b*) and (*c*) of Section 2.3 are already fulfilled for any $k_1 \geqslant 0$ and $k_2 \geqslant 0$. Arbitrarily high gains k_1 and k_2 can therefore be applied in each of the two principal loops without instability.

The effect of applying a gain k_1 is to multiply $\hat{q}_{11}$ and $\hat{q}_{12}$ by $\hat{k}_1 = k_1^{-1}$, and similarly for k_2. Consequently, as k_1 and k_2 are increased, the system becomes more and more nearly noninteracting. Eqn. 26 shows that the transfer function seen between input v_1 and output y_1, when the second loop is closed with gain k_2 and the first is open, is

$$r_{\bar{1}\bar{1}}^{-1}(s) = \hat{q}_{11}(s) - \frac{\hat{k}_2\hat{q}_{12}(s)\hat{q}_{21}(s)}{1 + \hat{k}_2\hat{q}_{22}(s)} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

so that

$$r_{\bar{1}\bar{1}}^{-1}(j\omega) = 1 + j\omega - \frac{1}{4(k_2 + 1 + j\omega)} \quad . \quad . \quad . \quad (33)$$

When k_2 becomes large, this is approximately $1 + j\omega$, which is $\hat{q}_{11}(j\omega)$. A similar result holds for the second loop when it is open and the first loop is closed with k_1 large. In other words, no attempt need be made to reduce the interaction by elementary row operations on the I.N.A.; large gains k_1 and k_2 can be used without instability; and when high gains are used the system behaves effectively as though it were noninteracting, with each loop having transfer function $1/(1 + s)$.

This example is exceptional because there is no difficulty in exerting control. Nevertheless, it illustrates the ability of the I.N.A. to suggest simple control schemes. Current alternative methods for designing multivariable controllers are the use of a state observer and either modal analysis[5] or optimal-control theory.[6] Those methods would give a simple answer for this example, but it would be easy to devise systems of higher order, having similar $G(j\omega)$, for which the I.N.A. would suggest the same control scheme whereas the alternatives would suggest much more complicated schemes.

The second example is

$$G(s) = \begin{bmatrix} \frac{1-s}{(1+s)^2} & \frac{2-s}{(1+s)^2} \\ \frac{1-3s}{3(1+s)^2} & \frac{1-s}{(1+s)^2} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (34)$$

which has been considered previously.[2] The inverse of G is

$$\hat{G}(s) = (1+s)\begin{bmatrix} 3(1-s) & -3(2-s) \\ -(1-3s) & 3(1-s) \end{bmatrix} \quad . \quad . \quad (35)$$

from which the I.N.A. shown in Fig. 4 is obtained.

An obvious first step in the design procedure is to bring

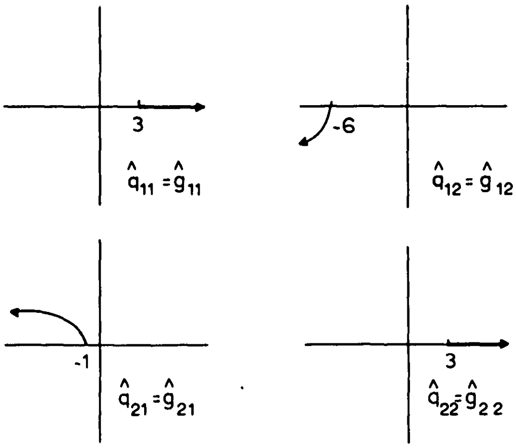

Fig. 4
Inverse Nyquist array for eqn. 35

$\hat{Q}(0)$ to diagonal form by premultiplying $\hat{G}(s)$ by the constant matrix $\hat{G}^{-1}(0) = G(0)$, which is nonsingular. [If $G(0)$ were singular, this would indicate that the inputs were badly chosen for the given outputs, since it would then be impossible to find inputs which would achieve arbitrarily chosen steady-state outputs.] The result is

$$\hat{Q}(s) = (1+s)\begin{bmatrix} 1 & 2 \\ 1/3 & 1 \end{bmatrix}\begin{bmatrix} 3(1-s) & -3(2-s) \\ -(1-3s) & 3(1-s) \end{bmatrix} \quad . \quad (16)$$

$$= (1+s)\begin{bmatrix} 1+3s & -3s \\ 2s & 1-2s \end{bmatrix} \quad . \quad . \quad . \quad . \quad (37)$$

from which the I.N.A. of Fig. 5 is obtained. This shows that

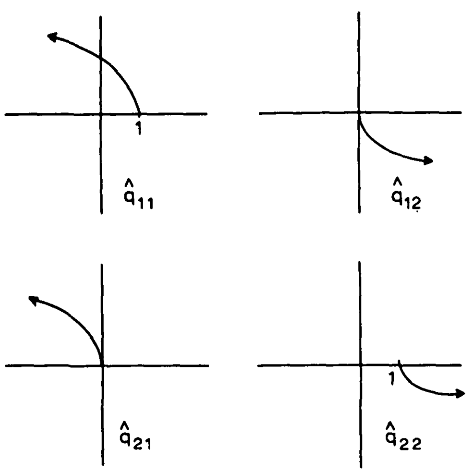

Fig. 5
Inverse Nyquist array for eqn. 37

$\hat{q}_{11}$ satisfies condition (*a*) of Section 2.3 for all $k_1 \geqslant 0$. Moreover, the d_1 calculated from eqn. 28 shows that conditions (*b*) and (*c*) will be fulfilled, so far as the first principal loop is concerned, for $k_1 \geqslant 0$.

On the other hand, $\hat{q}_{22}$ has a zero at $s = \frac{1}{2}$, and so at the point $(\alpha, 0)$ on D,

$$\left|\frac{\hat{q}_{22}(s)}{1+s}\right| = |1 - 2\alpha| = 2\alpha - 1 < 2\alpha = \left|\frac{\hat{q}_{21}(s)}{1+s}\right| \quad (38)$$

showing that condition (*b*) is not fulfilled for the second loop. To correct this, we may subtract a multiple of row 1 from row 2. To achieve the desired effect, the multiplier must exceed $\frac{2}{3}$, but must not exceed 1. Accordingly we choose $\frac{5}{6}$, giving a new $\hat{Q}$:

$$\hat{Q}(s) = (1+s)\begin{bmatrix} 1 & 0 \\ -\frac{5}{6} & 1 \end{bmatrix}\begin{bmatrix} 1+3s & -3s \\ 2s & 1-2s \end{bmatrix} \quad . \quad (39)$$

$$= (1+s)\begin{bmatrix} 1+3s & -3s \\ -\frac{5}{6}-\frac{1}{2}s & 1+\frac{1}{2}s \end{bmatrix} \quad . \quad . \quad . \quad . \quad (40)$$

The I.N.A. is shown in Fig. 6, and it follows from Section 2.3

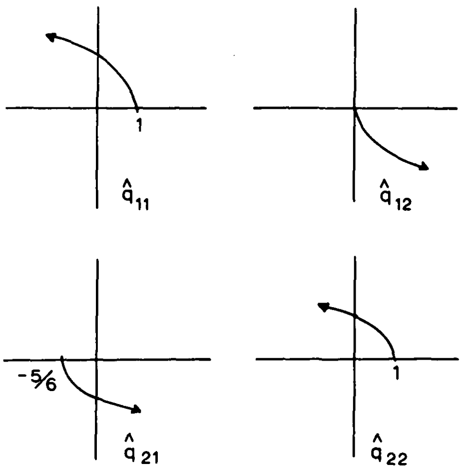

Fig. 6
Inverse Nyquist array for eqn. 40

that the closed-loop system is stable for all $k_1 \geqslant 0$ and $k_2 \geqslant 0$.

As k_1 and k_2 are increased, the system now approximates over an increasingly wide frequency band to a diagonal system. At high enough frequencies interaction remains important, and eqn. 26 gives, with the first loop open,

$$r_{11}^{-1}(s) = (1+s)(1+3s) - \frac{3s(1+s)^2(\frac{5}{6}+\frac{1}{2}s)}{k_2 + (1+s)(1+\frac{1}{2}s)} \quad . \quad (41)$$

and with the second loop open,

$$r_{22}^{-1}(s) = (1+s)(1+\frac{1}{2}s) - \frac{3s(1+s)^2(\frac{5}{6}+\frac{1}{2}s)}{k_1 + (1+s)(1+3s)} \quad . \quad (42)$$

The design may therefore proceed on this basis, and compensating networks may be used if desired in the two loops to improve their response. The importance of the stability theorem of Section 2.3 in permitting us to proceed in this way will be clear. In a previously suggested method,[2] resembling this in some respects, the assurance of stability was absent.

The final matrix $\hat{K}$ is

$$\hat{K} = \begin{bmatrix} 1 & 0 \\ -\frac{5}{6} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{2}{3} \end{bmatrix} \quad . \quad . \quad . \quad (43)$$

from which

$$K = \begin{bmatrix} -2 & -6 \\ \frac{3}{2} & 3 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$Q(s) = G(s)K = \frac{1}{(1+s)^2}\begin{bmatrix} 1+\frac{1}{2}s & 3s \\ \frac{5}{6}+\frac{1}{2}s & 1+3s \end{bmatrix} \quad . \quad (45)$$

and (with a slight rearrangement) the system appears as in Fig. 7. This system was simulated on an analogue computer, and the step responses for $k_1 = 50$ and $k_2 = 50$ are shown in Fig. 8.

An interesting point arises from the above procedure. If

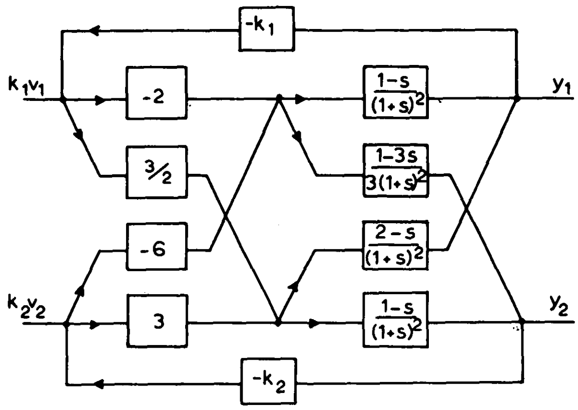

Fig. 7
System corresponding to eqn. 45 as simulated

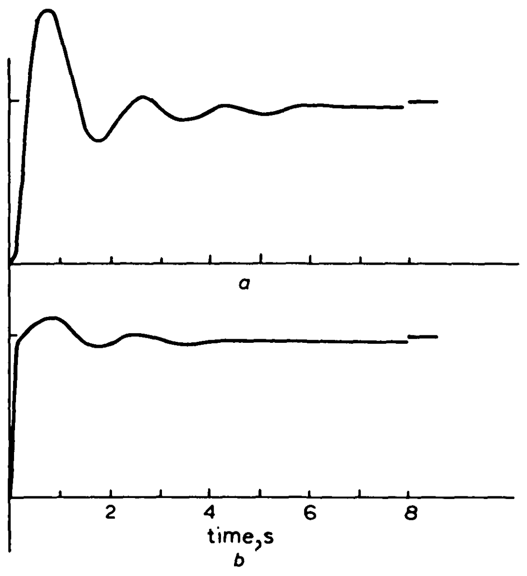

Fig. 8
Closed-loop responses of system in Fig. 7 with $k_1 = k_2 = 50$
a y_1 for unit step in v_1
b y_2 for unit step in v_2

the multiplier in eqn. 39 is reduced from $\frac{5}{6}$ towards $\frac{2}{3}$, the second time constant occurring in $\hat{q}_{22}^{-1}$ is reduced, becoming zero when the multiplier is $\frac{2}{3}$. The temptation to exploit this should probably be resisted, because it clearly makes the system more sensitive to variations in the form of G. Moreover, the response of the system can always be improved by the use of compensating networks in the two principal loops. Such considerations are easily incorporated in the present method.

This second example is not a trivial one. The original system, eqn. 34, allows four different single control loops to be set up. All of these have nonminimum phase, and they all present considerably more difficulty than either of the two loops in Fig. 7. Attempts to set up two loops simultaneously around the original system are equally unpromising. Yet only a matrix of constant interconnections is needed, as in Fig. 7, to allow two simultaneous loops of relatively good, and easily improved, performance. The method described allows this matrix to be obtained by a systematic procedure, which can take account of engineering constraints.

4 Generalisations and further work

So far it has been assumed that the control action will be exerted by the controller K preceding the plant. It has been pointed out elsewhere[2] that there is sometimes an advantage in using the more general system illustrated in Fig. 9. This is particularly true when $|G(s)|$ has right half-plane zeros, because the more general system may confine the resulting difficulties to fewer loops.

There are two contexts in which Fig. 9 may apply. In many

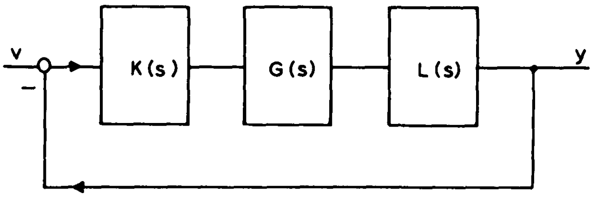

Fig. 9
More general multivariable control scheme

industrial regulator problems, the matrix L may actually be implemented; it will then most probably be restricted to be independent of s. For example, if two temperatures are to be controlled, it may be equally satisfactory to implement a scheme which controls the sum and the difference of the two temperatures. This may allow better control than a scheme in which the two temperatures are themselves treated as the controlled variables.

The matrix L may also be useful as a conceptual device, even when it is not implemented physically. The transformation of Fig. 10 shows that the dynamic behaviour of the

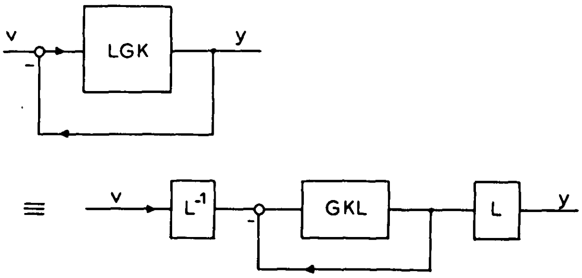

Fig. 10
Illustrating the matrix identity

multivariable closed-loop system can be analysed in terms of LGK, even when the system which is implemented uses GKL. Suppose, for example, that in the previous illustration the difference $(\theta_1 - \theta_2)$ between the two temperatures could be well controlled, while $\theta_1 + \theta_2$ could only be poorly controlled. Then both θ_1 and θ_2 will in general contain some component of $\theta_1 + \theta_2$, and so will show poor response. However, if the system is designed with K alone, θ_1, θ_2 and also $\theta_1 - \theta_2$ might all show poor response. Then the component of any disturbance affecting $\theta_1 - \theta_2$ would be less well controlled than before. By designing in terms of LGK, and implementing GKL, this possibility may be avoided.

It seems possible that the method illustrated by the examples in Section 3 may allow the describing function to be applied to multivariable systems. It may also be possible to adapt root-locus techniques as an alternative to the inverse Nyquist array. These are subjects for further study.

In the simple examples treated in Section 3, the modification of $\hat{Q}$ was carried out algebraically. If $G(j\omega)$ is obtained by measurement, it will be necessary to compute $\hat{Q}(j\omega)$ numerically, frequency by frequency. This may also be the simplest thing to do in more complicated examples when a computer is used, even when G is given algebraically, because the manipulation of polynomials in a computer is not easy to organise.

It seems, therefore, that for practical applications it will be desirable to have computer facilities which will allow the I.N.A. to be visually displayed. The effect of any proposed operation on the I.N.A. could then be computed by the machine and displayed, allowing the designer to assess its effect. Without such facilities, the labour of implementing the method would in many cases become excessive.

5 Acknowledgments

The assistance of P. D. McMorran in simulating the system shown in Fig. 7 is acknowledged.

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7 Appendix

Theorem 1

Let $K(s)$ be a nonsingular rational polynomial matrix. Let all the poles of $K(s)$, and all the zeros of $|K(s)|$, lie in the open left halfplane. Then $K(s)$ can be written as a product $K_aK_b(s)K_c(s)$ where

(*a*) K_a is a permutation matrix
(*b*) $K_b(s)$ is the product of elementary column operations, each consisting of the addition of a multiple $\alpha_{ij}(s)$ of column i to column j; α_{ij} is a rational polynomial function having all its poles in the open left halfplane
(*c*) $K_c(s)$ is a nonsingular diagonal matrix. All the poles and all the zeros of the principal diagonal elements of $K_c(s)$ lie in the open left halfplane.

Proof

Let $p_i(s)$ be the least common multiple of the denominators of the ith column of $K(s)$. Clearly each $p_i(s)$ has its zeros in the open left halfplane. In the polynomial matrix $K(s)$ diag$\{_ip(s)\}$, let $D_1(s)$ be the greatest common factor of elements in the first column. Let $D_2(s)$ be the greatest common factor of all 2×2 minors formed from the first two columns. In general, let $D_i(s)$ be the greatest common factor of all $i \times i$ minors formed from the first i columns. It follows from the Laplace expansion of a determinant that $D_i(s)$ divides D_{i+1}, $i = 1, 2, \ldots, m-1$. Then because a zero of $D_m(s)$ is a zero of $|K(s)|$ or of a $p_i(s)$, it follows that each $D_i(s)$ has all its zeros in the open left halfplane.

Multiply the first column of $K(s)$ by $p_1(s)/D_1(s)$. The resulting column will have polynomial entries with no common factor. Using a known procedure,[7] the first column can therefore be reduced to the form $(a_1, 0, 0, \ldots, 0)^T$, where a_1 is a nonzero constant, by successive operations of the following types

(i) transpose two rows
(ii) add to row i a multiple by $\alpha_{ij}(s)$ of row j, where $\alpha_{ij}(s)$ is a polynomial in s.

These operations do not change the greatest common factor $D_2(s)$ of 2×2 minors formed from the first two columns. It follows that if the second column is multiplied by $p_2(s)$ the elements in positions (2, 2), (3, 2), . . . , (m, 2) are then polynomials with highest common factor $D_2(s)/D_1(s)$. Divide the resulting second column by $D_2(s)/D_1(s)$ to give $m-1$ polynomial elements with no common factor other than 1. As before, reduce this $(m-1)$-vector to the form $(a_2, 0, 0, \ldots, 0)^T$ by row operations of the two types applied to the last $m-1$ rows. The element in position (1, 2) is a rational polynomial function. Add a suitable multiple of row 2 to row 1 to reduce the element (1, 2) to zero. This does not affect the element (1, 1), so that the matrix now has the form

$$\left[\begin{array}{cc|c} a_1 & 0 & \\ 0 & a_2 & T_1(s) \\ \hline & 0 & T_2(s) \end{array}\right]$$

where T_1 is $2 \times (m-2)$, and T_2 is $(m-2) \times (m-2)$.

Proceeding in this way, the matrix can be reduced to the form diag (a_i). Write $D_0(s) = 1$, and put

$$K_c(s) = \text{diag}\left\{\frac{a_iD_i(s)}{p_i(s)D_{i-1}(s)}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The above argument shows that

$$K_d(s)K(s)K_c^{-1}(s) = I_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where $K_d(s)$ is a matrix generated by row operations of the types

(*a*) transpose two rows
(*b*) add to row i a multiple by $\alpha_{ij}(s)$ of row j, where $\alpha_{ij}(s)$ is a rational function of s having as denominator either 1, or one of the polynomials D_1/D_0, D_2/D_1, . . . , D_m/D_{m-1}.

Each of these two types of operation has an inverse which is of the same type. Moreover if K_e represents a transposition of two rows, and K_f a row operation described by (*b*) above, it is easy to see that $K_fK_e = K_eK_f'$, where K_f' is a (different) matrix of the same type as K_f. For example

$$\begin{bmatrix} 1 & 0 & \alpha_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{12} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_{12} & 1 \end{bmatrix} \quad . \quad . \quad . \quad (48)$$

Further, a row operation of type (*b*) above is represented by a matrix such as the last matrix in eqn. 48. When operating on the right of another matrix this represents a column operation of the type used to define $K_b(s)$. By virtue of these remarks $K_d^{-1}(s)$ can be written as a product $K_aK_b(s)$, where K_a, $K_b(s)$ have the properties stated in the theorem. Eqn. 47 then gives

$$K(s) = K_aK_b(s)K_c(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

which completes the proof.

Theorem 2

Let $G(s)$ be a nonsingular rational polynomial matrix. Let all the poles of $G(s)$, and all the zeros of $|G(s)|$, lie in the open left halfplane. Then K_a and $K_b(s)$ can be found, defined as in theorem 1, such that $G(s)K_aK_b(s)$ is nonsingular, diagonal, and has all the poles and zeros of its principal diagonal elements in the open left halfplane.

Proof

Interchanging the roles of rows and columns in the proof of theorem 1, it follows that $G(s)$ can be written

$$G(s) = G_c(s)G_b(s)G_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Here G_a is a permutation matrix; $G_b(s)$ represents the product of elementary row operations each consisting of the addition of a multiple of row j by $\alpha_{ij}(s)$ to row i, where α_{ij} is a rational polynomial function having all its poles in the open left halfplane; and $G_c(s)$ is nonsingular, diagonal, and with all the poles and zeros of its principal diagonal elements in the open left halfplane. The choice

$$K_a = G_a^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$K_b(s) = G_b^{-1}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

gives the desired result.

Theorem 3

Let $G(s)$ satisfy the conditions in theorem 2, except that $|G(s)|$ has one or more zeros in the closed right halfplane. Then K_a and $K_b(s)$ can be found, defined as in theorem 1, such that $G(s)K_aK_b(s)$ is triangular (upper or lower as desired). In general $G(s)K_aK_b(s)$ cannot be made diagonal.

Proof

Let $p_i(s)$ be the least common multiple of the denominators of the ith row of $G(s)$. Each $p_i(s)$ has its zeros in the open left halfplane. In the polynomial matrix diag $[p_i(s)]G(s)$, let $D_i(s)$ be the greatest common factor of all $i \times i$ minors formed from the first i rows and put $D_0(s) = 1$. Let

$$D_i(s) = D_{il}(s)D_{ir}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where $D_{il}(s)$ has all its zeros in the open left halfplane and $D_{ir}(s)$ has all its zeros in the closed right halfplane. It follows, from the Laplace expansion of a determinant, that $D_{il}(s)$ [resp. $D_{ir}(s)$] divides $D_{i+1,l}(s)$ [resp. $D_{i+1,r}(s)$], $i = 1, 2, \ldots, m-1$.

Multiply the first row of $G(s)$ by $p_1(s)$. The resulting row has polynomial entries with greatest common factor $D_1(s)$. Using a known procedure,[7] the first row can therefore be brought to the form $\{a_1D_1(s), 0, 0, \ldots, 0\}$, where a_1 is a nonzero constant, by successive operations of the types

(*a*) transpose two columns
(*b*) add to column j a multiple by $\alpha_{ij}(s)$ of column i, where $\alpha_{ij}(s)$ is a polynomial in s.

These operations do not change the greatest common factor $D_2(s)$ of the 2×2 minors formed from the first two rows. Multiply the second row by $p_2(s)$. The elements in positions (2, 2), (2, 3), . . . , (2, m) are then divisible by $D_2(s)/D_1(s)$. By operations of the two types described above the last $m-1$ elements in the second row can be brought to the form $\{a_2D_2(s)/D_1(s), 0, 0, \ldots, 0\}$. Now

$$\frac{D_2(s)}{D_1(s)} = \frac{D_{2l}(s)}{D_{1l}(s)}\frac{D_{2r}(s)}{D_{1r}(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

so that a multiple [by a rational polynomial expression having as denominator the polynomial $D_{2l}(s)/D_{1l}(s)$] of column 2 can be subtracted from column 1 in such a way that the degree of the elements in position (2, 1) is less than the degree of the polynomial $D_{2r}(s)/D_{1r}(s)$.

Proceeding in this way we generate a lower triangular polynomial matrix $T_1(s)$. As in the proof of theorem 1, it follows that

$$G(s) = \text{diag}\{1/p_i(s)\}T_1(s)G_b(s)G_a \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where G_a is a permutation matrix; $G_b(s)$ is a product of column operations, each consisting of adding a multiple (by a rational polynomial expression having its poles, if any, in the open left halfplane) of one column to another; and $T_1(s) = [\tau_{ij}(s)]$ is a lower triangular polynomial matrix in which the degree of $\tau_{ij}(s)$ when $i > j$ is less than the degree of $D_{ir}(s)/D_{i-1,r}(s)$. Setting $K_a = G_a^{-1}$ and $K_b(s) = G_b^{-1}(s)$, it follows that

$$G(s)K_aK_b(s) = T(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where $T(s)$ is a lower triangular matrix. With minor changes in the proof, an upper triangular matrix can be generated.

That reduction to diagonal form is not generally possible when K_a and $K_b(s)$ are as defined in theorem 1 is shown by the example

$$G(s) = \begin{bmatrix} \dfrac{1-s}{(1+s)^2} & \dfrac{1}{1+2s} \\ 0 & \dfrac{1}{1+s} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Attempt to find $K(s)$ such that

$$G(s)K(s) = \begin{bmatrix} \dfrac{1-s}{(1+s)^2} & \dfrac{1}{1+2s} \\ 0 & \dfrac{1}{1+s} \end{bmatrix} \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix} \quad . \quad (58)$$

is diagonal. Then it follows that $k_{21}(s) = 0$, and

$$\frac{1-s}{(1+s)^2}k_{12}(s) + \frac{1}{1+2s}k_{22}(s) = 0 \quad . \quad . \quad . \quad . \quad (59)$$

Consequently

$$|K(s)| = k_{11}(s)k_{22}(s) = -k_{11}(s)k_{12}(s)\frac{(1+2s)(1-s)}{(1+s)^2} \quad . \quad (60)$$

No choice of $k_{11}(s)$ and $k_{12}(s)$ with left halfplane poles can avoid a factor $1-s$ in $|K(s)|$. But with the definition of theorem 1, $|K_a| = \pm 1$ and $|K_b(s)| = 1$, so that $K(s)$ cannot be represented by $K_aK_b(s)$.

Theorem 4

Let $G(s)$ be as in theorem 3. Find K_a and $K_b(s)$ as in that theorem so that eqn. 56 is true. Then $T(s)$ may be diagonal. If not, let $K_e(s)$ be any nonsingular rational polynomial matrix having all its poles in the open left halfplane and such that $G(s)K_e(s)$ is diagonal. Then $|K_e(s)|$ has at least one zero in the closed right halfplane.

Proof

Assume $T(s)$ is lower triangular. We have

$$G(s)K_aK_b(s)\{K_b^{-1}(s)K_a^{-1}K_e(s)\} = T(s)K_f(s) = D(s) \quad . \quad . \quad . \quad (61)$$

where $D(s) = [d_{ij}(s)]$ and $K_f(s) = [\kappa_{ij}(s)]$. Hence

$$K_f(s) = T^{-1}(s)D(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and so $K_f(s)$ is lower triangular. It is also nonsingular, and has all its poles in the open left halfplane.

If $T(s)$ is not diagonal, there is a $t_{ii}(s)$ having a zero in the closed right halfplane, and a $t_{ij}(s) \neq 0$ with $i > j$ such that

$$t_{i,j+1}(s) = t_{i,j+2}(s) = \ldots = t_{i,i-1}(s) = 0 \quad . \quad (63)$$

Then in eqn. 61

$$0 = d_{ij}(s) = t_{ij}(s)\kappa_{jj}(s) + t_{ii}(s)\kappa_{ij}(s) \quad . \quad . \quad . \quad . \quad (64)$$

and so

$$\kappa_{jj}(s) = -\frac{\kappa_{ij}(s)t_{ii}(s)}{t_{ij}(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

It follows from the proof of theorem 3 that

$$t_{ii}(s) = \frac{a}{p_i(s)}\{D_{il}(s)/D_{i-1,l}(s)\}\{D_{ir}(s)/D_{i-1,r}(s)\} \quad . \quad (66)$$

and $$t_{ij}(s) = \frac{\tau_{ij}(s)}{p_i(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where $\tau_{ij}(s)$ is an element of $T_1(s)$. Because the polynomial $D_{ir}(s)/D_{i-1,r}(s)$ has all its zeros in the closed right halfplane and exceeds $\tau_{ij}(s)$ in degree, it follows from eqn. 65 and the properties of $\kappa_{ij}(s)$ and the $D_{il}(s)$ that $\kappa_{jj}(s)$ has a zero in the closed right halfplane. Consequently $|K_f(s)|$ has a zero there, and by eqn. 61 and the properties of K_a and $K_b(s)$, so does $|K_e(s)|$. The proof holds, with minor changes, if $T(s)$ is upper triangular, and it is exemplified by eqn. 60.

Theorem 5

Let a system S_0 be described by a set of linear ordinary differential equations with constant coefficients, and after Laplace transformation with zero initial conditions let its equations be

$$T(s)z(s) = U(s)e(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$y(s) = V(s)z(s) + W(s)e(s) \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where $T(s)$, $U(s)$, $V(s)$ and $W(s)$ are polynomial matrices, respectively $r \times r$, $r \times m$, $m \times r$, and $m \times m$, let $|T(s)| \not\equiv 0$. Let S_0 be asymptotically stable and let its transfer function matrix be $Q(s) = V(s)T^{-1}(s)U(s) + W(s)$, where $|Q(s)|$ has no zero on any finite part of the imaginary axis. Let feedback be applied to S_0 according to the equation

$$e(s) = v(s) - Fy(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where F is as defined in Section 2, so that the transfer function matrix of the resulting system S_c is $R(s) = \{I_m + Q(s)F\}^{-1}Q(s)$. Let D be a contour consisting of the imaginary axis from $-i\alpha$ to $+i\alpha$ and a semicircle of radius α in the right halfplane. Here α is large enough to ensure that every finite pole or zero of $|Q|$, $|R|$, q_{ij}, $\hat{q}_{ij}$, r_{ij} and $\hat{r}_{ij}$ $(i, j = 1, 2, \ldots, m)$, which is in the open right halfplane lies within D, and every finite imaginary pole or zero of these functions lies on D. Let $|Q(s)|$ (resp. $|\hat{Q}(s)|$) map D into Γ_0 (resp. $\hat{\Gamma}_0$) and let $|R(s)|$ (resp. $|\hat{R}(s)|$) map D into Γ_c (resp. $\hat{\Gamma}_c$). Then S_c is asymptotically stable if and only if $|R(s)|$ (resp. $|\hat{R}(s)|$) has no pole (resp. zero) on any finite part of the imaginary axis and Γ_c (resp. $\hat{\Gamma}_c$) encircles the origin as often in a clockwise (resp. counterclockwise) direction as Γ_0 (resp. $\hat{\Gamma}_0$).

Proof

The system of eqns. 68–70 can be written

$$\begin{bmatrix} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I_m \\ 0 & I_m & F \end{bmatrix} \begin{bmatrix} -z(s) \\ e(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v(s) \end{bmatrix} \quad . \quad (71)$$

so that S_c is asymptotically stable if and only if the roots of

$$|P_c(s)| = \begin{vmatrix} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I_m \\ 0 & I_m & F \end{vmatrix} = 0 \quad . \quad . \quad . \quad (72)$$

are all in the open left halfplane. Similarly, on putting $F = 0$ in eqn. 72, it follows from the known asymptotic stability of S_0 that all the roots of

$$\begin{vmatrix} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I_m \\ 0 & I_m & 0 \end{vmatrix} = |T(s)| = 0 \quad . \quad . \quad . \quad (73)$$

lie in the open left halfplane.

From eqn. 71

$$\begin{bmatrix} -z(s) \\ e(s) \\ y(s) \end{bmatrix} = P_c^{-1}(s) \begin{bmatrix} 0 \\ 0 \\ v(s) \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and because

$$y(s) = R(s)v(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

it follows that $|R(s)|$ is the minor, formed from the elements of $P_c^{-1}(s)$ in rows $r + m + 1, r + m + 2, \ldots, r + 2m$ and columns $r + m + 1, r + m + 2, \ldots, r + 2m$. By a known result,[8] this is

$$|R(s)| = \begin{vmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{vmatrix} \div \begin{vmatrix} T(s) & U(s) & 0 \\ -V(s) & W(s) & -I_m \\ 0 & I_m & F \end{vmatrix} \quad (76)$$

The corresponding formula for $|Q(s)|$, which has been given previously,[9] is obtained by putting $F = 0$ in eqn. 76 and after simplification is

$$|Q(s)| = \begin{vmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{vmatrix} \div |T(s)| \quad . \quad . \quad . \quad . \quad (77)$$

Let $|R(s)|$ have z_c finite zeros and p_c finite poles in the closed right halfplane, and let $|Q(s)|$ have z_0 finite zeros and p_0 finite poles there. Since $|T(s)|$ has all its zeros in the open left halfplane, it follows that $p_0 = 0$, and z_0 is the number of zeros of

$$\begin{vmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

in the open right halfplane. This expression has no imaginary zeros because $|Q(s)|$ has no finite imaginary zeros. Comparison of eqns. 72 and 76 now shows that S_c is asymptotically stable if and only if

$$z_c - p_c = z_0 = z_0 - p_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

By eqn. 76, $|R(s)|$ has no finite zero on the imaginary axis. If it also has no finite pole on the imaginary axis, it follows that $z_c - p_c$ is the number of clockwise circuits of Γ_c about the origin. Also $z_0 - p_0$ is the number of clockwise circuits of Γ_0 about the origin. The system S_c is therefore asymptotically stable if and only if $|R(s)|$ has no finite pole on the imaginary axis and Γ_c, Γ_0 make the same number of clockwise circuits of the origin. The corresponding statements in terms of $|\hat{R}(s)|$, $\hat{\Gamma}_c$ and $\hat{\Gamma}_0$, follow at once.

Theorem 6

Let the contour D be as defined in theorem 5, and let the element $\hat{q}_{ii}(s)$ of $\hat{Q}(s)$ map D into $\hat{\Gamma}_{0i}$. Similarly let $\hat{r}_{ii}(s)$ map D into $\hat{\Gamma}_{ci}$. Let $\hat{\Gamma}_{0i}$ and $\hat{\Gamma}_{ci}$ encircle the origin n_{0i} times and n_{ci} times respectively (the counterclockwise direction being taken as positive). Define $d_i(s)$ [resp. $\delta_i(s)$] by

$$d_i(s) = \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ij}(s)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

$$\delta_i(s) = \sum_{\substack{j=1 \\ j \neq i}}^{m} |\hat{q}_{ji}(s)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Then if S_0, S_c are as defined in theorem 5, a sufficient condition for the asymptotic stability of S_c is that the following conditions are fulfilled

(*a*) $\sum_{i=1}^{m} n_{0i} = \sum_{i=1}^{m} n_{ci}$

(*b*) For all s on D and for $i = 1, 2, \ldots, m$,

$|\hat{q}_{ii}(s)| - d_i(s) \geqslant \epsilon > 0$ {resp. $|\hat{q}_{ii}(s)| - \delta_i(s) \geqslant \epsilon > 0$}

(*c*) For all s on D and for $i = 1, 2, \ldots, m$,

$|\hat{r}_{ii}(s)| - d_i(s) \geqslant \epsilon > 0$ {resp. $|\hat{r}_{ii}(s)| - \delta_i(s) \geqslant \epsilon > 0$}

Proof

Because $\hat{q}_{ii}(s) = Q_{ii}(s)/|Q(s)|$, it follows from the properties of $Q(s)$ that $\hat{q}_{ii}(s)$ has no finite imaginary pole. Also by condition (*b*), $\hat{q}_{ii}(s)$ has no finite imaginary zero. Let $\hat{Q}(\theta, s)$ be the matrix having elements

$$\left.\begin{aligned} \hat{q}_{ii}(\theta, s) &= \hat{q}_{ii}(s) \\ \hat{q}_{ij}(\theta, s) &= \theta \hat{q}_{ij}(s), \; i \neq j \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

where $\hat{Q}(s) = [\hat{q}_{ij}(s)]$ is the matrix defined in theorem 5, and $0 \leqslant \theta \leqslant 1$. The function $|\hat{Q}(0, s)| = \prod_{i=1}^{m} |\hat{q}_{ii}(s)|$ has no finite pole or zero on the imaginary axis; let it map D into $\hat{\Gamma}_p$. Also $|\hat{Q}(1, s)| = |Q(s)|^{-1}$ has no finite pole or zero on the imaginary axis, by the properties of $Q(s)$. It maps D into $\hat{\Gamma}_0$. If s_0 is a point on D, the function $|\hat{Q}(\theta, s_0)|$ of θ defines a continuous curve γ joining a point on $\hat{\Gamma}_p$ to a point on $\hat{\Gamma}_0$. As s_0 traces out D, starting from $s_0 = 0$, so γ sweeps out a region of the complex plane, and returns at last to its initial position.

Assume, contrary to what is to be proved, that $\hat{\Gamma}_p$ and $\hat{\Gamma}_0$ do not encircle the origin the same number of times. Then the region swept out by γ includes the origin; i.e. there is some θ and some s on D for which $|\hat{Q}(\theta, s)| = 0$. This implies that $\hat{Q}(\theta, s)$ has a zero eigenvalue, which is impossible by Gershgorin's theorem[3] and condition (*b*) of the theorem. Therefore $\hat{\Gamma}_0$ makes the same number of circuits of the origin as $\hat{\Gamma}_p$, which is $\sum_{i=1}^{m} n_{0i}$ because $\hat{Q}(0, s)$ is diagonal.

In the same way, $\hat{r}_{ii}(s)$, which is either $\hat{q}_{ii}(s)$ or $1 + \hat{q}_{ii}(s)$, has no finite pole on the imaginary axis. Also by condition (*c*), $\hat{r}_{ii}(s)$ has no finite zero on the imaginary axis. Therefore $\prod_{i=1}^{m} |\hat{r}_{ii}(s)|$ has no finite imaginary pole or zero. Further, $|\hat{R}(s)| = |F + \hat{Q}(s)|$ has no finite imaginary pole by the properties of $\hat{Q}(s)$: it has no finite imaginary zero by condition (*c*) and Gershgorin's theorem. Hence, as before, the number of circuits of the origin by $\hat{\Gamma}_c$ is $\sum_{i=1}^{m} n_{ci}$. It follows from condition (*a*) of the theorem, that $\hat{\Gamma}_0$ and $\hat{\Gamma}_c$ make the same number of circuits about the origin, and it has already been seen that $|\hat{R}(s)|$ has no finite imaginary zero. By theorem 5, S_c is therefore asymptotically stable.

A corresponding theorem with Q, R etc., in place of $\hat{Q}$, $\hat{R}$ etc., can be stated, but it seems to be less useful.