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The Characteristic Locus Design Method*

La Méthode Caractéristique de Design de Lieu

Die Methode des Entwurfes des charakteristischen Ortes

Метод расчета характеристической фазовой траектории

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Multivariable feedback systems may be designed using a technique which is a vector generalisation of the classical frequency-response methods introduced by Bode and Nyquist.

Summary—The classical work of Nyquist and Bode on the frequency–response analysis of scalar feedback systems leads to flexible and useful design procedures because it enables the conflicting requirements of stability and accuracy to be handled simultaneously via a single form of system representation—the open-loop frequency response function of a complex variable. State-space methods derive their elegance and power from the systematic exploitation of the algebraic and geometric properties of linear vector spaces. The basic idea underlying the Characteristic Locus Method developed here is the combination of the essence of these two approaches by exploiting the properties of linear vector spaces defined over base fields of functions of a complex variable. What then emerges is a general vector feedback theory in which the classical Bode–Nyquist technique is a special case, and from which a frequency–response based design technique called the Characteristic Locus Method is developed.

1. INTRODUCTION

BODE [1] and NYQUIST'S [2] classical work on the frequency–response analysis of scalar (single-input, single-output) feedback systems led to a flexible and useful design technique because it enabled the conflicting requirements of stability and accuracy to be handled simultaneously via a single form of representation—the open-loop frequency–response function of a complex variable. State-space methods derive their elegance and power from the systematic exploitation of the algebraic and geometric properties of linear vector spaces. The basic idea underlying the methods developed here is the combination of the essential features of these two approaches. This is done by introducing and exploiting the properties of linear vector spaces defined over base fields of functions of a complex variable. What then emerges is that the classical Bode–Nyquist theory is essentially the special scalar case of a completely general vector theory. Furthermore, this theory is constructed around a few basic and well-established algebraic and geometric properties of linear operators.

Apart from its intrinsic interest, there are two important reasons for the development of such an approach.

(i) It provides a useful technique for the design of a wide range of practical multivariable controllers for industrial plants described by a limited amount of experimentally obtained data.

(ii) It provides a bridge between the recently developed state-space methods used in optimal multivariable control [3] and optimal multivariable filtering [4] and the well-established classical frequency–response methods [5] hitherto largely restricted to single-input single-output systems.

2. FUNDAMENTAL FEEDBACK RELATIONSHIPS

The multivariable feedback configuration which most often arises in control studies is shown in Fig. 1 where

$\mathbf{r}(s)$ =vector of reference input transforms, of order m
$\mathbf{e}(s)$ =vector of error transforms, of order m
$\mathbf{u}(s)$ =vector of plant input transforms, of order l
$\mathbf{y}(s)$ =vector of output transforms, of order m
$\mathbf{K}(s)$=$l\times m$ matrix of controller transfer functions
$\mathbf{G}(s)$=$m\times l$ matrix of plant transfer functions
$\mathbf{H}(s)$=$m\times m$ matrix of feedback-transducer transfer functions
$\mathbf{I}_m$ =$m\times m$ identity matrix

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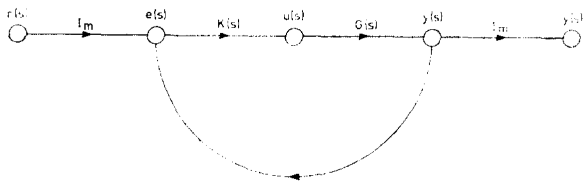

FIG. 1. Multivariable feedback system.

The closed-loop transfer function matrix $\mathbf{R}(s)$ for this system may be written in the form

$$\mathbf{R}(s)=[\mathbf{I}_m+\mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)\mathbf{K}(s). \quad (2\text{-}1)$$

Suppose that in Fig. 1 all the feedback loops are broken at $\mathbf{y}(s)$, and that a signal transform vector $\boldsymbol{\alpha}(s)$ is injected at this point. Then it is easily seen that the returned signal transform vector is

$$-\mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s)\boldsymbol{\alpha}(s)$$

so that the difference between injected and returned signals is given by

$$[\mathbf{I}_m+\mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s)]\boldsymbol{\alpha}(s)=\mathbf{F}_y(s)\boldsymbol{\alpha}(s) \quad (2\text{-}2)$$

where

$$\mathbf{F}_y(s)=\mathbf{I}_m+\mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s) \quad (2\text{-}3)$$

is a square matrix defined as the system *return-difference matrix* [6] measured at the output side of the plant. This is a natural generalisation of the scalar return-difference quantity defined by BODE [1] The matrix

$$\mathbf{T}_y(s)=\mathbf{G}(s)\mathbf{K}(s)\mathbf{H}(s) \quad (2\text{-}4)$$

is defined as the system *return-ratio matrix* measured at the output side of the plant. This again generalises BODE'S [1] corresponding scalar quantity. It then follows that

$$\mathbf{F}_y(s)=\mathbf{I}_m+\mathbf{T}_y(s). \quad (2\text{-}5)$$

The above "loop-breaking" approach to the definition of return-ratio and return-difference quantities need not be restricted to the system output vertex $\mathbf{y}(s)$. For instance, if the feedback loops were broken at the points corresponding to the signals $\mathbf{u}(s)$ and $\mathbf{e}(s)$ respectively, and the above analysis repeated, the results would be

$\mathbf{T}_u(s)=\mathbf{K}(s)\mathbf{H}(s)\mathbf{G}(s)=$return-ratio matrix for plant input vertex. (2-6)

$\mathbf{T}_e(s)=\mathbf{H}(s)\mathbf{G}(s)\mathbf{K}(s)=$return-ratio matrix for system error vertex. (2-7)

The corresponding return-difference operators then become

$$\mathbf{F}_u(s)=\mathbf{I}_l+\mathbf{T}_u(s) \quad (2\text{-}8)$$

$$\mathbf{F}_e(s)=\mathbf{I}_m+\mathbf{T}_e(s). \quad (2\text{-}9)$$

An application of SCHUR'S formulae for partitioned determinants [7] show that

$$\det \mathbf{F}_y(s)=\det \mathbf{F}_u(s)=\det \mathbf{F}_e(s). \quad (2\text{-}10)$$

In terms of these return difference matrices, simple algebraic manipulations give the following equivalent forms for the closed-loop system transfer function matrix relating $\mathbf{y}(s)$ and $\mathbf{r}(s)$:

$$\begin{aligned}\mathbf{R}(s)&=\mathbf{F}_y^{-1}(s)\mathbf{G}(s)\mathbf{K}(s)\\&=\mathbf{G}(s)\mathbf{F}_u^{-1}(s)\mathbf{K}(s)\\&=\mathbf{G}(s)\mathbf{K}(s)\mathbf{F}_e^{-1}(s).\end{aligned} \quad (2\text{-}11)$$

It is convenient to denote the product $\mathbf{G}(s)\mathbf{K}(s)$ which occurs repeatedly in treatments of this standard configuration by

$$\mathbf{Q}(s)=\mathbf{G}(s)\mathbf{K}(s) \quad (2\text{-}12)$$

and to call $\mathbf{Q}(s)$ the system open-loop transfer function matrix.

3. BASIC COMPLEX VARIABLE RELATIONSHIPS

Our objective is to devise frequency–response methods with which to attack the problem of analysing the stability and performance of feedback systems and, in particular, to develop design techniques for the standard configuration shown in Fig. 1. Furthermore, we wish these techniques to be natural generalisations of the classical Bode–Nyquist approach. Since this is essentially based on transform and complex variable theory, the key step which must be taken is to set up appropriate links between complex variables and matrix representations of linear operators. The required approach emerges naturally from the fact that, when transform methods are used in the analysis of multivariable feedback systems, one is immediately confronted by vectors and matrices whose elements are functions of a complex variable s. Thus we are dealing with quantities which we may formally define in the following way.

(i) A vector-valued function of a complex variable, $\mathbf{x}(s)$ say, is a mapping

$$\mathbf{x}(s):\mathscr{C}\to\mathscr{C}^m$$

from the set of complex numbers $\mathscr{C}$ into the set of complex vectors, $\mathscr{C}^m$.

(ii) A matrix-valued function of a complex variable, $\mathbf{G}(s)$ say, is a mapping

$$\mathbf{G}(s): \mathscr{C} \rightarrow \mathbf{M}(\mathscr{C})$$

from the set of complex numbers $\mathscr{C}$ into the set of matrices over the field of complex numbers, $\mathbf{M}(\mathscr{C})$. If $\mathbf{G}(s)$ is square, and det $\mathbf{G}(s)$ vanishes for every value of s, $\mathbf{G}(s)$ is said to be identically singular. If det $\mathbf{G}(s) \not\equiv 0$, then an inverse $\mathbf{G}^{-1}(s)$ can be computed in the usual way for every value of s for which det $\mathbf{G}(s)$ does not vanish.

For every specific value $s_1 \varepsilon \mathscr{C}$, a square $m \times m$ matrix function of a complex variable s, $\mathbf{G}(s_1)$ say, is a matrix having complex number entries. It thus has a set of eigenvalues $\{g_i(s_1): i=1, 2, \ldots, m\}$ such that

$$g_i(s_1) \varepsilon \mathscr{C} \quad i=1, 2, \ldots, m \tag{3-1}$$

and a corresponding set of eigenvectors $\{\mathbf{w}_i(s_1): i=1, 2, \ldots, m\}$ such that

$$\mathbf{w}_i(s_1) \varepsilon \mathscr{C}^m \quad i=1, 2, \ldots, m. \tag{3-2}$$

Put simply, the eigenvalues of a matrix function of a complex variable are themselves functions of a complex variable, and the corresponding eigenvectors are vector-valued functions of a complex variable.

The situation can however be looked at from a much more general point of view. Algebraic functions of a complex variable s form a field; put crudely, this means that we can carry out the standard forms of arithmetical manipulation with transfer functions, just as we do with real or complex numbers. This aspect of complex variable theory has been studied in great depth, culminating in WEYL'S famous text [16]. A good introduction to the more mathematical aspects of this concept has been given by SPRINGER [17].

It is sufficient for our present purposes simply to state the fact that it is a direct consequence of this that the eigenvalues of a square matrix $G(s)$ whose elements are rational functions in s will lie in the field of algebraic functions of the complex variable s. These complex functions eigenvalues will be called *characteristic transfer functions*, in order to avoid over-use of the term eigenvalue, and the corresponding eigenvectors called *characteristic directions*. In general, of course, such quantities will be irrational and must consequently be discussed, as by WEYL and SPRINGER, in the context of Riemann surfaces [16, 17]. Since we shall only be concerned with their frequency response evaluation, this aspect of their behavior is not further considered here.

If the vector space $\mathscr{C}^m$ is equipped with an inner product defined by

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \bar{x}_i y_i \tag{3-3}$$

for all $\mathbf{x}, \mathbf{y} \varepsilon \mathscr{C}^m$ it is called a unitary space [8]. In this last expression, $\bar{x}_i$ denotes the complex conjugate of x_i, and this convention in inner product definition is needed to ensure that the inner product is non-degenerate, that is that:

$$(\mathbf{x}, \mathbf{x}) = 0 \text{ if and only if } \mathbf{x} = \mathbf{0}.$$

With the inner product so defined, it is now possible to perform geometrical investigations in a resulting metric space using the natural metric

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}. \tag{3-4}$$

In particular, the angle between $\mathbf{x}, \mathbf{y} \varepsilon \mathscr{C}^m$ can be defined via

$$\cos \theta = \frac{|(\mathbf{x}, \mathbf{y})|}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \tag{3-5}$$

giving a measure of angle with all the properties required for straightforward geometrical operations in the complex vector space.

4. PERFORMANCE ANALYSIS

Any useful design procedure for multivariable feedback systems must aim at securing a suitable compromise between four conflicting objectives: stability, integrity, interaction and accuracy. In order to do this, appropriate techniques for the analysis of the above system properties must be established. Since these properties are essentially concerned with closed-loop phenomena, and since the proposed design method is to be based on the frequency–response behaviour of system characteristic transfer functions and characteristic vectors, a set of appropriate open-loop to closed loop relationships between these important quantities must first be obtained.

Let the $m \times m$ open-loop transfer function $\mathbf{Q}(s)$ matrix have a set of distinct characteristic transfer functions and associated linearly independent characteristic vectors denoted respectively by $q_i(s)$ and $\mathbf{w}_i(s)$ for $i=1, 2, \ldots, m$. Form the identically non-singular matrix

$$\mathbf{W}(s) = [\mathbf{w}_1(s)\mathbf{w}_2(s) \ldots \mathbf{w}_m(s)] \tag{4-1}$$

and invert it to give

$$\mathbf{V}(s)=\mathbf{W}^{-1}(s)=\begin{bmatrix}\mathbf{v}_1^t(s)\\ \mathbf{v}_2^t(s)\\ \cdot\\ \cdot\\ \cdot\\ \mathbf{v}_m^t(s)\end{bmatrix} \tag{4-2}$$

where the symbol t denotes transposition so that $\{\mathbf{v}_i^t(s)\}$ are the rows of $\mathbf{V}(s)$. Then standard algebraic relationships give that $\mathbf{Q}(s)$ may be expressed in the form

$$\mathbf{Q}(s)=\mathbf{W}(s)[\text{diag }\{q_i(s)\}]\mathbf{V}(s). \tag{4-3}$$

Alternatively, $\mathbf{Q}(s)$ may be expressed in the dyadic form

$$\mathbf{Q}(s)=\sum_{i=1}^{m} q_i(s)\mathbf{w}_i\mathbf{v}_i^t(s). \tag{4-4}$$

For unity feedback systems, with $\mathbf{H}(s)=\mathbf{I}_m$, the closed-loop transfer function matrix as given in equation (2-1) may be written as

$$\mathbf{R}(s)=[\mathbf{I}_m+\mathbf{Q}(s)]^{-1}\mathbf{Q}(s) \tag{4-5}$$

from which it can readily be shown that

$$\mathbf{R}(s)=\mathbf{W}(s)[\text{diag }\{q_i(s)/1+q_i(s)\}]\mathbf{V}(s) \tag{4-6}$$

or, in dyadic form, that

$$\mathbf{R}(s)=\sum_{i=1}^{m}\left[\frac{q_i(s)}{1+q_i(s)}\right]\mathbf{w}_i(s)\mathbf{v}_i^t(s). \tag{4-7}$$

Thus, for the case of unity feedback, the characteristic transfer functions of the open and closed-loop systems are respectively $q_i(s)$ and $[q_i(s)/1+q_i(s)]$ for $i=1, 2, \ldots, m$. In addition, the characteristic vectors for both the open and closed-loop systems are the same, namely $\mathbf{w}_i(s)$, $i=1, 2, \ldots, m$. This interesting generalisation of the corresponding basic result for scalar feedback systems plays a vital role in the performance analysis which follows.

4.1. *Stability*

The fundamental multivariable stability theorem from which we will develop an encirclement criterion defining closed-loop stability is originally due to Popov [10] and is

$$\det \mathbf{F}(s)=\frac{CLCP}{OLCP} \tag{4-8}$$

where $\mathbf{F}(s)$ is the system return-difference matrix, which from relationship (2-10) can represent any of the corresponding quantities introduced in Section 2, and $CLCP$ and $OLCP$ are respectively the system open-loop and closed-loop characteristic polynomials which include all zeros associated with unobservable and uncontrollable modes. Since such modes are unaffected by feedback, the factors of $OLCP$ and $CLCP$ associated with such modes will cancel when det $F(s)$ is formed.

Let the characteristic transfer functions of the return-ratio matrix $\mathbf{T}(s)$ be $t_i(s)$ (note that these are simply $q_i(s)$ when $\mathbf{H}(s)=\mathbf{I}_m$) for $i=1, 2, \ldots, m$. Then

$$\det \mathbf{F}(s)=\det[\mathbf{I}_m+\mathbf{T}(s)] = \prod_{i=1}^{m}[1+t_i(s)]. \tag{4-9}$$

Let the $t_i(s)$ map the usual Nyquist contour into the set of m *characteristic loci* denoted by $t_i(j\omega)$, $i=1, 2, \ldots, m$. Then it may be shown [11, 12] that, if p_o is the number of right-half-plane zeros in $OLCP$, and $\sum_{i=1}^{m} n_{ti}$ is the net sum of clockwise encirclements of the critical point $(-1, 0)$ in the complex plane contributed by the characteristic loci of $\mathbf{T}(s)$, the closed-loop system is stable if and only if

$$\sum_{i=1}^{m} n_{ti}=-p_o \tag{4-10}$$

where clockwise encirclements are counted positive corresponding to a clockwise traversal of the Nyquist contour. The encirclement theorem is still valid when any *rhp* zeros of $OLCP$ are uncontrollable and/or unobervable. However, it will then be impossible in practice to attain the required number of encirclements for closed-loop stability.

The above encirclement theorem can be used to great advantage in determining system closed-loop stability boundaries. More specifically, let the return-ratio matrix for a given system be $\mathbf{T}(s)$. Now apply a gain of k to each loop so that the return-ratio matrix for the modified system is

$$\mathbf{T}_1(s)=k\mathbf{T}(s). \tag{4-11}$$

The characteristic loci corresponding to $\mathbf{T}_1(s)$ are equal to those of $\mathbf{T}(s)$ scaled by the factor k. In the complex plane then, the stability of the new system is quickly determined by applying the encirclement theorem to the original system characteristic loci with $(^{-1}/k, 0)$ as the critical point. It is then a simple matter to find the limiting gain factor k which preserves overall stability, when applied to each loop. Viewed in graphical terms the theorem then defines a stability boundary along the line $k_i=k$, $i=1, 2, \ldots, m$ in an m-dimensional gain space. This argument is easily extended to define other lines in the gain space such as the line corresponding to $k_i=\alpha_i k$, $i=1, 2, \ldots, m$ where the α_i are constants and k_i is the gain in the ith loop. This is done by simply post-multiplying $\mathbf{T}(s)$ corresponding to $k=\alpha_i=1$ by the matrix diag $\{\alpha_i\}$ and applying the encirclement theorem in the usual way to the resulting characteristic loci. Here the critical point is again $(^{-1}/k, 0)$. In theory, the stable operating region for all possible combinations of loops gains can be determined, a facility which can prove extremely useful in studies of the control of nonminimum phase systems.

4.2. *Integrity*

A multivariable feedback system is said to be of satisfactory integrity if it remains stable under all combinations of a stipulated set of failure conditions. The set of primary concern includes output transducer, error-monitoring channel and actuator failures. Clearly, any design technique aimed at devising feedback controllers for practical systems must incorporate a check for stability when such component breakdowns occur. Consequently, the following results have been established [11].

To ensure integrity against failure of the output transducer in loop j say, the characteristic loci of the principal sub-matrix of the return-ratio matrix $\mathbf{T}_y(s)$, obtained by deleting row j and column j, must satisfy a Nyquist-type stability criterion, as defined by equation (4-10). For the case of simultaneous transducer failure in two or more loops, the above result is applied with respect to the principal sub-matrix of $\mathbf{T}_y(s)$ obtained by deleting those rows and columns whose index numbers coincide with the failed loops. In general then, integrity against transducer failures in *all* possible combinations of loops is assured when the characteristic loci of all the principal sub-matrices of $\mathbf{T}_y(s)$ satisfy the encirclement theorem previously established.

For integrity against actuator and error-monitoring channel failures, similar considerations to those above apply to $\mathbf{T}_u(s)$ and $\mathbf{T}_e(s)$ respectively. For the special case when $\mathbf{H}(s)=\mathbf{I}_m$, $\mathbf{T}_e(s)=\mathbf{T}_y(s)$ so that integrity against transducer failures automatically insures integrity in the face of error-monitoring channel failures. Note that it is virtually impossible in practice to achieve integrity under all combinations of the above failure conditions [11]. However, the results do provide guidance for improving the integrity situation and so play a vital role in the design procedure of Section 5.

4.3. *Interaction*

The general term interaction is used here to denote the body of relationships influencing the way in which a reference input $r_i(s)$, applied to input i, affects the set of outputs $\{y_j(s): j\neq i\}$. In general, the designer will aim at ensuring that only one specific output $y_i(s)$ reponds to $r_i(s)$ and all the other outputs $y_j(s), j\neq i$, remain sufficiently small to satisfy some performance criterion imposed in the design specification.

Consider the characteristic dyadic expansion of the closed-loop operator $\mathbf{R}(s)$ given by equation (4-7) and evaluated for $s=j\omega$. This is

$$\mathbf{R}(j\omega)=\sum_{i=1}^{m}\left[\frac{q_i(j\omega)}{1+q_i(j\omega)}\right]\mathbf{w}_i(j\omega)\mathbf{v}_i^t(j\omega). \quad (4\text{-}12)$$

Now suppose that, at some frequency ω_l

$$|q_i(j\omega_l)|\gg 1,\ i=1, 2, \ldots, m. \quad (4\text{-}13)$$

Then

$$\mathbf{R}(j\omega_l)\rightarrow\sum_{i=1}^{m}\mathbf{w}_i(j\omega_l)\mathbf{v}_i^t(j\omega_l)=\mathbf{I}_m \quad (4\text{-}14)$$

and the closed-loop system is clearly noninteracting at this frequency. Condition (4-13) can usually be met at low frequencies by ensuring that high characteristic gains are imposed. Thus, at low frequencies, interaction can be suppressed to any required amount by ensuring that the moduli of all the characteristic loci are sufficiently large.

At high frequencies however, condition (4-13) cannot be met because of stability restrictions which in fact invariably require that.

$$|q_i(j\omega_h)|\ll 1,\ i=1, 2, \ldots, m \quad (4\text{-}15)$$

at the *high* angular frequency ω_h. Then from equation (4-12)

$$\mathbf{R}(j\omega_h)\rightarrow\sum_{i=1}^{m}q_i(j\omega_h)\mathbf{w}_i(j\omega_h)\mathbf{v}_i^t(j\omega_h)=\mathbf{Q}(j\omega_h) \quad (4\text{-}16)$$

which means that any high-frequency cross-couplings in $\mathbf{Q}(j\omega)$ will pass straight through to $\mathbf{R}(j\omega)$ *despite the action of the feedback*. It follows from this that one available method of suppressing high frequency interaction is to ensure that $\mathbf{Q}(s)$ approaches diagonal form as $|s|\rightarrow\infty$. Another more effective method arises from geometrical considerations in that one can attempt to *align* the characteristic direction set of $\mathbf{Q}(j\omega)$ with the standard basis set To. show how this approach works, apply a reference input $r_j(j\omega)$ to input j so that

$$\mathbf{r}(j\omega)=r_j(j\omega)\mathbf{e}_j \quad (4\text{-}17)$$

where $\mathbf{e}_j$ is a standard basis vector, the jth column of a unit matrix of appropriate order. If it were the case that $\mathbf{w}_j(j\omega)=\mathbf{e}_j$ so that the system output vector

$$\begin{aligned}\mathbf{y}(j\omega)&=\mathbf{R}(j\omega)\mathbf{r}(j\omega)\\&=r_j(j\omega)\mathbf{R}(j\omega)\mathbf{w}_j(j\omega)\\&=r_j(j\omega)\left[\frac{q_j(j\omega)}{1+q_j(j\omega)}\right]\mathbf{e}_j\end{aligned} \quad (4\text{-}18)$$

then only element $y_j(j\omega)$ would respond to input $r_j(j\omega)$. Thus a convenient measure of alignment is the *angle*, as a function of frequency, between the vectors $\mathbf{w}_j(j\omega)$ and the vectors $\mathbf{e}_j$ for $j=1, 2, \ldots, m$. This is given by equation (3-5) for $j=1, 2, \ldots, m$ as

$$\cos\theta_j(j\omega)=\frac{|(\mathbf{w}_j(j\omega), \mathbf{e}_j)|}{\|\mathbf{w}_j(j\omega)\|} \quad (4\text{-}19)$$

where $\mathbf{w}_j(j\omega)$ is that characteristic direction which produces the minimum $\theta_j(j\omega)$ at frequency ω. Thus if $\theta_j(j\omega)$ is sufficiently small at high frequencies, interaction effects arising from the jth input will be correspondingly small. It is important to note however that this requirement is equivalent to making $\mathbf{Q}(j\omega)$ diagonal *only when* all the θ_j are in *exact* alignment with standard basis vectors. This means therefore that small misalignment angles need *not* necessarily arise from a $\mathbf{Q}(j\omega)$ which is nearly diagonal or diagonal dominant. An excellent example of this is the system

$$\mathbf{Q}(s)=\frac{1}{0{\cdot}99(s+1)}\begin{bmatrix}9 & 9\\ -9 & 99{\cdot}9\end{bmatrix}$$

which produces $\theta_1(j\omega)=\theta_2(j\omega)=5{\cdot}7$ deg at all frequencies. From the above criterion the feedback system is essentially noninteracting although $\mathbf{Q}(s)$ is not diagonal dominant. Further discussion of this is given in [9].

In summary, an assessment of interaction over the frequency range of system operation can be made on the basis of an inspection of both the characteristic loci and the characteristic directions. The simplest way to do this is via logarithmic plots where $|q_j(j\omega)|$ vs ω and $\theta_j(j\omega)$ vs ω are plotted for all j.

4.4. *Accuracy*

In a general sense, accuracy can be loosely defined as the degree to which actual system outputs follow desired system outputs. That is we wish to have

$$\mathbf{y}(s)\simeq\mathbf{r}(s),\ s=j\omega. \quad (4\text{-}20)$$

As shown in the discussion on interaction, condition (4-20) can be satisfied at low frequencies providing

$$|q_i(j\omega)|\gg 1,\quad i=1, 2, \ldots, m. \quad (4\text{-}21)$$

The system accuracy will be high providing the moduli of the characteristic loci are suitably large at low frequencies.

As will now be shown, it is possible to establish upper and lower bounds on closed-loop accuracy at any frequency. For the system in Fig. 1,

$$\mathbf{e}(j\omega)=\mathbf{F}^{-1}(j\omega)\mathbf{r}(j\omega). \quad (4\text{-}22)$$

Thus we have

$$\frac{\|\mathbf{e}(j\omega)\|^2}{\|\mathbf{r}(j\omega)\|^2}=\frac{\mathbf{e}^t(-j\omega)\hat{\mathbf{F}}^t(-j\omega)\hat{\mathbf{F}}(j\omega)\mathbf{e}(j\omega)}{\mathbf{e}^t(-j\omega)\mathbf{e}(j\omega)} \quad (4\text{-}23)$$

where $\hat{\mathbf{F}}(j\omega)=\mathbf{F}^{-1}(j\omega)$. This last expression may obviously be used as a measure of accuracy for a multivariable feedback system. Since the matrix $\hat{\mathbf{F}}^t(-j\omega)\hat{\mathbf{F}}(j\omega)$ is positive definite Hermitian for each value of ω, its eigenvalues will be real and positive. Denoting these eigenvalues by

$$\{\mu_i^2(j\omega):\ i=1, 2, \ldots, m\}$$

and invoking the standard Courant–Fisher min–max relationships for Hermitian matrices [13] gives

$$\min_i \mu_i^2(j\omega)\leqslant\frac{\mathbf{e}^t(-j\omega)\hat{\mathbf{F}}^t(-j\omega)\hat{\mathbf{F}}(j\omega)\mathbf{e}(j\omega)}{\mathbf{e}^t(-j\omega)\mathbf{e}(j\omega)}\leqslant\max_i \mu_i^2(j\omega). \quad (4\text{-}24)$$

Combining expressions (4-24) and (4-23)

$$\min_i \mu_i(j\omega)\leqslant\frac{\|\mathbf{e}(j\omega)\|}{\|\mathbf{r}(j\omega)\|}\leqslant\max_i \mu_i(j\omega) \quad (4\text{-}25)$$

where the quantities $\mu_i(j\omega)$ are the positive square roots of the eigenvalues of $\hat{\mathbf{F}}^t(-j\omega)\hat{\mathbf{F}}(j\omega)$ and are usually called the singular values of $\mathbf{F}(j\omega)$ [14].

Taking the overall percentage error of the closed-loop system to be $\dfrac{\|\mathbf{e}(j\omega)\|}{\|\mathbf{r}(j\omega)\|}\times 100$ per cent gives the minimum and maximum possible errors to be $\min \mu_i(j\omega)\times 100$ per cent and $\max \mu_i(j\omega)\times 100$ per cent at any frequency ω.

Using Browne's theorem [14] and noting that the characteristic loci of $\hat{\mathbf{F}}(j\omega)$ for $\mathbf{H}(s)=\mathbf{I}_m$ are

$\{1/[1+q_i(j\omega)]: i=1, 2, \ldots, m\}$ gives

$$\min_i \mu_i(j\omega) \leqslant \left[\frac{1}{1+q_i(j\omega)}\right] \leqslant \max_i \mu_i(j\omega) \quad (4\text{-}26)$$

so that the moduli of the characteristic loci of $\hat{\mathbf{F}}(j\omega)$ are always bounded by the singular values of $\hat{\mathbf{F}}(j\omega)$. The last expression again verifies that if all the $|q_i(j\omega)| \gg 1$ high accuracy will ordinarily be obtained.

5. CHARACTERISTIC LOCUS METHOD

In the last section it was shown how the four conflicting properties of stability, integrity, interaction and accuracy could be analysed using appropriate sets of system characteristic loci and characteristic directions. This, coupled with the desire to extend the Bode–Nyquist approach to the multivariable case, leads to the following design philosophy—the designer of multivariable controllers should strive to attain a required closed-loop stability and performance specification by appropriate manipulations of sets of open-loop characteristic loci and characteristic directions. In order to simplify the following discussion, unity feedback (i.e. $\mathbf{H}(s)=\mathbf{I}_m$) is assumed; then the transition from open-loop to closed-loop characteristic loci and characteristic directions is straightforward as described in Section 4. Now the entire design effort may be focused on the synthesis of the forward path controller matrix, $\mathbf{K}(s)$, which along with the plant transfer matrix $\mathbf{G}(s)$, is assumed to be square. It should be noted that non-square systems are currently under investigation and it is hoped that they will be discussed in a future paper.

The main forms of manipulation which must be performed via $\mathbf{K}(s)$ includes:

(i) Modifying the phases of appropriate sets of characteristic loci in order to achieve acceptable stability and integrity;
(ii) aligning the characteristic directions at high frequencies and balancing the gains the characteristic of loci at low frequencies in order to get acceptable interaction;
(iii) injecting gain into the phase-compensated and aligned system in order to achieve a satisfactory overall performance.

It is thus obvious that the final system controller, $\mathbf{K}(s)$ has many criteria to satisfy simultaneously. This strongly suggests that $\mathbf{K}(s)$ be designed as the cascaded combination of several sub-controllers, $\mathbf{K}_i(s)$, so that

$$\mathbf{K}(s) = \prod_{i=1}^{p} \mathbf{K}_i(s) \qquad (5\text{-}1)$$

where each $\mathbf{K}_i(s)$ has a *specific* task to handle during the sequential synthesis. The number of sub-controllers needed to produce $\mathbf{K}(s)$, denoted above by p, will of course be different for different problems.

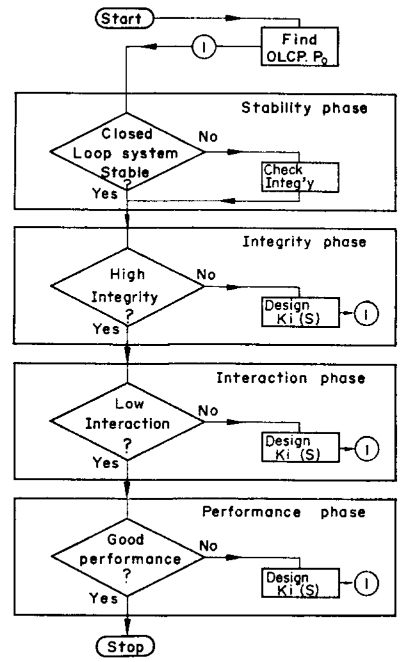

FIG. 2. The design strategy.

Before looking in detail at some of the more useful forms of controller factors, $\mathbf{K}_i(s)$, it will be helpful at this stage to outline the design process or strategy which makes use of this proposed controller factorisation in such a way as to satisfy stability, integrity, interaction and accuracy requirements. This design procedure is summarised in the flow diagram shown in Fig. 2 to which the following comments apply:

(i) Controller design is separated into four distinct phases whose order depends on the relative importance of each system property. For instance, as shown in the figure, closed-loop stability is given the highest priority, followed by high integrity and so on.
(ii) Poor closed-loop stability margins are attributed to either poor integrity or severe interaction, or a combination of both, so that no compensating $\mathbf{K}_i(s)$ need be designed during the stability phase. This means that if there is initially instability, the implementation of controllers which achieve satisfactory integrity and interaction properties automatically ensure overall stability.
(iii) The method is an iterative one and, like any usual engineering design technique, alternates continuously between the steps of *system analysis* and *design decision* until the final specifications are met.

(iv) The final controller $\mathbf{K}(s)$ emerges as the product of the individual factors $\mathbf{K}_i(s)$, so that it is the cumulative effect of the $\mathbf{K}_i(s)$ which produces the required overall control action.

(v) It is assumed that all the calculations and graphics required to assess system properties are performed by a digital computer with graphic display facilities. This then allows the designer to analyse the effects of each $\mathbf{K}_i(s)$ by inspection and thus to concentrate entirely on controller synthesis.

The flow diagram of Fig. 2 provides a guiding framework within which any of a large number of specific types of controller factor $\mathbf{K}_i(s)$, can be used to achieve the required objectives at each phase of the design process. For practical reasons, each $\mathbf{K}_i(s)$ should be as simple as possible; in particular $\mathbf{K}_i(s)$ should be synthesised as much as is practicable from wholly constant factors. For obvious reasons, all the dynamical elements of $\mathbf{K}_i(s)$ must be rational functions in s, and det $\mathbf{K}_i(s)$ must be identically non-singular. Furthermore, all poles of $\mathbf{K}_i(s)$ should lie in the open left-half plane. Finally, det $\mathbf{K}_i(s)$ must not have any right-half plane zeros if non-minimum phase difficulties are to be avoided. Listed below are several types of controller factors which provide considerable assistance in achieving desired manipulations of the systems characteristic loci and characteristic directions.

(i) *Elementary transformation matrix controllers*

(a)

$$\mathbf{K}_i(s)=\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & \mathbf{0} & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & k_{jj}(s) & & & \\ & & & & & \cdot & & \\ & \mathbf{0} & & & & & \cdot & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \tag{5-2}$$

where the diagonal element $k_{jj}(s)$ is a rational function in s with all its poles and zeros in the open left-half plane.

(b)

$$\mathbf{K}_i(s)=\begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & & & & & | \\ | & & \cdot & & & & | \\ | & & & \cdot & & & | \\ 0 & - & - & 1 & k_{jk}(s) & \cdots & 0 \\ | & & & & \cdot & & | \\ | & & & & & \cdot & | \\ | & & & & & \cdot & | \\ | & & & & & \cdot & | \\ 0 & - & - & - & - & - & 1 \end{bmatrix} \tag{5-3}$$

where the off-diagonal entry $k_{jk}(s)$ is also a rational function in s having all its poles in the open left-half plane. The way in which these factors modify the system characteristic loci has been discussed elsewhere [11] and so will not be repeated here. It will suffice to say that they are particularly useful in improving integrity where only sub-systems are to be modified, and in reducing interaction by diminishing the magnitudes of the off-diagonal elements of $\mathbf{G}(s)$.

(ii) *Scalar matrix controller*

$$\mathbf{K}_i(s)=k(s)\mathbf{I}_m. \tag{5-4}$$

This controller has the special property of multiplying the plant characteristic loci by the scalar $k(s)$ while leaving the characteristic directions unaltered.

(iii) *Permutation matrix controller*

$$\mathbf{K}_i(s)=[\mathbf{e}_1 \ldots \mathbf{e}_{q-1}\mathbf{e}_p\mathbf{e}_{q+1} \ldots \mathbf{e}_{p-1}\mathbf{e}_q\mathbf{e}_{p+1} \ldots \mathbf{e}_m] \tag{5-5}$$

where $p>q$ and $\mathbf{e}_j$ is the jth column of $\mathbf{I}_m$. This controller has the effect of interchanging columns p and q of $\mathbf{G}(s)$, and is used in the initial design phase when, for some specific technical reason, such as to improve integrity, it is required that the inputs to $\mathbf{G}(s)$ be reordered.

(iv) *d.c. plant inverse controller matrix*

$$\mathbf{K}_i(s)=\mathbf{G}^{-1}(0). \tag{5-6}$$

This controller diagonalises the plant at d.c. and therefore has the effects of:

(a) aligning the plant characteristic directions with the standard basis vectors at d.c., thereby removing any d.c. interaction;

(b) normalising the characteristic loci by assigning a gain of one to each locus at d.c. This makes it easier to meet low frequency performance specifications for cases where some of the plant characteristic loci are large at d.c. and others relatively small.

(v) *Matrix P.I. controller*

$$\mathbf{K}_i(s)=\mathbf{K}_\gamma+\frac{\mathbf{K}_\delta}{s} \tag{5-7}$$

where $\mathbf{K}_\gamma$ and $\mathbf{K}_\delta$ are $m \times m$ non-singular matrices of constants so that $\mathbf{K}_i(s)$ is a matrix generalisation of the scalar P.I. controller. Following ROSENBROCK [15] the controller may be designed by putting

$$\mathbf{K}_\gamma=\mathbf{K}_\infty \mathbf{D}_1 \tag{5-8}$$

$$\mathbf{K}_\delta=\mathbf{K}_o \mathbf{D}_2 \tag{5-9}$$

where $\mathbf{K}_o$ and $\mathbf{K}_\infty$ respectively diagonalise the system at zero and very high frequencies. The diagonal matrices $\mathbf{D}_2$ and $\mathbf{D}_1$ are used to allow an appropriate amount of freedom in adjusting the weighting between zero and infinite frequencies in each column of $\mathbf{G}(s)\mathbf{K}_i(s)$.

An obvious choice for $\mathbf{K}_o$ is

$$\mathbf{K}_o=\mathbf{G}^{-1}(0). \tag{5-10}$$

To find $\mathbf{K}_\infty$, multiply $\mathbf{g}_i(s)$, i.e. row i of $\mathbf{G}(s)$, by s^{p_i} where the integer p_i is chosen so that as $|s|\to\infty$ no element of $s^{p_i}\mathbf{g}_i(s)$ tends to infinity and not every element tends to zero. Now define the row vector.

$$\mathbf{b}_i=\lim_{s\to\infty} s^{p_i}\mathbf{g}_i(s). \tag{5-11}$$

Repeat the above procedure for all the remaining rows of $\mathbf{G}(s)$ and then assemble the vectors $\mathbf{b}_i$, $i=1, 2, \ldots, m$ into a matrix $\mathbf{B}$. Then

$$\mathbf{K}_\infty=\mathbf{B}^{-1}. \tag{5-12}$$

Reflection on the above operations will show that the magnitudes of the off-diagonal elements in each row of $\mathbf{G}(s)\mathbf{K}_\infty$ become arbitrarily small relative to the diagonal elements as $|s|\to\infty$.

This form of matrix P.I. controller therefore performs the following set of functions:

(1) it eliminates steady-state error and low frequency interaction by virtue of the integral action since;

$$|q_i(s)| \gg 1 \text{ as } |s|\to 0;$$

(2) it reduces high frequency interaction by ensuring that $\mathbf{Q}(s)=\mathbf{G}(s)\mathbf{K}_i(s)$ approaches diagonal form as $|s|\to\infty$. It therefore tends to align the characteristic vectors with the standard basis vectors at high frequencies.

(vi) *Eigenvalue adjustment controller matrix*

Given a plant whose dyadic form at *one specific frequency*, ω_1, is

$$\mathbf{G}=\sum_{i=1}^{m} g_i \mathbf{w}_i \mathbf{v}_i^t \tag{5-13}$$

then the controller

$$\mathbf{K}=\sum_{i=1}^{m} \frac{q_i}{g_i} \mathbf{w}_i \mathbf{v}_i^t \tag{5-14}$$

is a matrix of complex numbers such that the product $\mathbf{GK}$ has the same eigenvectors as $\mathbf{G}$ but has a new desired set of eigenvalues $\{g_k\}$. Now, determine a $\mathbf{K}_i(s)$ such that

$$\mathbf{K}_i(j\omega_1)=\mathbf{K}. \tag{5-15}$$

Thus a $\mathbf{K}_i(s)$ designed in this way will have the property of shifting the characteristic loci of $\mathbf{G}(s)$ to any desired location at one specific frequency without changing the characteristic vectors. The elements of this controller can usually be easily realised by phase advance or phase retard networks. In this case the controller is somewhat complicated.

(vii) *Eigenvector adjustment controller matrix*

The idea behind the synthesis of this form of controller factor is similar to the previous case except that

$$\mathbf{K}=\sum_{j=1}^{m} g_j(\mathbf{G}^{-1}\mathbf{l}_j)\mathbf{t}_j^t \tag{5-16}$$

where $\{\mathbf{l}_j\}$ is the set of desired characteristic directions at specific frequency ω_1 and $\{\mathbf{t}_i\}$ is the set of reciprocal eigenvectors. Again, finding a $\mathbf{K}_i(s)$ so that $\mathbf{K}_i(j\omega_1)=\mathbf{K}$ has the effect of changing the characteristic directions of $\mathbf{G}(s)$ at a specific frequency to any desired location without affecting the characteristic loci. The same rules apply for the realisation of this controller as for $\mathbf{K}_i(s)$ in (vi).

The last two controllers would in general be used at high frequencies to improve closed-loop stability margins and reduce interaction respectively.

6. DESIGN EXAMPLE

The system considered here as an illustrative example of the method is a pressurised flow-box, an important part of most modern paper-making machines. A state-space model for the system obtained by linearisation about the steady-state is given by [9]

$$\frac{d}{dt}\begin{bmatrix} H(t) \\ h(t) \end{bmatrix} = \begin{bmatrix} -0{\cdot}395 & 0{\cdot}01145 \\ -0{\cdot}011 & 0 \end{bmatrix}\begin{bmatrix} H(t) \\ h(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}03362 & 1{\cdot}038 \\ 0{\cdot}000966 & 0 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \qquad (6\text{-}1)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} H(t) \\ h(t) \end{bmatrix} \qquad (6\text{-}2)$$

where the system outputs are flow-box liquid level, $h(t)$, and total head of stock, $H(t)$. The inputs $u_1(t)$ and $u_2(t)$ are respectively stock inflow and air inflow, both of which enter the flow-box through valves whose dynamics are not included in the above model.

The open-loop characteristic polynomial is given by

$$OLCP = \det(s\mathbf{I}-\mathbf{A}) = (s+0{\cdot}3949)(s+0{\cdot}32\times10^{-3}) \qquad (6\text{-}3)$$

so that $p_o=0$ and closed-loop stability follows if and only if

$$\sum_{i=1}^{2} n_{ti}=0. \qquad (6\text{-}4)$$

The transfer function matrix is given by

$$\mathbf{G}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} \dfrac{0{\cdot}03362}{s+0{\cdot}3949} & \dfrac{1{\cdot}03s}{\alpha(s)} \\ \dfrac{9{\cdot}66\times10^{-4}s+1{\cdot}17\times10^{-5}}{\alpha(s)} & \dfrac{-0{\cdot}01141}{\alpha(s)} \end{bmatrix} \qquad (6\text{-}5)$$

where

$$\alpha(s)=s^2+0{\cdot}395s+1{\cdot}26\times10^{-4}. \qquad (6\text{-}6)$$

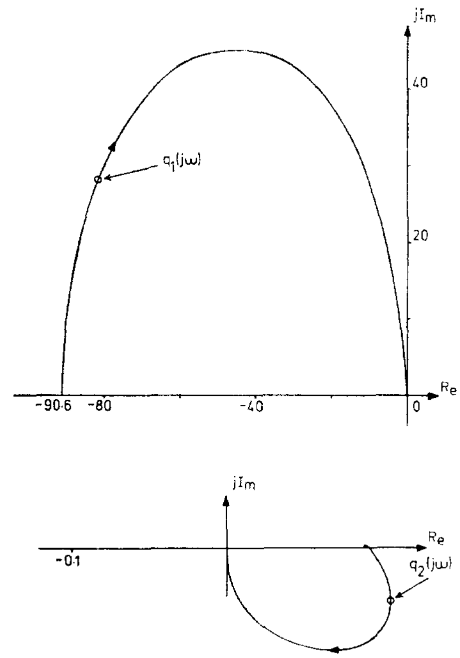

Fig. 3. Characteristic loci of $G(s)$.

Entering the design cycle of Fig. 2 the characteristic loci of $\mathbf{G}(s)$, shown in Fig. 3, reveal that for equal gains in each loop (i.e. $k_1=k_2=k$), the stability condition (6-4) is satisfied providing

$$k<1/90{\cdot}6=0{\cdot}0111. \qquad (6\text{-}7)$$

Clearly, these stability margins are extremely small and an attempt to improve the situation leads us to enter the integrity phase where stability in the face of loop failures, i.e. transducer failures, is analysed. Using the results of Section 4, system stability when loop 1 fails, i.e. $k_1=0$, is guaranteed if element 22 of $\mathbf{G}(s)$ satisfies the Nyquist stability criterion. Thus, using any of the classical methods, it is easily shown that the system with loop 1 open and loop 2 closed is stable for the loop gain values

$$k_1=0,\ k_2<0{\cdot}0111. \qquad (6\text{-}8)$$

Applying the same argument to failure in loop 2 produces the stability condition

$$k_2=0,\ k_1\geqslant0. \qquad (6\text{-}9)$$

The above results show that lack of suitable stability margins is primarily due to poor integrity when loop 1 is open as indicated in Fig. 4. To remedy this situation, a type (a) elementary transformation matrix controller factor

$$\mathbf{K}_1=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (6\text{-}10)$$

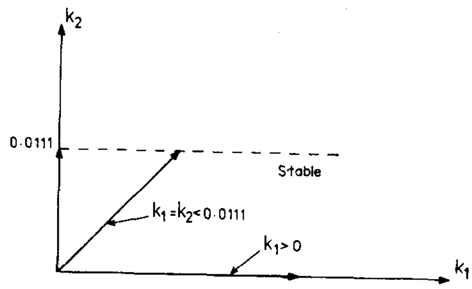

FIG. 4. Stable operating region.

is used to change the sign of the feedback round $g_{22}(s)$ without affecting $g_{11}(s)$. This now produces the stability margins

$$k_1=0,\ k_2\geqslant 0 \tag{6-11}$$

and

$$k_2=0,\ k_1\geqslant 0 \tag{6-12}$$

and defines

$$\mathbf{Q}_1(s)=\mathbf{G}(s)\mathbf{K}_1$$

$$=\begin{bmatrix} \dfrac{0{\cdot}03362}{s+0{\cdot}3949} & \dfrac{-1{\cdot}03s}{\alpha(s)} \\ \dfrac{9{\cdot}66\times10^{-4}s+1{\cdot}17\times10^{-5}}{\alpha(s)} & \dfrac{0{\cdot}01141}{\alpha(s)} \end{bmatrix}. \tag{6-13}$$

Returning to the stability phase to see if $\mathbf{K}_1$ has improved overall stability with all loops closed gives

$$\sum_{i=1}^{2} n_{ti}=0 \text{ for } k_1=k_2=k>0 \tag{6-14}$$

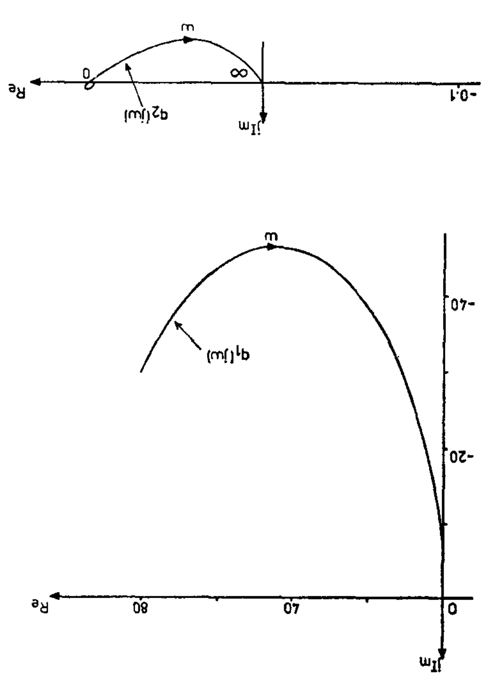

FIG. 5. Characteristic loci of $Q_1(s)$.

upon inspection of the characteristic loci of $\mathbf{Q}_1(s)$ shown in Fig. 5. From these results, it can safely be assumed that stability will be maintained for all combinations of loop gain between zero and arbitrarily high values.

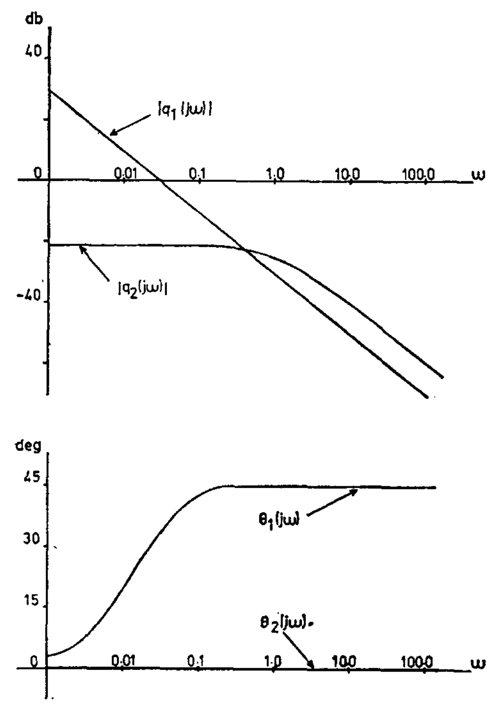

FIG. 6. Interaction analysis for $Q_1(s)$.

Having successfully dealt with the stability and integrity phases of the design process, attention is now turned to interaction. Figure 6 shows plots of the angular misalignment of the characteristic directions of $\mathbf{Q}_1(s)$ vs frequency. From an inspection of these plots, it is concluded that interaction problems are present at both low and high frequencies. Furthermore, general performance characteristics at low frequencies are not expected to be good since $|q_2(j\omega)|\ll 1$. Thus a reasonable indication for choice of the next controller factor $\mathbf{K}_2(s)$ is the matrix P.I. controller $\mathbf{K}_2(s)=\mathbf{K}_\infty\mathbf{D}_1+\mathbf{K}_0\mathbf{D}_2/s$. Selecting

$$\mathbf{D}_1=\begin{bmatrix} 0{\cdot}1 & 0 \\ 0 & 1\times10^{-4} \end{bmatrix},\ \mathbf{D}_2=\mathbf{I}_2 \tag{6-15}$$

gives

$$\mathbf{K}_2(s)=\begin{bmatrix} \dfrac{11{\cdot}72}{s} & 10{\cdot}3 \\ -9{\cdot}66-\dfrac{0{\cdot}012}{s} & 0{\cdot}3362+\dfrac{0{\cdot}011}{s} \end{bmatrix} \tag{6-16}$$

where $\mathbf{K}_\infty$ and $\mathbf{K}_0$ are determined according to the procedure laid down in the previous section. This gives

$$\mathbf{Q}_2(s)=\mathbf{Q}_1(s)\mathbf{K}_2(s). \tag{6-17}$$

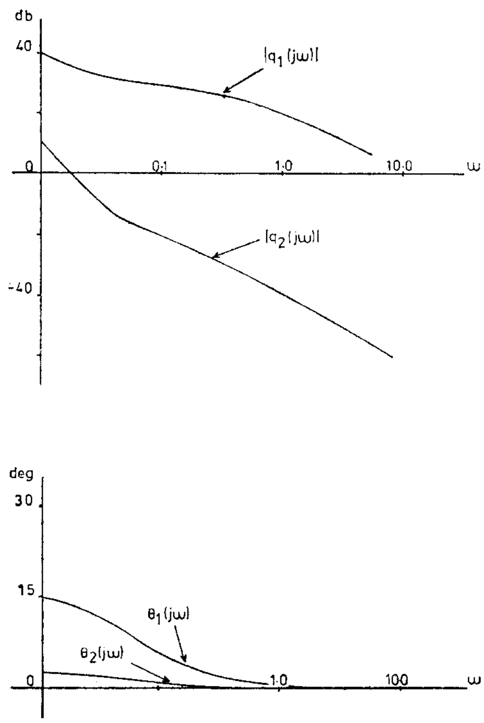

Fig. 7. Interaction analysis for $Q_2(s)$.

Repeating the interaction analysis with $\mathbf{Q}_2(s)$ produces the plots in Fig. 7 where it is seen that interaction is now negligible. Checking the characteristic loci of $\mathbf{Q}_2(s)$, shown in Fig. 8, reveals that system stability is not impaired by the addition of $\mathbf{K}_2(s)$. The same comment is valid for integrity. All that now remains is to adjust the overall closed-loop performance by tuning the design values of loop gain. This is done on the basis of the diagonal elements of $\mathbf{Q}_2(s)$ and produces

$$\mathbf{K}_3=\begin{bmatrix}10 & 0\\ 0 & 100\end{bmatrix} \tag{6-18}$$

so that the overall controller becomes

$$\mathbf{K}(s)=\mathbf{K}_1\mathbf{K}_2(s)\mathbf{K}_3$$

$$=\begin{bmatrix}\dfrac{117\cdot 2}{s} & 1030\\ 96\cdot 6+\dfrac{0\cdot 12}{s} & -33\cdot 62-\dfrac{1\cdot 1}{s}\end{bmatrix}. \tag{6-19}$$

The transient responses for unit step changes in total head $H(t)$ and level $h(t)$ are shown in Fig. 9 where it is seen that the closed-loop responses are fast and that interaction is negligible.

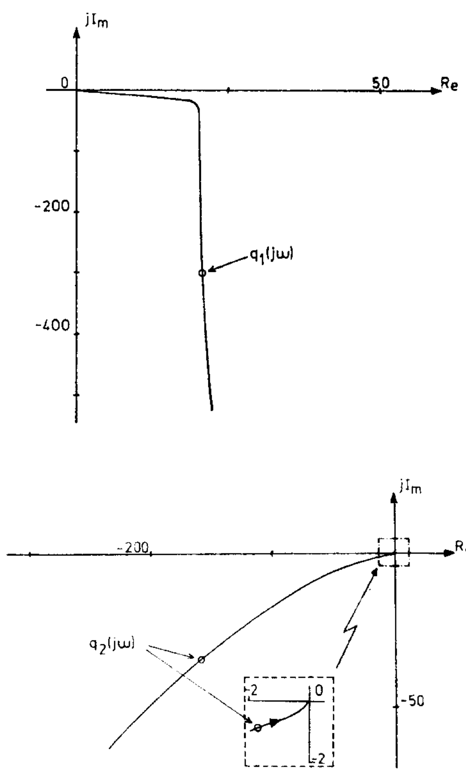

Fig. 8. Characteristic loci of $Q_2(s)$.

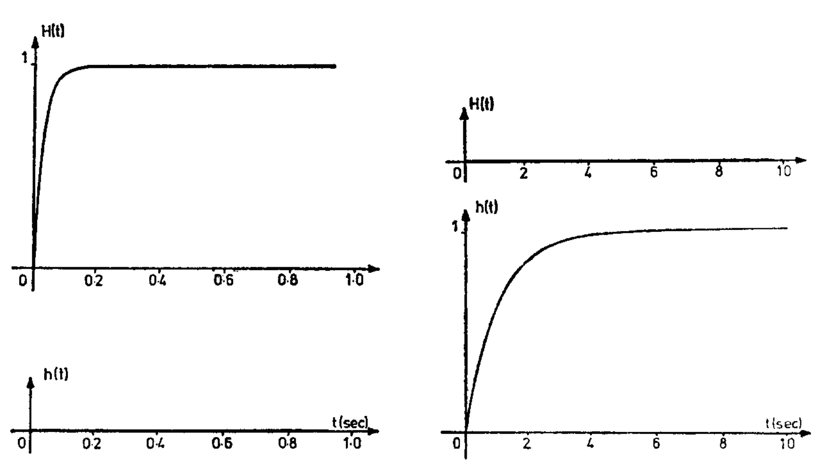

Fig. 9. Transient responses for compensated system.

Before closing the discussion of the pressurised-flow-box, the effects of the control valve dynamics, previously ignored, may be considered, the purpose being to determine if the controller $\mathbf{K}(s)$ which has just been devised can, with slight modifications, adequately control the more accurate system model

$$\mathbf{G}'(s)=\mathbf{G}(s)\mathbf{G}_v(s) \qquad (6\text{-}20)$$

where the extra plant factor

$$\mathbf{G}_v(s)=\begin{bmatrix} \dfrac{1}{1+5s} & 0 \\ 0 & \dfrac{1}{1+2s} \end{bmatrix} \qquad (6\text{-}21)$$

accounts for the valve dynamics. By slight modifications is meant the addition of single-loop controllers chosen to compensate the diagonal elements of $\mathbf{G}'(s)\mathbf{K}(s)$. Such an investigation is of great interest since it will show if the previously designed controller $\mathbf{K}(s)$ is a robust controller capable of handling large changes in system dynamics. It is easy to see that the inclusion of valve dynamics can seriously affect the behaviour of the system, since both valve time constants are of the same order of magnitude as the fastest time constants associated with the poles of $\mathbf{G}(s)$.

Following the above procedure leads to the incorporation of the phase-advance controller factor

$$\mathbf{K}_v(s)=\begin{bmatrix} \dfrac{1+0{\cdot}2s}{1+0{\cdot}02s} & 0 \\ 0 & \dfrac{0{\cdot}5(1+2s)}{(1+0{\cdot}2s)} \end{bmatrix} \qquad (6\text{-}22)$$

so that the overall controller now becomes

$$\mathbf{K}'(s)=\mathbf{K}(s)\mathbf{K}_v(s). \qquad (6\text{-}23)$$

The transient responses for unit step changes in total head and level are shown in Fig. 10 and are seen to be quite acceptable. If better responses were desired then the full design procedure could be carried through for the complete model in which valve dynamics were included right from the start.

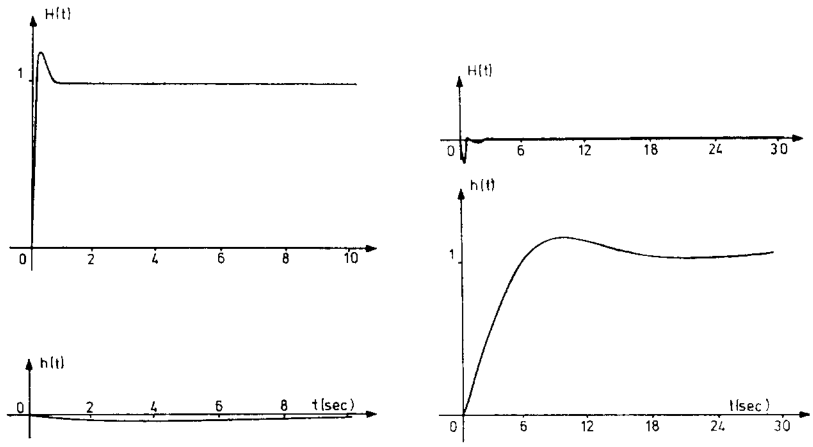

FIG. 10. Transient responses for compensated system. Including valve dynamics.

7. CONCLUSIONS

The work outlined above indicates that it should be possible to develop, for multivariable feedback systems, approaches which are natural generalisations of the classical Bode–Nyquist methods for the frequency–response analysis and design of scalar systems. Since the general multivariable problem is very complex, a fully-developed design technique capable of handling problems of arbitrary dimension in a completely systematic way will require a great deal of further development. Nevertheless, it is felt that the procedure described represents an encouraging step in the right direction. As it stands, it should give a flexible and useful design tool which may prove helpful for a range of practical industrial problems where plant data is available in an experimentally measured form from which the required frequency responses may be calculated. The technique described is not inherently limited by the dimensions of the system transfer function matrix. It is currently being investigated on systems of higher dimension than that of the simple illustrative example, and it is hoped to report on this work in a future paper. The chief problem to be overcome in an extension to much higher, say greater than 4, dimensions will lie in the need to devise algorithms which allow a computer to generate suitable controller factors.

The work described differs from state-space approaches in that its prime motivation is to extend the Bode–Nyquist frequency–response approach. Nevertheless, much of the mathematical apparatus is the same, using linear vector space theory which appears to be the natural tool for all multivariable problems. It differs from the frequency–response approach of Rosenbrock [15] in using loci which are direct generalisations of Nyquist loci, rather than approximating bands within which such loci lie. It differs from the approach of Mayne [19] in its strong emphasis on and exploitation of the geometrical structure of the linear vector space operators involved in the analysis. The links with state-space approaches, and a fuller discussion of other frequency–response methods have been given in [12]. The multivariable feedback problem is too complex for any one approach to serve all purposes. Hopefully however it has been shown that the combination of complex variable theory with the methodology of linear vector spaces has a great deal to offer in this area.

REFERENCES

[1] H. W. Bode: *Network Analysis and Feedback Amplifier Design.* Van Nostrand, New York (1945).
[2] H. Nyquist: Regeneration theory. *Bell. Syst. Tech. J.* **11,** 126–147 (1932).
[3] B. D. O. Anderson and J. B. Moore: *Linear Optimal Control.* Prentice Hall, Englewood Cliffs, N.J. (1971).
[4] R. S. Bucy and P. D. Joseph: *Filtering for Stochastic Processes with Applications to Guidance.* Interscience, New York (1968).
[5] I. M. Horowitz: *Synthesis of Feedback Systems.* Academic Press, New York (1963).
[6] A. G. J. MacFarlane: Return-difference and return-ratio matrices and their use in analysis and design of multivariable feedback control systems. *Proc. IEE* **117,** 2037–2049 (1970).
[7] F. R. Gantmacher: *Theory of Matrices,* Vol. 1. Chelsea, New York (1959).
[8] G. Fano: *Mathematical Methods of Quantum Mechanics.* McGraw-Hill, New York (1971).
[9] J. J. Belletrutti: Theory and Design of Multivariable Feedback Systems. Ph.D. Thesis, U.M.I.S.T. (1971).
[10] V. M. Popov: *Hyperstability and Optimality of Automatic Systems with Several Control Functions.* Revue Roumaine de Sciences Techniquies—Electrotechnics and Energetics, Vol. 9, Bucharest (1964).
[11] J. J. Belletrutti and A. G. J. MacFarlane: Characteristic loci techniques in multivariable control system design. *Proc. IEE* **118,** 1291–1297 (1971).
[12] A. G. J. MacFarlane: A survey of some recent results in linear multivariable feedback theory. *Automatica* **8,** 455–492 (1972).
[13] E. Beckenbach and R. Bellman: *Inequalities.* Springer, Berlin (1961).
[14] M. Marcus and H. Minc: *A Survey of Matrix Theory and Matrix Inequalities.* Allyn & Bacon, Boston (1964).
[15] H. H. Rosenbrock: Progress in the design of multivariable control systems. *Trans. Inst. Measure. Control* **4,** 9–11 (1971).
[16] H. Weyl: *The Concept of a Riemann Surface,* 3rd Edition. Addison-Wesley, Reading, Mass. (1955).
[17] G. Springer: *Introduction to Riemann Surfaces.* Addison-Wesley, Reading, Mass. (1957).
[18] D. C. Youla: On the factorisation of rational matrices. *IRE Trans. Inform. Theory* **IT-7,** 172–189 (1961).
[19] D. Q. Mayne: The Design of Linear Multivariable Systems. Preprints of the 5th IFAC Congress, Paris, Part 4a, Paper No. 29.1 (1972). Also *Automatica* **9,** 201–207 (1973).

Résumé—L'ouvrage classique de Nyquist et Bode sur l'analyse de réponse de fréquence de systèmes rétroactifs scalaires mène à des procédures de design souples et utiles car il permet de manipuler simultanément les demandes opposées de stabilité et de précision par une représentation de système de forme unique—la fonction de réponse de fréquence à boucle ouverte d'une variable complexe. Des méthodes d'espace d'état dérivent leur élégance et leur puissance de l'exploitation systématique des propriétés algébriques et géométriques d'espaces de vecteurs linéaires. L'idée fondamentale de la Méthode Caractéristique de Lieu développée ici est la combinaison des idées principales de ces deux abordages en exploitant les propriétés d'espaces de vecteurs linéaires définis sur des domaines de base de fonctions d'une variable complexe. Ce qui émerge est une théorie générale de rétroaction de vecteur dans laquelle le technique classique Bode-Nyquist est un cas spécial, et de laquelle une technique de design fondée sur la réponse de fréquence est développée et connue sous le nom de Méthode Caractéristique de Lieu.

Zusammenfassung—Das klassische Werk von Nyquist und Bode über die Frequenzganganalysis von skalaren Rückkopp lungssystemen führt zu flexiblen und nützlichen Entwurfssprozeduren, weil es ermöglicht, die einander widersprechenden Erfordernisse von Stabilität und Genauigkeit über eine einzige Form der Systemdarstellung - nämlich die Frequenzfunktion einer komplexen Variablen des aufgeschnittenen Kreises - zu behandeln. Zustandsraum-Methoden leiten ihre Eleganz und Kraft aus der systematischen Erforschung der algebraischen und geometrischen Eigenschaften linearer Vektorräume her. Die Grundidee der hier entwickelten Methode des charakteristischen Ortes liegt in der Kombination dieser beiden Approximationen durch Ausnutzung der Eigenschaften linearer Vektorräume, die über Grundfelder von Funktionen einer komplexen Variablen definiert sind. Was sich dann ergibt, ist eine allgemeine Vektor-Rückkopplungstheorie, von der die klassische Bode–Nyquist-Technik ein Spezialfall ist und aus der eine auf dem Frequenzgang basierende Technik, die sogenannte Methode des charakteristischen Ortes, entwickelt wurde.

Резюме—Из классической работы Найквиста и Бодэ, посвященной анализу скалярной системы с обратной связью вытекает гибкая и полезная рассчетная процедура. Она получается в результате того, что конфликтные требования устойчивости и точности можно одновременно учитывать с помощью единой формы представления - частотной функции комплексного переменного для разомкнутой цепи.

Методы фазовых пространств получаются мощными и изящными в результате систематического использования алгебраических и геометрических свойств линейных векторных пространств. Основная идея развитого здесь метода фазовой траектории заключается в объединении существа этих двух подходов путем использования свойств линейных векторных прос транств, определяемых на основ ном множестве функций компле ксного переменного. В результате получается общая теория векторной обратной связи в которой классический метод Бодэ-Найквиста является особым случаем и из которой выводится метод названный методом характеристической фазовой траектории.