Return-difference and return-ratio matrices and their use in analysis and design of multivariable feedback control systems

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Abstract

Bode's concepts of return difference and return ratio are shown to play a fundamental role in the analysis of multivariable feedback control systems. Matrix transfer functions are regarded as operators on linear vector spaces over the field of rational functions in the complex variable s. The eigenvalues of such operators are identified as characteristic transfer functions. The corresponding characteristic frequency responses provide a simple and natural link between classical single-loop design techniques and multivariable-system feedback theory. These concepts then serve as a unifying thread in a coherent and systematic discussion of multivariable-feedback-system design techniques.

List of principal symbols

s, z = complex variables
$F(s)$ = return-difference matrix
$\det F(s)$ = return-difference determinant
$T(s)$ = return-ratio matrix
$Q(s)$ = matrix whose elements are rational functions in s, see also below
$\hat{Q}(s)$ = inverse of matrix $Q(s)$
$\det Q(s)$ = determinant of matrix $Q(s)$
$|\det Q(s)|$ = modulus of complex number $\det Q(s)$
$G(s)$ = matrix of plant transfer functions
$K(s)$ = matrix of controller transfer functions
$Q(s) = G(s)K(s)$
$R(s)$ = closed-loop transfer-function matrix
$\hat{G}(s), \hat{K}(s), \hat{R}(s)$ = inverses of $G(s), K(s), R(s)$
u = plant input
y = plant output
$\rho(s), \mu(s), \nu(s), \gamma(s)$ = eigenvalues of matrices over the field of rational functions in s; characteristic transfer functions
$\langle u, Ru\rangle, \langle y, Qy\rangle$ = quadratic forms in input and output
$B\begin{pmatrix} i_1\, i_2 \ldots i_p \\ k_1\, k_2 \ldots k_p \end{pmatrix}$ = minor formed from rows $i_1, \ldots, i_p$ and columns $k_1, \ldots, k_p$ of matrix B
$\hat{R}_{ij}$ = cofactor of $\hat{r}_{ij}$ in $\hat{R}$

1 Introduction

The concepts of return difference and return ratio have been shown by Bode[1] to be of fundamental importance in the study of feedback systems. The purpose of this paper is to show that the appropriate generalisation of these concepts to the multivariable case is equally fundamental to multiple-loop feedback control studies, and gives a valuable unification of all the known techniques in the analysis of linear multivariable feedback control.

2 Return-difference and return-ratio matrices

For the feedback control system of Fig. 1, let

$r(s) = m \times 1$ matrix of reference input transforms
$e(s) = m \times 1$ matrix of error transforms
$y(s) = m \times 1$ matrix of plant output transforms
$u(s) = r \times 1$ matrix of plant input transforms
$K(s) = r \times m$ matrix of controller transfer functions
$G(s) = m \times r$ matrix of plant transfer functions
$H(s) = m \times m$ matrix of feedback-transducer transfer functions.

$K(s)$, $G(s)$ and $H(s)$ are matrices over the field of rational functions in the complex variable s. (The algebraic implications of this are discussed in Section 8.)

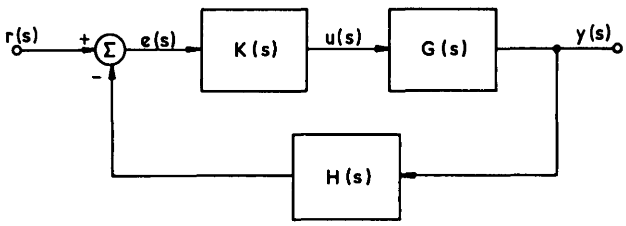

Fig. 1
Vector feedback system

The closed-loop-system transfer-function matrix is given by

$$R(s) = \{I_m + G(s)K(s)H(s)\}^{-1}G(s)K(s) \quad . \quad . \quad . \quad (1)$$

Suppose all the feedback loops are broken as shown in Fig. 2, and a signal transform vector $\alpha(s)$ is injected at point a. The transform of the signal returned at a' is then

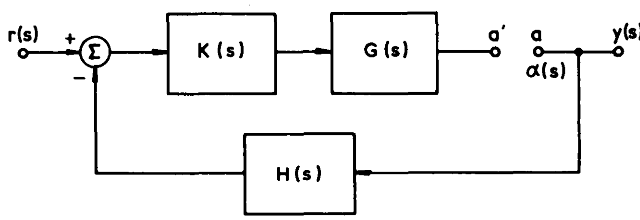

Fig. 2
Calculation of return difference

$$-G(s)K(s)H(s)\alpha(s)$$

and the difference between injected and returned signals is thus

$$\{I_m + G(s)K(s)H(s)\}\alpha(s) = F(s)\alpha(s)$$

where $F(s) = \{I_m + G(s)K(s)H(s)\}$ . . . . . (2)

is defined as the system return-difference matrix. This is a natural generalisation of the scalar return-difference quantity introduced by Bode.[1] The matrix

$$T(s) = G(s)K(s)H(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is defined as the system return-ratio matrix, so that we have

$$F(s) = I_m + T(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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This, again, is a natural generalisation of the equivalent scalar concept introduced by Bode.

Matrix generalisations of Bode's work have been applied to linear networks by Tasny-Tschiassny.[18]

3 Fundamental relationships between open-loop and closed-loop behaviour

Using arguments given by Rosenbrock,[2] and by Hsu and Chen,[12] it can be shown that (as in Appendix 15.1)

$$\det F(s) = \frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}} \qquad (5)$$

This is the fundamental equation relating open- and closed-loop behaviour in multiple-loop control systems.

4 Complex-plane mapping criterion for multiple-loop system stability in terms of return-difference matrix

Assume that the system is open-loop stable. The open-loop characteristic polynomial will then have no zeros in the closed right half complex plane. Thus it follows from eqn. 5 that the closed-loop characteristic polynomial will not vanish in the closed right-half complex plane if, and only if, $\det F(s)$ does not vanish in the closed right half complex plane. Let D be a contour in the complex plane consisting of the imaginary axis from $-j\,\alpha$ to $+j\,\alpha$ and a semicircle centred on the origin of radius α in the right halfplane. Further, let α be large enough to ensure that every zero and pole of $\det GK(s)$ and $\det R(s)$ which is in the open right halfplane lies within D.

Suppose D maps into a closed curve Γ in the complex plane under the mapping $\det F(s)$. Then the system is closed-loop stable if no point within D maps on to the origin of the complex plane under the mapping $\det F(s)$.

Thus the system is closed-loop stable if Γ does not enclose the origin of the complex plane. If $|\det F(s)| \to 1$ as $|s| \to \infty$, then, taking α as arbitrarily large, we can conveniently refer to Γ as the locus $\det F(j\omega)$. This gives the multiple-loop Nyquist-type criterion for stability shown in Fig. 3.

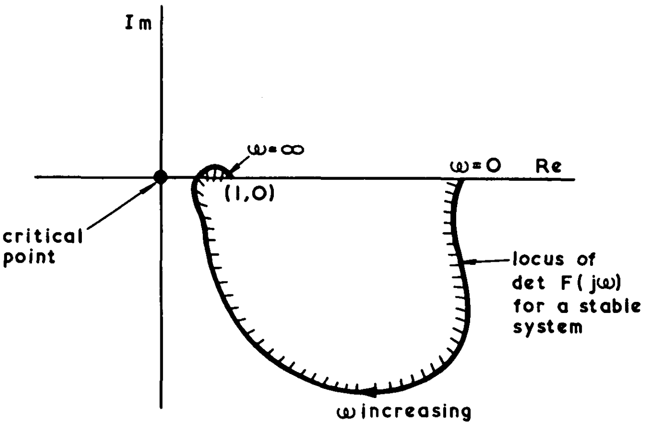

Fig. 3
Simple multivariable Nyquist criterion

Let the eigenvalues of $F(s)$ be $\{\rho_j(s): j = 1, 2, \ldots, m\}$; the algebraic implications of this are discussed in Section 8. We then have that

$$\det F(s) = \prod_{j=1}^{m} \rho_j(s) \qquad (6)$$

Therefore, $\det F(s)$ will not vanish for any s enclosed by D if none of $\{\rho_j(s): j = 1, 2, \ldots, m\}$ vanish for any s enclosed by D. Let D map into Δ_j in the complex plane under $\{\rho_j(s): j = 1, 2, \ldots, m\}$. Then Γ will not enclose the origin of the complex plane if none of Δ_j enclose the origin of the complex plane for $j = 1, 2, \ldots, m$. Thus the system will be stable with all loops closed if none of Δ_j enclose the origin of the complex plane for $j = 1, 2, \ldots, m$. We thus have the following result.

Fundamental stability property of complex-plane loci of the return-difference matrix eigenvalues: The system is closed-loop stable if all the eigenvalue loci $\rho_j(j\omega)$ for $j = 1, 2, \ldots, m$ satisfy the Nyquist criterion as illustrated in Fig. 4.

This criterion can equally well be stated in terms of the return-ratio matrix. Since

$$F(s) = I_m + T(s)$$

the eigenvalues of $F(s)$ and $T(s)$ are simply related via the

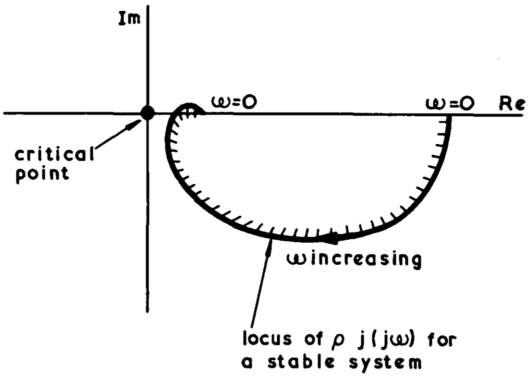

Fig. 4
Extended Nyquist criterion for characteristic frequency responses

eigenvalue-shift theorem.[17] This shows that, if $\{\nu_j(s): j = 1, \ldots, m\}$ are the eigenvalue of $T(s)$, then

$$\rho_j(s) = 1 + \nu_j(s) \qquad j = 1, 2, \ldots, m \qquad (7)$$

In terms of the return-ratio matrix, therefore, we simply get (as in the scalar case) a unit shift in the location of the critical point. The system is thus closed-loop stable if all the eigenvalue loci $\nu_j(j\omega)$ for $j = 1, 2, \ldots, m$ satisfy the Nyquist criterion as illustrated in Fig. 5.

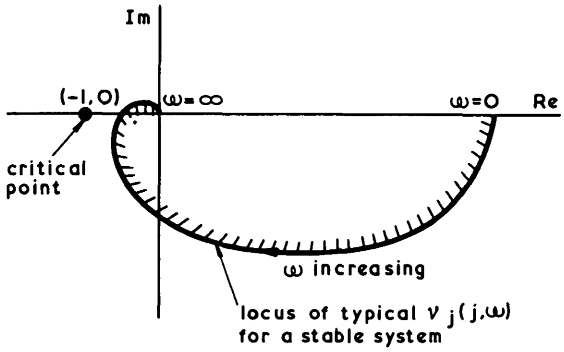

Fig. 5
Change in critical point

A consideration of the behaviour of $\det F(s)$ along the imaginary axis in the complex plane shows that Chen's alternative return-difference stability criterion[13] for scalar systems is simply extended to the multiple-loop case. In what follows, however, we will use the more familiar Nyquist type of frequency-response-stability criterion.

5 Sensitivity of closed-loop-system behaviour in terms of return-difference matrix

Bode's classical studies[1] involved the characterisation of sensitivities, as well as stability, in terms of return differences. The fundamental sensitivity relationship may be generalised to the multivariable case as follows:

Let $G(s)K(s) = Q(s)$ (8)

Then the closed-loop transfer-function matrix is given by

$$R(s) = \{I_m + Q(s)H(s)\}^{-1}Q(s) \qquad (9)$$

Inverting both sides of this equation gives, where $R(s)$ and $Q(s)$ are invertible,

$$\begin{aligned} R^{-1}(s) &= Q^{-1}(s)\{I_m + Q(s)H(s)\} \\ &= Q^{-1}(s) + H(s) \qquad (10) \end{aligned}$$

Suppose, owing to small changes in system-parameter values, that the forward-path transfer-function matrix $Q(s)$ is perturbed to $Q(s) + \delta Q(s)$, and that $R(s)$ is correspondingly perturbed to $R(s) + \delta R(s)$. Then, from eqn. 10, we have

$$\{R(s) + \delta R(s)\}^{-1} = \{Q(s) + \delta Q(s)\}^{-1} + H(s) \quad (11)$$

Expanding the inverses in series now gives

$$R^{-1}(s) + R^{-2}(s)\delta R + \ldots = H(s) + Q^{-1}(s) + Q^{-2}(s)\delta Q + \ldots$$

Using eqn. 10 and neglecting terms of higher than first order in δR and δQ then leads to

$$R^{-2}(s)\delta R = Q^{-2}(s)\delta Q \quad (12)$$

Multiplying both sides of eqn. 12 by R and substituting for R on the right-hand side from eqn. 9 then gives

$$R^{-1}(s)\,\delta R(s) = \{I_m + Q(s)\,H(s)\}^{-1}Q^{-1}(s)\delta Q(s)$$

so that

$$R^{-1}(s)\delta R(s) = F^{-1}(s)Q^{-1}(s)\delta Q(s) \quad (13)$$

which is the matrix generalisation of Bode's scalar equation (in our notation)

$$\frac{d\ln R}{d\ln Q} = \frac{\frac{dR}{R}}{\frac{dQ}{Q}} = \frac{1}{F} \quad (13a)$$

6 Complex-plane mapping criterion for multiple-loop-system optimality in terms of return-difference matrix[5]

Suppose a controller input is to be manipulated in such a way that the performance index

$$V(\infty) = \int_0^\infty (\langle y, \bar{Q}y\rangle + \langle u, \bar{R}u\rangle)dt \quad (14)$$

is minimised for a plant having state-space equations

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx \quad (15)$$

where (A, C) is an observable pair. The corresponding optimal-control action is given by[6]

$$u = -\bar{R}^{-1}B^TPx \quad (16)$$

where P is the unique positive definite solution of the steady-state matrix Riccati equation[6]

$$-PA - A^TP + PB\bar{R}^{-1}B^TP = C^T\bar{Q}C \quad (17)$$

(This is the 'negative-feedback convention' form of the Riccati equation.) Let the optimal-feedback matrix corresponding to this value of P be denoted by

$$K = \bar{R}^{-1}B^TP \quad (18)$$

We now wish to link the preceding discussion of multiple-loop feedback theory to optimal control theory by first expressing the matrix Riccati equation in terms of the return-difference matrix and then deducing the properties of the eigenvalue loci and determinantal loci of $F(s)$ for the optimal case. This requires some fairly involved juggling with the matrix Riccati equation along the lines introduced by Kalman in his original treatment[7] of the frequency response of a single-loop optimal system.

First, add sP to and subtract sP from the left-hand side of eqn. 17. This gives

$$P(sI - A) + (-sI - A^T)P = C^T\bar{Q}C - PB\bar{R}^{-1}B^TP \quad (19)$$

Now multiply both sides of eqn. 19 on the left-hand side by $B^T(-sI - A^T)^{-1}$ and on the right by $(sI - A)^{-1}B$.
This gives

$$\begin{aligned} &B^T(-sI - A^T)^{-1}PB + B^TP(sI - A)^{-1}B \\ &= B^T(-sI - A^T)^{-1}C^T\bar{Q}C(sI - A)^{-1}B \\ &\quad - B^T(-Is - A^T)^{-1}PB\bar{R}^{-1}B^TP(sI - A)^{-1}B \quad (20) \end{aligned}$$

Rearranging the terms in eqn. 20 and adding $\bar{R}$ to both sides gives

$$\begin{aligned} &B^T(-sI - A^T)^{-1}C^T\bar{Q}C(sI - A)^{-1}B + \bar{R} \\ &= \bar{R} + B^T(-sI - A^T)^{-1}PB + B^TP(sI - A)^{-1}B \\ &\quad + B^T(-sI - A^T)^{-1}PB\bar{R}^{-1}B^TP(sI - A)^{-1}B \quad (21) \end{aligned}$$

Consider the right-hand side of eqn. 21. We have

$$\begin{aligned} &\bar{R} + B^T(-sI - A^T)^{-1}PB + B^TP(sI - A)^{-1}B \\ &\quad + B^T(-sI - A^T)^{-1}PB\bar{R}^{-1}B^TP(sI - A)^{-1}B \\ &= \bar{R} + \bar{R}\bar{R}^{-1}B^TP(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}B \\ &\quad + B^T(-sI - A^T)^{-1}PB\bar{R}^{-1}\bar{R}\bar{R}^{-1}B^TP(sI - A)^{-1}PB\bar{R}^{-1}\bar{R} \\ &= \bar{R} + \bar{R}K(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}K^T\bar{R} \\ &\quad + B^T(-sI - A^T)^{-1}K^T\bar{R}K(sI - A)^{-1}B \\ &= \{I + B^T(-sI - A^T)^{-1}K^T\}\{\bar{R} + \bar{R}K(sI - A)^{-1}B\} \\ &= \{I + B^T(-sI - A^T)^{-1}K^T\}\bar{R}\{I + K(sI - A)^{-1}B\} \\ &= F^T(-s)\bar{R}F(s) \quad (22) \end{aligned}$$

where $F(s)$ is the return-difference matrix for the optimally controlled feedback system.

The left-hand side of eqn. 21 can be written as

$$G^T(-s)\bar{Q}G(s) + \bar{R}$$

where, as usual, $G(s)$ is the plant transfer-function matrix. The steady-state matrix Riccati equation thus leads to the following equation relating $\bar{Q}$, $\bar{R}$, $G(s)$ and the optimal return-difference matrix:

$$F^T(-s)\bar{R}F(s) = \bar{R} + G^T(-s)\bar{Q}G(s) \quad (23)$$

in which $\bar{Q}$ is a positive definite and $\bar{R}$ a positive semidefinite matrix.

Now let $w(s)$ be an eigenvector of $F(s)$ corresponding to the eigenvalue $\rho(s)$. We then have

$$F(s)w(s) = \rho(s)w(s) \quad (24)$$

$$w^T(-s)F^T(-s) = \rho(-s)w(-s) \quad (25)$$

Multiply both sides of eqn. 23 on the left by $w^T(-s)$ and on the right by $w(s)$. Then we have

$$\begin{aligned} &w^T(-s)\bar{R}w(s) + w^T(-s)G^T(-s)\bar{Q}G(s)w(s) \\ &= w^T(-s)F^T(-s)\bar{R}F(s)w(s) \\ &= \rho(-s)w^T(-s)\bar{R}\rho(s)w(s) \\ &= \rho(-s)\rho(s)w^T(-s)Rw(s) \\ &= |\rho(s)|^2w^T(-s)\bar{R}w(s) \qquad \text{when } s = j\omega \quad (26) \end{aligned}$$

When both $\bar{R}$ and $\bar{Q}$ are positive definite, we deduce from eqn. 26 that

$$|\rho(s)|^2w^T(-s)\bar{R}w(s) \geqslant w^T(-s)\bar{R}w(s) \geqslant 0 \quad (27)$$

for all $s = j\omega$.

Thus $|\rho(s)| \geqslant 1$ for all $s = j\omega$ (28)

This extremely interesting result, first obtained by Cumming,* means that Kalman's criterion[7] for the optimality of single-loop systems in terms of the frequency response of the return difference generalises to the multiple-loop case via the eigenvalues of the return-difference matrix, each of which satisfies the criterion for a single-loop system, just as in the stability case.

From eqn. 28 we immediately obtain the frequency-response criteria for optimality:

$$|\rho_j(j\omega)| \geqslant 1 \qquad j = 1, 2, \ldots, m \quad (29)$$
$$0 \leqslant \omega \leqslant \infty$$

* CUMMING, S. D. G.: Private communication

where, as before, $\{\rho_j(s): j = 1, 2, \ldots, m\}$ are the eigenvalues of $F(s)$. Thus, using the fact that

$$\det F(s) = \prod_{j=1}^{m} \rho_j(s) \qquad (30)$$

we obtain the criterion for optimality

$$|\det F(j\omega)| \geqslant 1 \quad \text{for all } \omega \qquad (31)$$

These results have the graphical interpretations shown in Figs. 6 and 7, namely that the complex-plane plots of

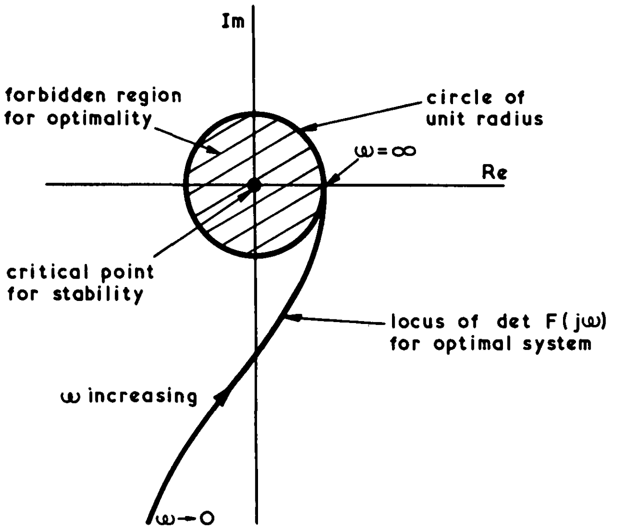

Fig. 6
Necessary condition for optimality satisfied by return-difference determinant

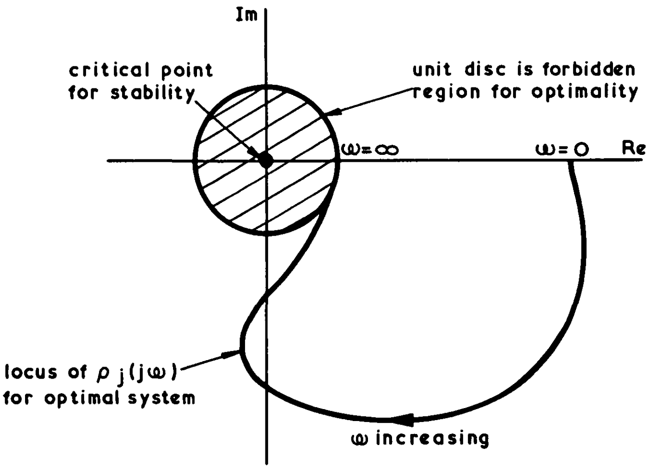

Fig. 7
Necessary condition for optimality satisfied by characteristic frequency-response loci

$|\det F(j\omega)|$ and $|\rho_j(j\omega)|$ must not penetrate the interior of the unit disc. These results enable a direct comparison of optimal and nonoptimal designs to be made in terms of frequency-response plots.

7 Use of return-difference matrix to generate modal control action

The eigenvalue-shift techniques of modal control[8] can be easily derived from the determinant of the appropriate return-difference matrix.

The transfer-function matrix for a plant having, as state-space equations, eqns. 15 is

$$G(s) = C(sI - A)^{-1}B \qquad (32)$$

Now, suppose that the system A matrix eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. Let U be a matrix whose columns are the system A matrix eigenvectors, V be a matrix whose rows are the reciprocal eigenvectors of A and Λ be a diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$UV = \text{a unit matrix} \qquad (33)$$

$$UAV = \Lambda \qquad (34)$$

and it follows that

$$G(s) = CU(sI - \Lambda)^{-1}VB \qquad (35)$$

If we have complete freedom over the measurement of state variables, we may put

$$C = V$$

and thus obtain

$$G(s) = (sI - \Lambda)^{-1}VB \qquad (36)$$

Using this transfer-function matrix in eqns. 2 and 5 gives

$$\det\{I_m + (sI - \Lambda)^{-1}VBK(s)H(s)\} = \frac{\prod_i (s - \gamma_i)}{\prod_j (s - \lambda_j)} \qquad (37)$$

where γ_i and λ_j denote the set of closed-loop and open-loop characteristic frequencies, respectively.

If we put

$$K(s)H(s) = fd^T \qquad (38)$$

the outer product of an $n \times 1$ matrix f and a $1 \times n$ matrix d^T, then simple determinantal manipulations give

$$1 + d^T(sI - \Lambda)^{-1}BVf = \frac{\prod_i (s - \gamma_i)}{\prod_j (s - \lambda_j)} \qquad (39)$$

This gives a set of equations from which d^T may be calculated for a specified f to give any desired closed-loop eigenvalue positions. It shows that the resulting loop gains are directly proportional to the eigenvalue shifts imposed. The chief implication of this result is that it shows that arbitrary closed-loop frequency locations can be achieved, given access to all the system states. The technique of altering characteristic frequencies in this way is called modal control.[8] The argument may easily be extended to the case where the system A matrix has repeated eigenvalues.

8 Algebraic and vector-space relationships in terms of which design techniques may be discussed

The most important aspect of the discussion so far is that it shows that the properties of the eigenvalues of the matrices $F(s)$ and $T(s)$ hold the key to the generalisation of classical frequency-response design techniques to the multivariable case. The algebraic and physical aspects of these eigenvalues are thus now examined in more detail, followed by a discussion of design techniques.

All the matrices considered here, namely $G(s)$, $K(s)$, $H(s)$, $T(s)$, $F(s)$ and I_m, are matrices over the field of rational functions. They may therefore be discussed within the framework of the theory of general linear operators on vector spaces over fields,[9,10] and thus in terms of transformations of basis sets, eigenvalues, eigenvectors etc. In general, the eigenvalues of a matrix of rational functions (regarded as an operator on a finite-dimensional vector space over the field of rational functions) may not lie in the field of rational functions. They may, for example, be irrational functions. This difficulty is overcome by following the standard algebraic practice[9] of defining a suitable extension field within which the field of rational functions is imbedded. The transfer-function matrices are then regarded as representing operators on vector spaces over the extension field. In practice, any eigenvalue $\rho(s)$ in this extension field may be approximated to, to any required degree of accuracy, by a rational function, just as any real irrational or transcendental number may be arbitrarily well approximated to by a real rational fraction number. For engineering purposes of visualisation, this extension field may be heuristically thought of as the field of all possible transfer functions arising from root-solving operations on polynomials whose coefficients are rational-fraction transfer functions. Actual calculations carried out for analysis or design purposes will normally be carried out for specific sets of values of the complex variable s; the operations considered then reduce to operations over the field of complex numbers.

The m-dimensional vector spaces over the field of rational functions in s which have thus been introduced can be thought of as spaces in which reside m-dimensional vectors of signal transforms, each transform (or vector component) being a rational fraction in s. An operator on the space, such as $Q(s)$, will convert one vector of signal transforms, such as a system-input transform set, into another vector of signal transforms, such as a system-output transform set. If $w(s)$ is an eigenvector of $Q(s)$ with corresponding eigenvalue $q(s)$, then

$$Q(s)w(s) = q(s)w(s) \quad \ldots \ldots \quad (40)$$

which means that the action of $Q(s)$ on $w(s)$ is such that every transform component of $w(s)$ is multiplied by the same scalar transfer function $q(s)$. This is the equivalent of the familiar 'vector stretching by a scalar multiplier' for the case of a linear vector space over the real number field. $q(s)$ can be thought of as a characteristic transfer function with an associated characteristic frequency response $q(j\omega)$. The analytical studies of previous Sections show that the m characteristic frequency responses associated with the return-difference and return-ratio matrices completely describe the system behaviour from the stability and optimality points of view. These characteristic loci are thus the key to a systematic and unified discussion of the multivariable control problem.

8.1 Controller closed-loop behaviour in terms of characteristic loci

Let the eigenvalues of the matrix $Q(s)$, defined in eqn. 8, be $q_1(s), q_2(s), \ldots, q_m(s)$, and suppose that they are distinct in the extension field. [If they are not distinct, an arbitrarily small set of perturbations in the elements of $Q(s)$ will make this so.] Let a corresponding set of eigenvectors be $w_1(s), w_2(s), \ldots, w_m(s)$. From standard algebraic theory,[10] this vector set will constitute a basis for a linear vector space of dimension m over the field of rational functions. Define a matrix $W(s)$ by

$$W(s) = [w_1(s)w_2(s) \ldots w_m(s)] \quad \ldots \quad (41)$$

Since the vectors $w_1(s)$ through $w_m(s)$ are linearly independent, $W(s)$ is invertible, and we have

$$W^{-1}(s)Q(s)W(s) = \mathrm{diag}\,[q_1(s)q_2(s) \ldots q_m(s)]$$
$$= \Lambda^Q(s) \text{ say} \quad \ldots \quad (42)$$

Let

$$W^{-1}(s) = V(s) = \begin{bmatrix} v_1^T(s) \\ v_2^T(s) \\ \vdots \\ v_m(s) \end{bmatrix} \quad \ldots \quad (43)$$

where $v_1^T(s), \ldots, v_m^T(s)$ are the rows of $V(s)$.

Then $\quad V(s)Q(s)W(s) = \Lambda^Q(s)$

$$W(s)\Lambda^Q(s)V(s) = Q(s)$$

and the system closed-loop transfer-function matrix $R(s)$ may be written in the form

$$R(s) = \{I_m + W(s)\Lambda^Q(s)V(s)H(s)\}^{-1}W(s)\Lambda^Q(s)V(s) \quad \ldots \quad (44)$$

and we have

$$R^{-1}(s) = W(s)\{\Lambda^Q(s)\}^{-1}V(s) + H(s) \quad \ldots \quad (44a)$$

and, putting $s = j\omega$ to discuss frequency response,

$$R^{-1}(j\omega) = W(j\omega)\{\Lambda^Q(j\omega)\}^{-1}V(j\omega) + H(j\omega) \quad . \quad (45)$$

Suppose that, at some specific frequency ω_1, all the eigenvalues of $Q(j\omega)$ have arbitrarily large modulus. Then all the elements of $\Lambda^Q(j\omega_1)$ will be arbitrarily large, and we shall have

$$R^{-1}(j\omega_1) \simeq H(j\omega_1)$$
$$R(j\omega_1) \simeq H^{-1}(j\omega_1)$$

This leads to the important conclusion that arbitrarily good control action will be exercised over any frequency range over which the moduli of all the eigenvalues of $Q(j\omega)$ are arbitrarily large.

Conversely, suppose that, at some other specific frequency ω_h, all the eigenvalues of $Q(j\omega)$ have arbitrarily small modulus. Then all the elements of $\Lambda^Q(j\omega_h)$ will be arbitrarily small and we will have

$$R^{-1}(j\omega_h) \simeq Q^{-1}(j\omega_h)$$
$$R(j\omega_h) \simeq Q(j\omega_h)$$

For any form of design procedure to achieve a multivariable feedback control, two sets of conditions must be satisfied by the eigenvalues of $Q(j\omega)$:

(a) The moduli of all the eigenvalues of $Q(j\omega)$ must be large over any frequency range for which it is desired to exercise good control. Specifically, all the eigenvalues of $Q(0)$ must be large if good control action is required at d.c. and very low frequencies.
(b) To satisfy the closed-loop-stability criteria of Nyquist type imposed on the characteristic frequency-response loci, all the eigenvalues of $Q(j\omega)$ must have small moduli at high frequencies.

Now consider the response, under feedback control, of a system satisfying both these conditions to a set of step changes in the reference inputs. Thus, let the reference input-transform vector be

$$r(s) = \frac{1}{s}\bar{r} \quad \ldots \ldots \quad (46)$$

where $\bar{r}$ is a vector whose elements are real constants. The above arguments on the behaviour of $R(j\omega)$ at low and high frequencies, together with use of the final-value and initial-value theorems of Laplace transform theory, then enable us to infer the initial and final values of system-controlled outputs under feedback control as

$$y(0+) = Q(\infty)\bar{r} \quad \ldots \ldots \quad (47)$$
$$y(\infty) = H^{-1}(0)\bar{r} \quad \ldots \ldots \quad (48)$$

8.2 Commutative controller

The general problem of multivariable-feedback-control-system design can now be looked on in terms of choosing the controller matrix $K(s)$ so that the eigenvalues of the matrix $G(s)K(s)$ have certain prescribed properties. This is the viewpoint taken in the general discussion of design techniques given in the following Section. The crux of the design is choosing the individual elements of $K(j\omega)$ to achieve the desired modification of the characteristic loci of $G(j\omega)$; this is difficult since little is known about the location of the eigenvalues of the product of two matrices in terms of the eigenvalues of the individual matrices which are multiplied together. One particular situation may be handled, however, in a simple way; namely that in which the two matrices commute, and therefore have a common set of eigenvectors.[10] This leads to the theoretical concept of what may be called a commutative controller. A commutative controller would be designed by synthetising m classical single-loop controllers in the eigenframework of the plant transfer-function matrix, and then 'transforming back' to the original reference basis to get the required controller matrices.

Let $\gamma_1(s), \ldots, \gamma_m(s)$ be the eigenvalues (characteristic transfer functions) of the plant transfer-function matrix $G(s)$, with a corresponding eigenvector set $w_1(s), \ldots, w_m(s)$, and associated matrices $W(s)$, $V(s)$ as defined by eqns. 41 and 43. For a commutative controller, the eigenframework of $G(s)$ and $Q(s)$ will be the same. $G(s)$ can then be expressed in dyadic form as

$$G(s) = \sum_{j=1}^{m} w_j(s)\gamma_j(s)v_j^t(s) \quad \ldots \ldots \quad (49)$$

Let $\quad K(s) = \sum_{k=1}^{m} w_k(s)k_k(s)v_k^t(s) \quad \ldots \ldots \quad (50)$

and $\quad H(s) = \sum_{i=1}^{m} w_i(s)h_i(s)v_i^t(s) \quad \ldots \ldots \quad (51)$

Then $\quad G(s)K(s) = \sum_{j=1}^{m} w_j(s)\gamma_j(s)k_j(s)v_i^t(s) \quad \ldots \quad (52)$

$$\text{since} \quad v_i^t(s) w_j(s) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \ldots \ldots \quad (53)$$

as we have

$$V(s)W(s) = I_m \quad \ldots \ldots \quad (54)$$

Simple manipulations then give

$$\begin{aligned} R(s) &= \{I_m + G(s)K(s)H(s)\}^{-1}G(s)K(s) \\ &= \sum_{j=1}^{m} w_j(s) \left\{ \frac{\gamma_j(s)k_j(s)}{1 + \gamma_j(s)k_j(s)h_j(s)} \right\} v_j^t(s) \\ &= W(s)\Lambda(s)V(s) \quad \ldots \ldots \quad (55) \end{aligned}$$

where

$$\Lambda(s) = \operatorname{diag} \frac{\gamma_j(s)k_j(s)}{1 + \gamma_j(s)k_j(s)h_j(s)} \quad \ldots \ldots \quad (56)$$

and

$$K(s) = W(s)\{\operatorname{diag} k_i(s)\}V(s) \quad \ldots \ldots \quad (57)$$

$$H(s) = W(s)\{\operatorname{diag} h_i(s)\}V(s) \quad \ldots \ldots \quad (58)$$

From eqn. 55, it follows that

$$R(s) \to \sum_{j=1}^{m} w_j v_j^t = I_m \text{ as } k_j \to \infty \; j = 1, \ldots, m$$

if $h_j(s) = 1$ and $j = 1, 2, \ldots, m$.

The above relationships exhibit the idea of the commutative controller. In simple physical terms, the transformation of basis to the plant eigenframework means expressing general signal-transform vectors as linear combinations of those vectors [the eigenvectors of $G(s)$] to which the plant appears as a set of simple single scalar transfer functions, namely the characteristic transfer functions or eigenvalues $\gamma_j(s)$. In this basis, we would try to carry out m single-loop designs to choose the set of single-loop controllers $k_j(s)$ and feedback operators $h_j(s)$, and finally transform back to obtain the actual controller $K(s)$ and feedback operator $H(s)$ by means of eqns. 57 and 58.

9 General discussion of multivariable-feedback-control-system design techniques

We now have available all the material required to give a unified and systematic discussion of linear multivariable-control-system design techniques. The stability and optimality characteristics of the m characteristic frequency-response loci serve as the link between classical single-loop frequency-response methods and multivariable-system design techniques.

9.1 Noninteracting control techniques

In the so-called noninteracting controller-design technique,[15] $H(s)$ is taken to be a diagonal matrix and the controller matrix $K(s)$ is chosen so that $G(s)\,K(s)$ is diagonal. It follows immediately that $R(s)$ is diagonal; hence the name noninteracting control. The noninteracting control is achieved by making the open-loop transfer-function matrix $Q(s)$ diagonal, i.e. 'noninteracting'. Since $Q(s)$ and $H(s)$ are diagonal, it follows that the return-difference and return-ratio matrices are diagonal and thus their diagonal elements are, trivially, the appropriate characteristic transfer functions. The design problem therefore reduces to the design of m 'uncoupled' single-loop controllers. The disadvantages of this technique are

(i) Much of the design freedom in choosing the elements of $K(s)$ is used up in making $G(s)\,K(s)$ diagonal, leaving little room for manoeuvre in compensating for the resulting characteristic frequency-response loci.
(ii) The forms of characteristic frequency-response loci achieved by this simple diagonalisation procedure often require considerable compensation to give a satisfactory form of closed-loop response. It has been shown by Rosenbrock[14] that this procedure suffers a considerable disadvantage when det $G(s)$ has right-halfplane zeros, the technique then giving poor or unstable control.

9.2 Modal-control techniques[8]

Section 7 shows how, given access to all the states of a system described in state-space terms, a linear proportional-feedback controller may be built up which places the system closed-loop state-space eigenvalues in any desired position. The disadvantages of the use of this algorithm as a design technique are

(a) It leads to a simple proportional controller, and gives no guidance as to the choice of dynamic compensating elements.
(b) No means are provided for correlating closed-loop transient response (which involves the zeros of all the transfer functions concerned) with the positions achieved for the poles of the closed-loop transfer functions.
(c) Linear combinations of system states (the system modes) are controlled, and not the states themselves. In certain circumstances, this can lead to a situation where the modally controlled system has poor disturbance-rejection properties, and relatively large disturbances of states occur even with high loop gains in the controller.

9.3 Optimal-control techniques

The discussion of Section 6 shows that the characteristic frequency-response loci of an optimal proportional-feedback controller have infinite gain margin and at least 60° phase margin. This follows from simple geometrical consequences of the fact that the characteristic loci do not penetrate the unit disc surrounding the critical point in the complex plane. The disadvantages of the optimal-control approach are therefore:

(i) It requires access to all the system states.
(ii) It provides gain margins far in excess of these actually required for stability.
(iii) It offers no means of providing dynamic compensation.

Reflection on the mechanism of operation of an optimal controller in terms of its characteristic frequency-response loci will show that the accession of all system states is what gives the degree of phase advance required to achieve the arbitrarily high gain margins of the optimal characteristic frequency-response loci.

9.4 Commutative-controller technique

The commutative-controller technique is at first sight a conceptually appealing one, since it operates directly on the eigenvalues which control the feedback system behaviour. It has, however, several severe disadvantages:

(a) The eigenvalues of $G(s)$ will normally be irrational. This leads to severe computational difficulties which are only partially alleviated by approximation of the irrational quantities by rational functions in s.
(b) At high frequencies, the stability considerations become paramount, and the eigenvalue moduli must be made small. As shown in the discussion leading up to eqn. 47, the closed-loop response immediately after a transient input is largely determined by $Q(s)$. This means that the transient response of the closed-loop system cannot be completely controlled by designing a set of m scalar systems as in Section 8.2.

If $Q(s)$ has significant off-diagonal terms at high frequencies, interaction terms cannot be suppressed by the design of m scalar systems which specify the eigenvalue behaviour. This is simply because the stability requirements reduce high-frequency gain to the point at which high-frequency cross-couplings cannot be suppressed. The only straightforward way to eliminate such interaction is to make $Q(s)$ of diagonally dominated form, as discussed below.

9.5 Eigenvalue-locator controller techniques

A practicable approach to the design problem is to use methods of approximately locating the eigenvalues of $T(s)$ and $F(s)$ without carrying out a spectral analysis. Such techniques offer a very promising line of development, and are considered in detail in the following Section.

10 Eigenvalue-locator design techniques

The simplest approach to the frequency-response design of a multiple-loop dynamically compensated feedback control system is to use the frequency-response locus $\det F(j\omega)$. The disadvantage of using this locus for the design of compensators is that it changes size and shape with every change in every element of the controller matrix $K(j\omega)$. We would like to be able to do design syntheses which exploited the properties of the m characteristic frequency-response loci without having to calculate them in detail. The results established in this Section give the conditions under which this may be done. They will be called indirect multiple-loop complex-plane stability criteria, since they enable us to infer the stability of the complete system with all feedback loops closed from the stability of a set of m single-loop frequency-response loci which are not the characteristic frequency-response loci.

10.1 Indirect multiple-loop stability criterion in terms of return-difference matrix F(s)

Let s trace out the contour D described in Section 4. If:

(i) $$|f_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |f_{ij}(s)| \quad i = 1, 2, \ldots, m \quad . \quad . \quad (59)$$

for all s on D,

(ii) none of Γ_i ($i = 1, 2, \ldots, m$) enclose the origin of the complex plane, where

$$D \text{ maps into } \Gamma_i \text{ under } f_{ii}(s)\ i = 1, 2, \ldots, m \quad . \quad (60)$$

the system is stable with all feedback loops closed.

A system satisfying both exprs. 59 and 60 will be said to be diagonal-dominated.

Proof of indirect stability criterion: The proof depends on establishing the fact that exprs. 59 and 60 automatically ensure that the locus Γ obtained by mapping D under $\det F(s)$ does not enclose the origin. This is done by exploiting the two facts that

(*a*) $\det F(s)$ is the product of the eigenvalues of $F(s)$

(*b*) the eigenvalues of a matrix lie in the union of its set of Gershgorin discs.[4]

The eigenvalues of $F(s)$ are $\{\rho_j(s): j = 1, 2, \ldots, m\}$ and we have

$$\det F(s) = \prod_{j=1}^{m} \rho_j(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The eigenvalues $\rho_j(s)$ are contained in the union of Gershgorin discs for $F(s)$, which are the set of circles[4] having centres

$$f_{ii}(s) \quad i = 1, 2, \ldots, m$$

and radii

$$\sum_{\substack{j=1\\j\neq i}}^{m} |f_{ij}(s)| \qquad i = 1, 2, \ldots, m$$

Consider the Gershgorin-disc set generated by moving along the imaginary axis part of the contour D defined in Section 4

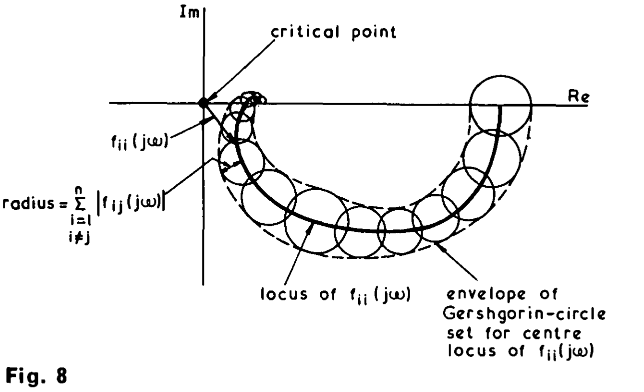

Fig. 8
Indirect stability criterion
If the diagonal-dominance condition is satisfied, no circle can sweep over the origin

for the particular element $f_{ii}(j\omega)$ as shown in Fig. 8. Gershgorin's theorem ensures that every eigenvalue locus $\rho_j(j\omega)$ lies within the union of envelopes of the type illustrated in Fig. 8. It follows from this line of argument that exprs. 59 and 60 ensure that none of the loci Δ_j defined in Section 4 can enclose the origin of the complex plane. It then immediately follows from the argument in Section 4 that the system is thus stable with all loops closed if exprs. 59 and 60 are satisfied.

10.1.1 Restatement in terms of return-ratio matrix T(s)

$$\text{Since } f_{ii}(s) = 1 + t_{ii}(s) \qquad i = 1, 2, \ldots, m \qquad (62)$$

$$\text{and } f_{ij}(s) = t_{ij}(s) \qquad j = 1, 2, \ldots, m \text{ and } j \neq i$$

the indirect stability criterion may be expressed in terms of the return-ratio matrix $T(s)$. In these terms it is most conveniently expressed in the following way. If

(*i*) $$|1 + t_{ii}(s)| > \sum_{\substack{j=1\\j\neq 1}}^{m} |t_{ij}(s)| \qquad i = 1, 2, \ldots, m \quad . \quad (63)$$

for all s on D

(*ii*) all $t_{ii}(j\omega)$, $i = 1, 2, \ldots, m$, satisfy Nyquist's stability criterion with critical point at $(-1, 0)$, then the system is stable with all loops closed.

10.1.2 Dual statement of results

Since the eigenvalues of a matrix and its transpose are equal, Gershgorin discs may be constructed for either row sums or column sums of off-diagonal elements.[4]

Thus exprs. 59 and 63 may equally well be replaced by

$$|f_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |f_{ji}(s)|$$

$$|1 + t_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |t_{ji}(s)|$$

for all s on D.

10.2 Rosenbrock's stability theorems in terms of det Q(s) and det R(s) and their inverses

For reasons of convenience in relating open- and closed-loop responses, it is often found useful to work in terms of $Q(s)$ and $R(s)$ rather than $F(s)$. It follows from eqns. 1, 2 and 8 that

$$\det F(s) = \frac{\det Q(s)}{\det R(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and so stability criteria can be generated in terms of $\det Q(s)$ and $\det R(s)$. These criteria have the further great advantage that they can easily be re-expressed in terms of inverse mappings in the complex plane which, as shown in a later Section, are of great utility in the proposed design methods. In terms of $\det Q(s)$ and $\det R(s)$, Rosenbrock has proved the following stability theorem.[2]

10.2.1 Rosenbrock's stability theorem

Let $\det Q(s)$ map the contour D into Γ_0 and let $\det R(s)$ map D into Γ_c. Then the system is closed-loop stable if, and only if, $\det R(s)$ has no pole on any finite part of the imaginary axis and Γ_c encircles the origin as often, in a clockwise direction, as Γ_0.

Alternative statement: It is often convenient to work in terms of inverse matrices, as will be seen later. If

$$Q^{-1} = \hat{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$R^{-1} = \hat{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

then the theorem may be stated in the following equivalent way.

Let $\det \hat{Q}(s)$ map D into $\hat{\Gamma}_0$ and let $\det \hat{R}(s)$ map D into $\hat{\Gamma}_c$. Then the system is closed-loop stable if, and only if, $\det \hat{R}(s)$ has no zero on any finite part of the imaginary axis and $\hat{\Gamma}_c$ encircles the origin as often in an anticlockwise direction as $\hat{\Gamma}_0$.

10.2.2 Rosenbrock's indirect-stability theorem

In designing the controller, we frequently wish to arrange things so that the stability of the whole closed-loop system can be inferred from a consideration of the frequency-response characteristics of a set of m principal loops, where the principal loop transferences are the diagonal terms of the return-ratio matrix or open-loop-gain matrix. The stability theorem just given can be used to derive a result of this type. Since inverse matrices are most used in the design techniques based on these results, the alternative statement of the theorem in terms of inverse matrices is used. It is convenient to first derive some preliminary results.

10.2.2.1 Origin encirclements of det $\hat{Q}(s)$

Let det $\hat{Q}(s)$ map D into $\hat{\Gamma}_0$ and $\hat{q}_{ii}(s)$ map D into $\hat{\Gamma}_{0i}$ $(i = 1, 2, \ldots, m)$. Let the number of anticlockwise origin encirclements of $\hat{\Gamma}_0$ be n_0 and of $\hat{\Gamma}_{0i}$ be n_{0i} $(i = 1, 2, \ldots, m)$. Then, if

$$|\hat{q}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ij}(s)| \qquad i = 1, 2, \ldots, m \qquad (67)$$

or if $$|\hat{q}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ji}(s)| \qquad i = 1, 2, \ldots, m \qquad (67a)$$

for all s on D,

$$n_0 = \sum_{i=1}^{m} n_{0i} \qquad (68)$$

Proof: Let $\{\hat{\nu}_j(s): j = 1, 2, \ldots, m\}$ be the eigenvalues of $\hat{Q}(s)$. Then the eigenvalues $\{\hat{\nu}_j(s)\}$ lie within the union of Gershgorin discs for $\hat{Q}(s)$ which are circles with centres lying on $\hat{\Gamma}_{0i}$ with radii $\sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ij}(s)|$ or, alternatively, $\sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ji}(s)|$, as illustrated for a typical locus in Fig. 9. Exprs. 67 and 67a ensure that the

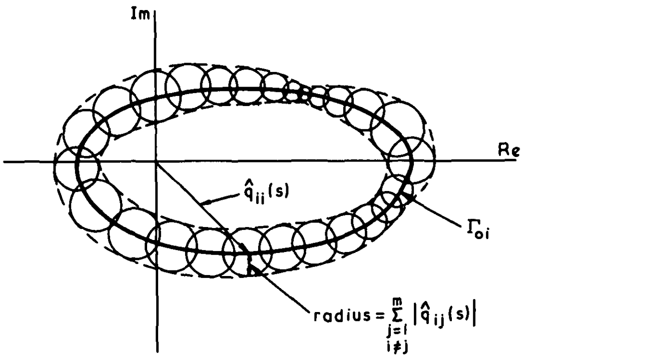

Fig. 9
Alternative indirect stability determination
If the diagonal-dominance condition is satisfied, no circle can sweep over the origin

envelope path swept out by the set of discs never covers the origin. Thus the loci traced out by the eigenvalues $\hat{\nu}_j(s)$ are trapped within a union of paths of the sort swept out by the discs in Fig. 9. It follows from this that

$$\sum_{i=1}^{m} n_{0i} = \text{sum of origin encirclements for all } \hat{\nu}_j(s)$$

Now $$\det \hat{Q}(s) = \prod_{j=1}^{m} \hat{\nu}_j(s) \qquad (69)$$

so that

$$n_0 = \text{sum of origin encirclements for all } \hat{\nu}_j(s)$$

and thus

$$n_0 = \sum_{i=1}^{m} n_{0i} \qquad (70)$$

10.2.2.2 Origin encirclements of det $\hat{R}(s)$

Let det $\hat{R}(s)$ map D into $\hat{\Gamma}_c$ and $\hat{r}_{ii}(s)$ map D into $\hat{\Gamma}_{ci}(i = 1, 2, \ldots, m)$. Let the number of anticlockwise origin encirclements of $\hat{\Gamma}_c$ be n_c and of $\hat{\Gamma}_{ci}$ be $n_{ci}(i = 1, 2, \ldots, m)$. Then, if

$$|\hat{r}_{ii}(s)| > \sum_{\substack{j=1\\j\neq j}}^{m} |\hat{r}_{ij}(s)| \qquad i = 1, 2, \ldots, m \qquad (71)$$

or if

$$|\hat{r}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{r}_{ji}(s)| \qquad i = 1, 2, \ldots, m \qquad (71a)$$

for all s and D,

$$n_c = \sum_{i=1}^{m} n_{ci} \qquad (72)$$

The proof is exactly as for det $\hat{Q}(s)$.

10.2.3 Statement of indirect-stability theorem

A sufficient condition for closed-loop stability is that the following three criteria are satisfied:

(i) $$\sum_{i=1}^{m} n_{0i} = \sum_{i=1}^{m} n_{ci} \qquad (73)$$

(ii) For all s on D and for $i = 1, 2, \ldots, m$

$$|\hat{q}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ij}(s)| \text{ or } |\hat{q}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{q}_{ji}(s)| \qquad (74)$$

(iii) For all s on D and for $i = 1, 2, \ldots, m$

$$|\hat{r}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{r}_{ij}(s)| \text{ or } |\hat{r}_{ii}(s)| > \sum_{\substack{j=1\\j\neq i}}^{m} |\hat{r}_{ji}(s)| \qquad (75)$$

Proof: From the preliminary results established for the origin encirclements of det $\hat{Q}(s)$ and det $\hat{R}(s)$ we see that, if conditions (ii) and (iii) are satisfied, condition (i) implies that $\hat{\Gamma}_0$ and $\hat{\Gamma}_c$ encircle the origin in an anticlockwise direction the same number of times.

The result then immediately follows from Rosenbrock's stability theorem.

10.3 Relationships between inverse transfer-function matrices for open- and closed-loop systems

Consider the system shown in Fig. 1. If, as before, we put

$$G(s)K(s) = Q(s)$$

then the closed-loop transfer-function matrix is

$$R(s) = \{I_m + Q(s)H(s)\}^{-1}Q(s) \qquad (76)$$

so that

$$R^{-1}(s) = Q^{-1}(s)\{I_m + Q(s)H(s)\}$$
$$= Q^{-1}(s) + H(s) \qquad (77)$$

and, putting $R^{-1} = \hat{R}$, $Q^{-1} = \hat{Q}$ we obtain

$$\hat{R}(s) = \hat{Q}(s) + H(s) \qquad (78)$$

Eqn. 78 is of great importance in design work, since it shows that the inverse transfer-function matrices for open- and closed-loop operation are related in a simple way. For frequency-response plots where H is a diagonal matrix of real constants

$$r_{ii}(j\omega) = q_{ii}(j\omega) + h_{ii} \qquad j = 1, 2, \ldots, m$$
$$r_{ij}(j\omega) = q_{ij}(j\omega)\ i \neq j \qquad i, j = 1, 2, \ldots, m \qquad (79)$$

and thus inverse frequency-response loci are obtained by a simple horizontal shift in the complex plane (which may be zero) for the change from open-loop to closed-loop working.

10.4 Relationships for R(s) between the inverses of the elements and the elements of the inverse

The system closed-loop transfer-function matrix is

$$R(s) = \{I_m + G(s)K(s)H(s)\}^{-1}\ G(s)K(s) \qquad (80)$$

If the moduli of the diagonal elements of K (for some specified region in the complex plane) all become large, then the moduli of all the elements of $G(s)K(s)H(s)$ will (in the same sense) become large, and thus

$$\{I_m + G(s)K(s)H(s)\} \to G(s)K(s)H(s) \quad . \quad . \quad (81)$$

so that

$$R \to \{G(s)K(s)H(s)\}^{-1}G(s)K(s) = H^{-1}(s) \quad . \quad . \quad (82)$$

Thus, if $H(s)$ is a diagonal matrix,

$$R \to \text{a diagonal matrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

so that

$$r_{ii}^{-1}(s) \to \hat{r}_{ii}(s)\ i = 1, 2, \ldots, m \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Thus, when sufficiently high values of static gain exist in all loops, and for sufficiently low frequencies, $\hat{r}_{ii}(s)$ approximates to $r_{ii}^{-1}(s)$. This means that we can use the diagonal elements of $\hat{R}(s)$ as an approximation to the inverses of the elements of the closed-loop transfer-function matrix $R(s)$, and thus, via eqn. 78, we can use the elements of $\hat{Q}(s)$ to design for a desired closed-loop response in terms of inverse Nyquist plots for the diagonal elements of $R(s)$ which define the system behaviour with the m principal loops closed.

For these reasons the relationship is required, in the general case, between the inverses of the elements of $R(s)$ and the elements of the inverse. Standard matrix formulas give

$$r_{ii}(s) = \frac{\hat{R}_{ii}(s)}{\det \hat{R}(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

so that

$$r_{ii}^{-1}(s) = \frac{\det \hat{R}(s)}{\hat{R}_{ii}(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Expanding $\det \hat{R}(s)$ by the ith column gives

$$\det \hat{R}(s) = \sum_{j=1}^{m} \hat{r}_{ji}(s)\hat{R}_{ji}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

Substituting eqn. 87 in eqn. 86 then gives

$$r_{ii}^{-1}(s) = \hat{r}_{ii}(s) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \hat{r}_{ji}(s)\frac{\hat{R}_{ji}(s)}{\hat{R}_{ii}(s)} \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Alternatively, $\det \hat{R}(s)$ may be expanded by the ith row to obtain

$$r_{ii}^{-1}(s) = \hat{r}_{ii}(s) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \hat{r}_{ij}(s)\frac{\hat{R}_{ij}(s)}{\hat{R}_{ii}(s)} \quad . \quad . \quad . \quad . \quad . \quad (89)$$

10.5 Inverse mappings and inverse Nyquist diagrams

The discussion in Sections 10.3 and 10.4 has shown that the properties of $\hat{Q}$ and $\hat{R}$ are of great importance in design studies, and we are thus led to consider inverse mappings in the complex plane.

The complex plane mapping $z \to z'$ where

$$z' = \frac{1}{z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

will be called an inversion. Since

$$|z'| = \frac{1}{|z|} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

and $\quad \arg z' = -\arg z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$

it is equivalent to a reflection in the real axis followed by a further reflection in the unit circle where 'reflection' is interpreted to ensure that eqns. 90 and 92 are satisfied. Since the inversion mapping is conformal, any stability condition deduced from a locus in the complex plane may be equally well deduced from the corresponding locus in the inverse plane. In particular, the Nyquist diagram and inverse Nyquist diagram are related as shown in Fig. 10. Gain margin and phase margin are related to the inverse Nyquist diagram as shown, and encirclement criteria for stability are easily translated from one diagram to the other.

10.6 Simple interpretation of Rosenbrock's indirect-stability criteria in the inverse Nyquist diagram

It has been shown above that, if $\hat{Q}(s)$ and $\hat{R}(s)$ are diagonal-dominant, then, for mappings of the contour D, the

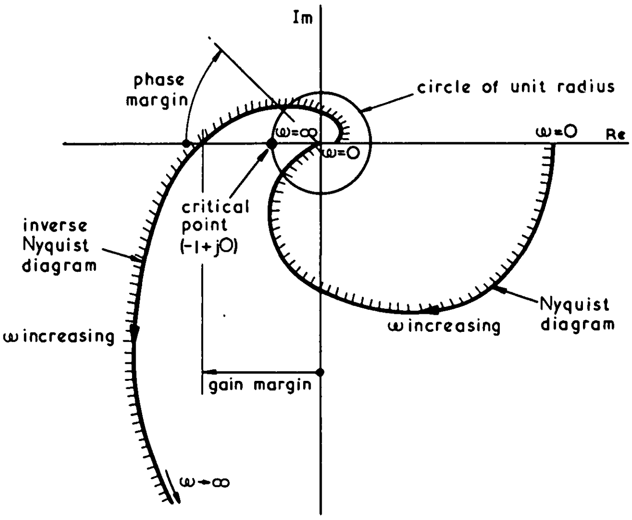

Fig. 10
Inverse Nyquist diagram

origin encirclements of the determinants of $\hat{Q}(s)$ and $\hat{R}(s)$ are equal to the sums of the encirclements by the mappings of their principal diagonal elements.

It has been further shown that $\hat{Q}(s)$ and $\hat{R}(s)$ are related in such a simple way, namely by eqn. 78, that the properties of the diagonal elements of $\hat{R}(s)$ can be immediately deduced from those of $\hat{Q}(s)$. We would thus expect, in the majority of situations of practical importance, to be able to deduce the stability for situations in which $\hat{Q}(s)$ is diagonal-dominant by an inspection of the inverse Nyquist plots $\hat{q}_{ii}(j\omega)$ ($i = 1, 2, \ldots, m$).

Suppose neither $\det G(s)$ nor $\det K(s)$ has any right-half-plane zeros. Then neither will $\det Q(s)$, since

$$\det Q(s) = \det G(s) \det K(s) \quad . \quad . \quad . \quad . \quad . \quad (93)$$

In these circumstances, for a diagonal-dominated $\hat{R}(s)$, a

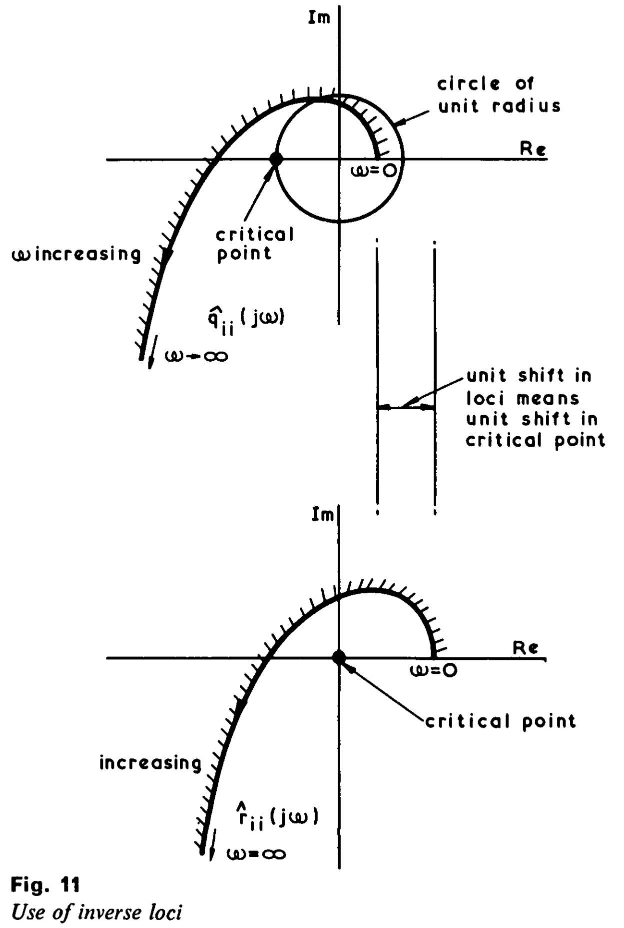

Fig. 11
Use of inverse loci

sufficient criterion for stability is that $\hat{q}_{ii}(j\omega)(i = 1, 2, \ldots, m)$ all satisfy a simple inverse Nyquist stability criterion with respect to the normal critical point $(-1, 0)$. This is illustrated by Fig. 11.

With $H = I_m$,

$$\hat{r}_{ii}(j\omega) = 1 + \hat{q}_{ii}(j\omega) \qquad i = 1, \ldots, m \quad . \quad . \quad (94)$$

Thus, if the origin of the complex plane always lies to the left of $\hat{r}_{ii}(j\omega)$ as ω increases from $0 \rightarrow \infty$, the critical point $(-1, 0)$ will always lie to left of $\hat{q}_{ii}(j\omega)$ as ω increases from $0 \rightarrow \infty$ for $i = 1, 2, \ldots, m$, as is illustrated in Fig. 11.

10.7 Outline of Rosenbrock's inverse-Nyquist-array design technique[2]

The essence of the material contained in the last few Sections is that if the open-loop transfer-function matrix is diagonal-dominated, the m principal loops can be designed using single-loop frequency-response techniques. Furthermore, it is useful to work in terms of inverse transfer functions in inverse Nyquist diagrams, since open- and closed-loop behaviour can then be related in a simple and efficient way.

Let the controller $K(s)$ be synthetised in three successive stages, which will be enlarged on below, so that

$$K(s) = K_a K_b(s) K_c(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Thus $\hat{Q}(s) = \{G(s)K_a K_b(s) K_c(s)\}^{-1}$

$$= \hat{K}_c(s)\hat{K}_b(s)\hat{K}_a\hat{G}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

and $\hat{R}(s) = H + \hat{K}_c(s)\hat{K}_b(s)\hat{K}_a\hat{G}(s) \quad . \quad . \quad . \quad . \quad . \quad (97)$

where, as previously, $\hat{K}$ denotes the inverse of K.

10.7.1 Inverse Nyquist array[2]

The set of m^2 loci in the complex plane corresponding to the entries of $\hat{G}(j\omega)$ is called the inverse Nyquist array.

The design method proposed by Rosenbrock[2] starts with eqn. 97. $\hat{K}_a$ is a combination of permutation operations such that, from the controller viewpoint, the ith input is regulated from the ith output for $i = 1, 2, \ldots, m$, and scaling operations such that inputs and outputs are related by convenient units.

A suitable $\hat{K}_a$ is selected, and $\hat{K}_b(s)$ is chosen to make $\hat{Q}(s)$ diagonal-dominant. It is now possible to determine the system closed-loop stability from the diagonal entries of $\hat{Q}(j\omega)$ alone.

The final stages of the design process are concerned with choosing the elements of the diagonal matrix $\hat{K}_c(s)$ which synthetises the m principal control loops. This can be done using conventional single-loop inverse-Nyquist-diagram techniques. As explained in Section 10.4, if the loops all have a sufficiently high gain, $\hat{r}_{ii}(j\omega)$ is a good approximation to $r_{ii}^{-1}(j\omega)$ over frequency ranges for which high gains are present in all the loops, and so the actual closed-loop behaviour can be shaped at this stage in the design by shaping $\hat{r}_{ii}(j\omega)$ $i = 1, \ldots, m$. Eqns. 88 and 89 can then be used for a check on the closed-loop response using the final values of $\hat{K}(j\omega)$ entries over the full operating-frequency range of the system.

10.8 Extension of return-difference stability theorem to nonlinear case via describing-function techniques

If nonlinearities are present which can be represented by describing functions,[11] then, under suitable conditions, the stability results of Section 10.1 can be extended to deal with nonlinear system stability.

Consider the system of Fig. 12 where $n_1(a_1)$, $n_2(a_2)$, ... $n_m(a_m)$ are frequency-independent describing functions for the set of nonlinearities shown. Suppose the plant frequency-response matrix $G(j\omega)$ is such that, as in the usual describing function hypothesis,[11] each plant output may be accurately represented by its first harmonic. Thus a sinusoidal vector input to the controller frequency-response matrix $K(j\omega)$ produces a sinusoidal response at the plant output.

For a sustained oscillation of frequency ω and of constant amplitudes $a_1, a_1, \ldots, a_m$ at the inputs to the nonlinearities, we shall have

$$G(j\omega)N(a)K(j\omega)H = -I_m \quad . \quad . \quad . \quad . \quad . \quad (98)$$

where $N(a) = \text{diag}\{n_1(a_1), n_2(a_2), \ldots, n_m(a_m)\} \quad . \quad . \quad (99)$

so that we shall have

$$\{I_m + G(j\omega)N(a)K(j\omega)H\} = 0$$

and thus

$$\det\{I_m + G(j\omega)N(a)K(j\omega)H\} = 0 \quad . \quad . \quad . \quad . \quad (100)$$

or $\quad \det F(a, j\omega) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$

where $K(a, j\omega)$ is the appropriately evaluated return-difference matrix and a is a vector with elements $a_1, a_2, \ldots, a_m$.

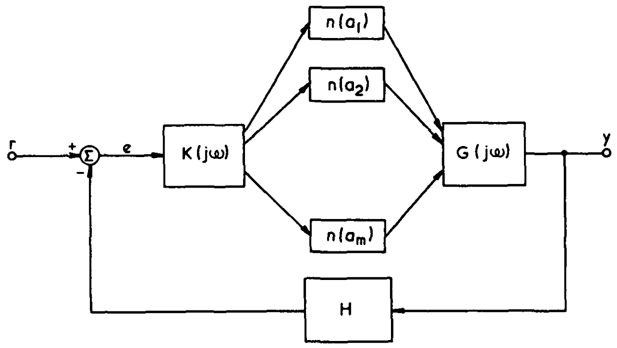

Fig. 12
Describing-function system

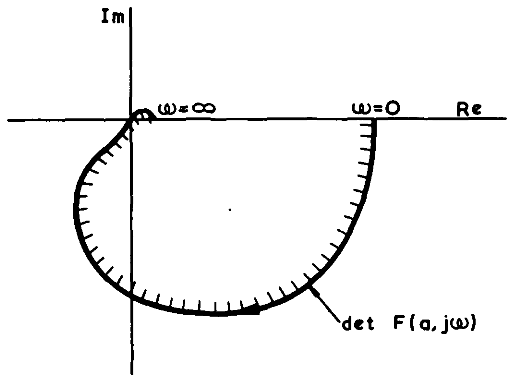

Fig. 13
Instability criterion for nonlinear system

We now argue heuristically that no oscillation will exist if there is no (a, ω) such that $\det F(a, j\omega)$ vanishes; i.e. such that there is no a for which the corresponding locus $\det F(a, j\omega)$ passes through the origin as shown in Fig. 13.

Now put

$$F(a, s) = \{I_m + G(s)N(a)K(s)H\} \quad . \quad . \quad . \quad . \quad (102)$$

and suppose that the following conditions are satisfied:

(i) $F(a, s)$ is diagonal-dominated for all a and for all s on the contour D.
(ii) None of $f_{ii}(a, j\omega)$ for $i = 1, 2, \ldots, m$ enclose the origin of the complex plane.

Then, by a straightforward extension of the argument used in the linear case, there will be no value of a so that $\det F(a, j\omega)$ vanishes.

Put

$$T(a, s) = G(s)N(a)K(s)H \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

If none of the $f_{ii}(a, j\omega)$ for $i = 1, 2, \ldots, m$ enclose the origin of the complex plane, then none of the diagonal elements $t_{ii}(a, j\omega)$ enclose the usual critical point $(-1 + j0)$ in the complex plane. This, however, is simply the usual describing-function stability criterion for the m principal loops considered as individual single loops.[11] Thus, if condition (i) above is satisfied, and the m principal loops are stable by the usual describing-function method for single loops considered one at a time, the nonlinear system of Fig. 12 will be stable with all loops closed.

11 Discrete-time systems

The complex-plane analysis and design techniques developed in the above Sections for continuous-time systems may be extended to deal with discrete-time systems. Most of the extensions are very straightforward, since the algebraic relationships involved are essentially the same.[4]

Discrete-time return-difference and return-ratio matrices are defined exactly as for the continuous-system case.

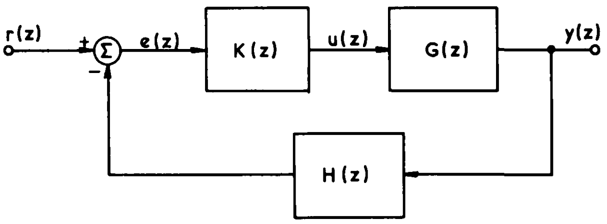

Fig. 14
Discrete vector feedback system

For the feedback control system of Fig. 14 let

$r(z) = m \times 1$ matrix of reference input z transforms
$e(z) = m \times 1$ matrix of error z transforms
$y(z) = m \times 1$ matrix of plant output z transforms
$u(z) = r \times 1$ matrix of plant input z transforms
$K(z) = r \times m$ matrix of controller discrete-time transfer functions
$G(z) = m \times r$ matrix of plant discrete-time transfer functions
$H(z) = m \times m$ matrix of feedback transducer discrete-time transfer functions

The closed-loop-system discrete-time transfer-function matrix $M(z)$ is given by

$$M(z) = \{I_m + G(z)K(z)H(z)\}^{-1}G(z)K(z) \quad . \quad . \quad (104)$$

$$= F^{-1}(z)G(z)K(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

where $F(z) = \{I_m + G(z)K(z)H(z)\}$ . . . . . (106)

is the system discrete-time return-difference matrix. A discrete-time return-ratio matrix $T(z)$ may be defined as

$$T(z) = G(z)K(z)H(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

so that

$$F(z) = I_m + T(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

The fundamental relationship between open-loop- and closed-loop-system behaviour is derived in Appendix 15.1 and is

$$\det F(z) = \frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}} \quad (109)$$

11.1 Complex-plane mapping criterion for discrete-time multiple-loop-system stability in terms of the return-difference matrix

Since (in Appendix 15.1) we have assumed that the system is open-loop stable, the open-loop characteristic polynomial will have no zeros outside the unit disc in the complex plane. (By the unit disc in the complex plane is meant the interior of the disc bounded by the circle of unit modulus in the complex plane.) Thus, it follows from eqn. 109 that the closed-loop characteristic polynomial will not vanish outside the unit disc in the complex plane if, and only if, $\det F(s)$ does not vanish outside the unit disc in the complex plane. Let D be the contour in the complex plane shown in Fig. 15. This consists of a unit-modulus circle and a circle of radius α, both centred at the origin of the complex plane, joined by a double path along the real axis as shown. Further, let α be large enough to ensure that every zero and pole of $\det Q(z)$ and $\det R(z)$ which lies outside the unit disc in the complex plane lies within D.

Suppose D maps into a closed curve Γ in the complex plane under the mapping $\det F(z)$. Then the system is closed-loop stable if no point within D maps on to the origin of the complex plane under the mapping $\det F(z)$.

Thus the system is closed-loop stable if Γ does not enclose the origin of the complex plane. If $|\det F(z)| \to 1$ as $|z| \to \infty$, then, taking α as arbitrarily large, we can conveniently refer to Γ as the mapping of the unit-modulus circle in the complex plane. This gives the discrete-system multiple-loop complex-plane mapping criterion for stability shown in Fig. 16.

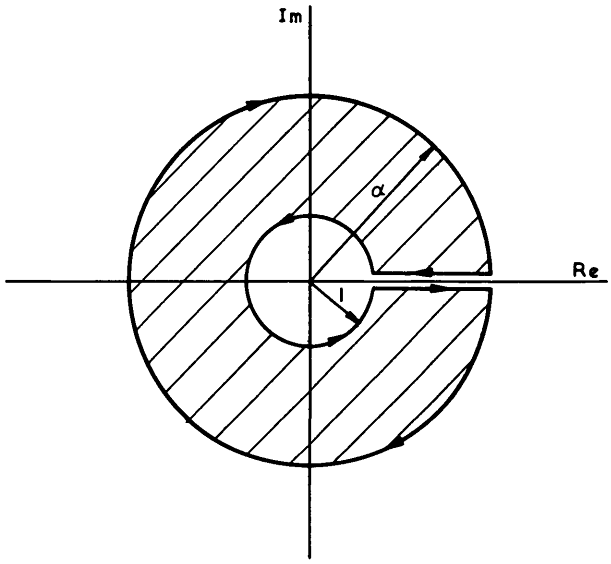

Fig. 15
Test contour in zplane

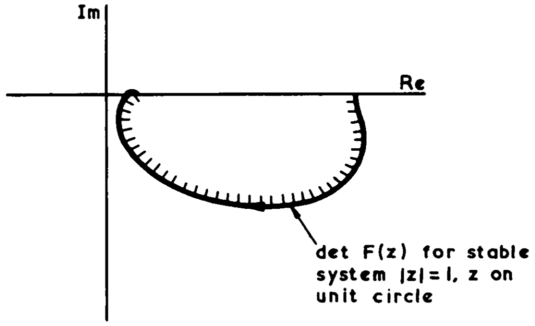

Fig. 16
Simple discrete-system stability criterion

Let the eigenvalues of $F(z)$ be $\{\rho_j(z): j = 1, 2, \ldots, m\}$. We then have that

$$\det F(z) = \prod_{j=1}^{m} \rho_j(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Therefore, $\det F(z)$ will not vanish for any z enclosed by D if none of $\{\rho_j(z): j = 1, 2, \ldots, m\}$ vanish for any z enclosed by D. Let D map into Δ_j in the complex plane under $\{\rho_j(z): j = 1, 2, \ldots, m\}$. Then Γ will not enclose the origin of the complex plane if none of Δ_j enclose the origin of the complex plane for $j = 1, 2, \ldots, m$. Thus the system will be stable with all loops closed if none of Δ_j enclose the origin of the complex plane for $j = 1, 2, \ldots, m$, giving the following result.

11.1.1 Fundamental stability property of complex-plane loci for discrete-time system return-difference-matrix eigenvalues

The system is closed-loop stable if all the eigenvalue loci $\rho_j(z)$, when z runs round the unit-modulus circle in the

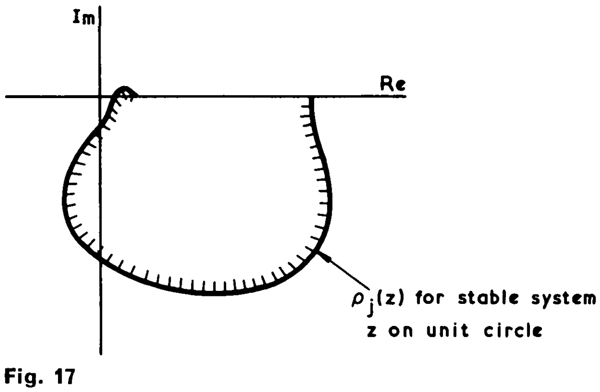

Fig. 17
Characteristic frequency response for discrete system

complex plane, satisfy the mapping criterion illustrated in Fig. 17.

This criterion can be equally well stated in terms of the return-ratio matrix. Since

$$F(z) = I_m + T(z) \qquad (111)$$

if $\{\nu_j(z): j = 1, 2, \ldots, m\}$ are the eigenvalues of $T(z)$, then

$$\rho_j(z) = 1 + \nu_j(z) \qquad j = 1, 2, \ldots, m \qquad (112)$$

In terms of the return-ratio matrix, therefore, we simply obtain a unit shift in the location of the critical point.

11.2 Extension to discrete-time case of indirect stability theorems and design techniques

The concepts of diagonal-dominance in indirect stability theorems, and of ideal commutative control, may now be extended to the discrete-time case in an obvious way. As in the case of single-loop theory, the unit-modulus circle in the complex plane plays the role in the discrete-time case which is played by the imaginary axis in the continuous-time case.[16]

12 Conclusions

The return-difference and return-ratio matrices play a central role in multivariable-feedback-control theory. Regarding matrix transfer functions as operators on linear vector spaces over the field of rational fractions in s leads to the concept of characteristic transfer functions which are the eigenvalues of such operators. The corresponding characteristic frequency responses then provide a simple and natural link between classical single-loop techniques and multivariable-control theory. This concept then serves as a unifying thread in a coherent and systematic discussion of multivariable-feedback-system design techniques.

13 Acknowledgments

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15 Appendix

15.1 Fundamental relationship between open-loop- and closed-loop-system behaviour

The continuous-time system relationship given by eqn. 5 may be deduced by means of the arguments presented by Rosenbrock[2] and Hsu and Chen.[12] The proof given by Rosenbrock is adapted here to derive the relationship for the discrete-time case:

$$\text{Let} \quad G(z)K(z) = Q(z) \qquad (113)$$

$$\text{and} \quad H(z) = H \qquad (114)$$

where H is a diagonal matrix of real constants, and consider the system shown in Fig. 18. $Q(z)$ is thus the discrete-time

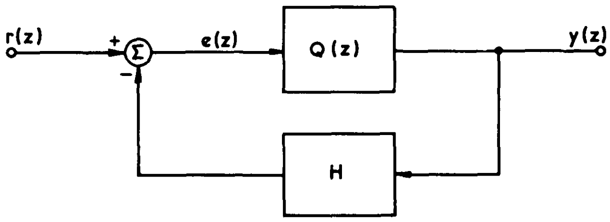

Fig. 18
Simplified discrete system

transfer-function matrix of the cascaded controller and plant, and there is no dynamic action in the feedback transducers.

Let the system in the box with discrete-time transfer-function matrix $Q(z)$ be such that its behaviour is defined by the discrete-time transformed equation set

$$X(z)q(z) = U(z)e(z) \qquad (115)$$

$$y(z) = V(z)q(z) + W(z)e(z) \qquad (116)$$

where $X(z)$, $U(z)$, $V(z)$ and $W(z)$ are polynomial matrices. Eqns. 115 and 116 can be thought of as constituting a physical description of the process, and would be obtained by taking z transforms of the difference equations defining the physical behaviour of plant and controller. From eqns. 115 and 116, we obtain

$$Q(z) = V(z)X^{-1}(z)U(z) + W(z) \qquad (117)$$

$$e(z) = r(z) - H(z)y(z) \qquad (118)$$

In what follows, we assume that det $Q(z)$ has no zero on any finite part of the unit circle in the complex plane, that det $X(z)$ is not identically zero (i.e. zero for all values of z), and that the system represented by eqns. 115 and 116 is asymptotically stable.

From eqns. 115, 116 and 118, we have

$$\begin{bmatrix} X(z) & U(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & H \end{bmatrix} \begin{bmatrix} -q(z) \\ e(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(z) \end{bmatrix} \qquad (119)$$

so that, for closed-loop behaviour,

$$\begin{bmatrix} -q(z) \\ e(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} X(z) & V(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & H \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ r(z) \end{bmatrix} \qquad (120)$$

$$= P_c^{-1}(z) \begin{bmatrix} 0 \\ 0 \\ r(z) \end{bmatrix} \qquad (121)$$

where the matrix $P_c(z)$ is given by

$$P_c(z) = \begin{bmatrix} X(z) & U(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & H \end{bmatrix} \qquad (122)$$

Since all the constituent matrices out of which $P_c(z)$ has been assembled are polynomial matrices, $P_c(z)$ will be a polynomial matrix. It follows from this that a necessary and sufficient condition for the closed-loop system to be stable is that all the roots of the equation

$$\det P_c(z) = 0 \qquad (123)$$

lie in the interior of the unit disc in the complex plane. Written out in full, eqn. 123 is

$$\det \begin{bmatrix} X(z) & U(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & H \end{bmatrix} = 0 \qquad (123a)$$

and thus the open-loop situation, which is obtained by putting the feedback matrix $H = 0$, gives

$$\det \begin{bmatrix} X(z) & U(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & 0 \end{bmatrix} = \det X(z) = 0 \qquad (124)$$

Since the open-loop system is asymptotically stable (by assumption), all the roots of eqn. 124 will lie in the interior of the unit disc in the complex plane.

Let the discrete-time transfer-function matrix for the closed-loop system be $R(z)$, so that

$$y(z) = R(z)r(z) \qquad (125)$$

In the manipulations which follow, it is helpful to use a formula which gives the minors of an inverse matrix in terms of the minors of the original matrix, and this formula is accordingly quoted here.

Minors of inverse-matrix formula: If $B = A^{-1}$ the minors B are expressible in terms of the minors of A as follows:

$$B\begin{pmatrix} i_1 i_2 \ldots i_p \\ k_1 k_2 \ldots k_p \end{pmatrix} = \frac{(-1)^{\sum_{v=1}^{p} i_v + \sum_{v=1}^{p} k_v} A\begin{pmatrix} k'_1 k'_2 \ldots k'_{n-p} \\ i'_1 i'_2 \ldots i'_{n-p} \end{pmatrix}}{A\begin{pmatrix} 1\,2 \ldots n \\ 1\,2 \ldots n \end{pmatrix}} \qquad (126)$$

where $i_1 < i_2 < \ldots < i_p$ and $i'_1 < i'_2 < \ldots < i'_{n-p}$

and $k_1 < k_2 < \ldots < k_p$ and $k'_1 < k'_2 < \ldots < i'_{n-p}$

form complete sets of indices $1, 2, \ldots, n$.

An inspection of eqns. 120, 121 and 125 shows that $R(z)$ is obtained from $P_c^{-1}(z)$ by striking out the first $r + m$ rows and columns of $P_c^{-1}(z)$. Therefore $\det R(z)$ is the minor formed from the elements of $P_c^{-1}(z)$ in rows $(r + m + 1) \ldots (r + 2m)$ and columns $(r + m + 1) \ldots (r + 2m)$. Eqn. 126 then gives

$$P_c^{-1}\begin{pmatrix} (r+m+1) \ldots (r+m+m) \\ (r+m+1) \ldots (r+m+m) \end{pmatrix} = \frac{P_c\begin{pmatrix} 12 \ldots (r+m) \\ 12 \ldots (r+m) \end{pmatrix}}{P_c\begin{pmatrix} 12 \ldots r+2m \\ 12 \ldots r+2m \end{pmatrix}} \qquad (127)$$

so that

$$\det R(z) = \frac{\det \begin{bmatrix} X(z) & U(z) \\ -V(z) & W(z) \end{bmatrix}}{\det \begin{bmatrix} X(z) & U(z) & 0 \\ -V(z) & W(z) & -I_m \\ 0 & I_m & H \end{bmatrix}} \qquad (128)$$

Since the open-loop system is obtained by putting $H = 0$, we can obtain $\det Q(z)$ from $\det R(z)$ by putting $H = 0$ in eqn. 128. This gives

$$\det Q(z) = \frac{\det \begin{bmatrix} X(z) & U(z) \\ -V(z) & W(z) \end{bmatrix}}{\det X(z)} \qquad (129)$$

Now the block diagram of Fig. 18 shows that

$$R(z) = \{I_m + Q(z)H\}^{-1}Q(z) \qquad (130)$$

so that

$$\det R(z) = [\det \{I_m + Q(z)H\}^{-1}]\{\det Q(z)\} \qquad (131)$$

$$= \frac{\det Q(z)}{\det \{I_m + Q(z)H\}} \qquad (132)$$

since the determinant of an inverse is the inverse of the determinant.

Thus, expressing the return-difference matrix as usual in the form $F(z)$, we have

$$F(z) = \{I_m + Q(z)H\} \qquad (133)$$

and thus

$$\det F(z) = \frac{\det Q(z)}{\det R(z)} \qquad (134)$$

Now the characteristic equations of the open- and closed-loop systems are given by

$$\det P_c(z) = \text{closed-loop characteristic equation} \qquad (135)$$

$$\det X(z) = \text{open-loop characteristic equation} \qquad (136)$$

Dividing both sides of eqn. 129 by the corresponding sides of eqn. 128 gives

$$\frac{\det Q(z)}{\det R(z)} = \frac{\det P_c(z)}{\det X(z)} \qquad (137)$$

Thus, using eqns. 134, 135 and 136, we have

$$\det F(z) = \frac{\det P_c(z)}{\det X(z)}$$

$$= \frac{\text{closed-loop characteristic polynomial}}{\text{open-loop characteristic polynomial}} \qquad (138)$$

This is the fundamental equation relating the open- and closed-loop behaviour of discrete-time multiple-loop control systems. The above argument applies, with the obvious minor modifications, to the continuous-time case.