APPENDIX B

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs)

The DF is given by (cf. Sec. 2.2)

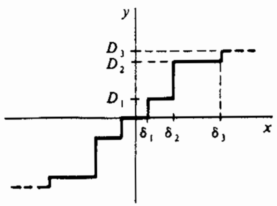
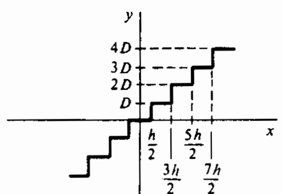
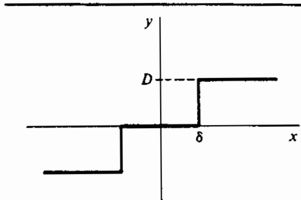
$$N(A,\omega) = n_p(A,\omega) + jn_q(A,\omega) = \frac{j}{\pi A}\int_0^{2\pi} y(A\sin\psi, A\omega\cos\psi)e^{-j\psi}\,d\psi$$

In this table we employ the "saturation function" (cf. Sec. 2.3) denoted by

$$\begin{aligned} f(\gamma) &= -1 & \gamma &< -1 \\ &= \frac{2}{\pi}\left(\sin^{-1}\gamma + \gamma\sqrt{1-\gamma^2}\right) & |\gamma| &\le 1 \\ &= 1 & \gamma &> 1 \end{aligned}$$

This function is plotted in Fig. C.1.

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
	$A < \delta_1$	$n_p = 0$ $n_q = 0$
1. General odd quantizer	$\delta_{n+1} > A > \delta_n$	$n_p = \dfrac{4}{\pi A}\sum_{i=1}^{n}(D_i - D_{i-1})\sqrt{1-\left(\dfrac{\delta_i}{A}\right)^2}$ $n_q = 0$
	$A < \dfrac{h}{2}$	$n_p = 0$ $n_q = 0$
2. Uniform quantizer or granularity	$\dfrac{2n+1}{2}h > A > \dfrac{2n-1}{2}h$ See Fig. B.1 and Sec. 2.3	$n_p = \dfrac{4D}{\pi A}\sum_{i=1}^{n}\sqrt{1-\left(\dfrac{2i-1}{2}\dfrac{h}{A}\right)^2}$ $n_q = 0$
	$A < \delta$	$n_p = 0$ $n_q = 0$
3. Relay with dead zone	$A > \delta$ See Fig. B.1	$n_p = \dfrac{4D}{\pi A}\sqrt{1-\left(\dfrac{\delta}{A}\right)^2}$ $n_q = 0$

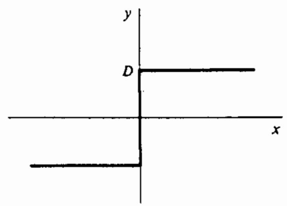
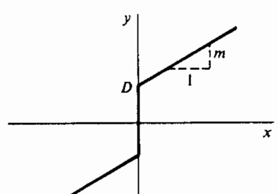
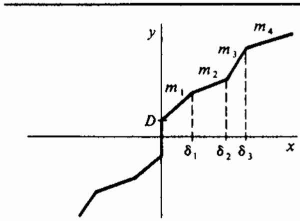
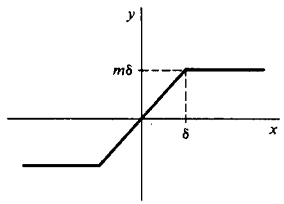
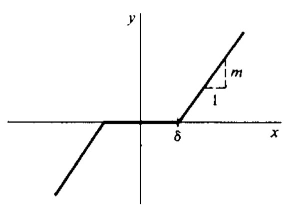
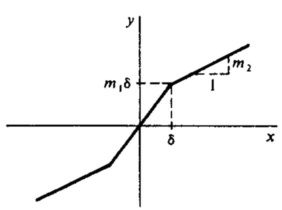
 4. Ideal relay	See Sec. 2.3	$n_p = \dfrac{4D}{\pi A}$ $n_q = 0$
 5. Preload	See Sec. 2.3	$n_p = \dfrac{4D}{\pi A} + m$ $n_q = 0$
 6. General piecewise-linear odd memoryless nonlinearity	$\delta_1 > A > 0$ $\delta_2 > A > \delta_1$ or, in a form valid for all A, $(\delta_{n+1} > A > \delta_n)$ See Sec. 2.3	$n_p = \dfrac{4D}{\pi A} + m_1$ $n_q = 0$ $n_p = \dfrac{4D}{\pi A} + (m_1 - m_2) f\left(\dfrac{\delta_1}{A}\right) + m_2$ $n_q = 0$ $n_p = \dfrac{4D}{\pi A} + \sum_{i=1}^{n} (m_i - m_{i+1}) f\left(\dfrac{\delta_i}{A}\right) + m_{n+1}$ $n_q = 0$

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
 7. Saturation or limiter	See Fig. B.2 and Sec. 2.3	$n_p = mf\left(\frac{\delta}{A}\right)$ $n_q = 0$
 8. Dead zone or threshold	See Fig. B.2 and Sec. 2.3	$n_p = m\left[1 - f\left(\frac{\delta}{A}\right)\right]$ $n_q = 0$
 9. Gain-changing nonlinearity	See Sec. 2.3	$n_p = (m_1 - m_2)f\left(\frac{\delta}{A}\right) + m_2$ $n_q = 0$

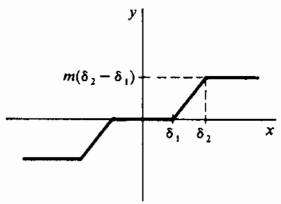
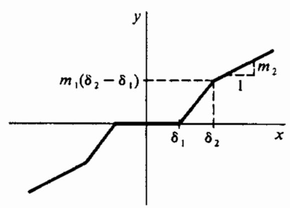
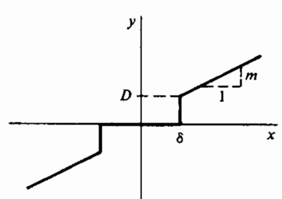
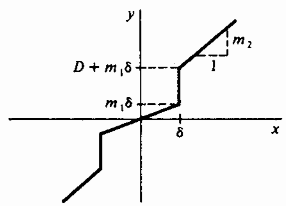
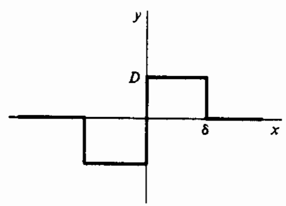
 10. Limiter with dead zone	See Sec. 2.3	$n_p = m\left[f\left(\frac{\delta_2}{A}\right) - f\left(\frac{\delta_1}{A}\right)\right]$ $n_q = 0$
 11. Gain-changing nonlinearity with dead zone		$n_p = -m_1 f\left(\frac{\delta_1}{A}\right) + (m_1 - m_2) f\left(\frac{\delta_2}{A}\right) + m_2$ $n_q = 0$
 12.	$A < \delta$ $A > \delta$	$n_p = 0$ $n_q = 0$ $n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2} + m\left[1 - f\left(\frac{\delta}{A}\right)\right]$ $n_q = 0$

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
 13.	$A < \delta$ $A > \delta$	$n_p = m_1$, $n_q = 0$ $n_p = (m_1 - m_2)f\left(\frac{\delta}{A}\right) + m_2 + \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2}$, $n_q = 0$
 14.	$A < \delta$ $A > \delta$	$n_p = \frac{4D}{\pi A}$, $n_q = 0$ $n_p = \frac{4D}{\pi A}\left[1 - \sqrt{1 - \left(\frac{\delta}{A}\right)^2}\right]$, $n_q = 0$
$y = c$ 15.		$n_p = 0$ $n_q = 0$
$y = x$ 16. Linear gain		$n_p = 1$ $n_q = 0$

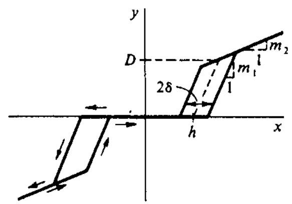
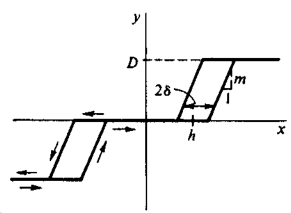
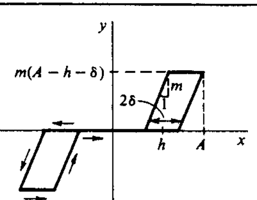
$y = x\,\lvert x\rvert$ 17. Odd square law	See Fig. B.3	$n_p = \dfrac{8}{3\pi} A$ $n_q = 0$
$y = x^3$ 18. Cubic characteristic	See Fig. B.3	$n_p = \dfrac{3}{4} A^2$ $n_q = 0$
$y = x^3\,\lvert x\rvert$ 19. Odd quartic characteristic		$n_p = \dfrac{32}{15\pi} A^3$ $n_q = 0$
$y = x^5$ 20. Quintic characteristic		$n_p = \dfrac{5}{8} A^4$ $n_q = 0$
$y = x^5\,\lvert x\rvert$ 21.		$n_p = \dfrac{64}{35\pi} A^5$ $n_q = 0$
$y = x^7$ 22.		$n_p = \dfrac{35}{64} A^6$ $n_q = 0$

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
$y = x^7 \lvert x \rvert$ 23.		$n_p = \dfrac{512}{315\pi} A^7$ $n_q = 0$
$y = x^n$ 24.	$n = 3, 5, 7, \ldots$ See Fig. B.3 and Sec. 2.3	$n_p = \dfrac{n(n-2)(n-4)\cdots(3)}{(n+1)(n-1)(n-3)\cdots(4)} A^{n-1}$ $n_q = 0$
$y = x^{n-1} \lvert x \rvert$ 25.	$n = 2, 4, 6, \ldots$ See Fig. B.3 and Sec. 2.3	$n_p = \dfrac{4}{\pi} \dfrac{n(n-2)(n-4)\cdots(2)}{(n+1)(n-1)(n-3)\cdots(3)} A^{n-1}$ $n_q = 0$
$y = \sqrt{x} \quad (x \geq 0)$ $= -\sqrt{-x} \quad (x < 0)$ 26. Odd square root	See Fig. B.3	$n_p = 1.11\, A^{-1/2}$ $n_q = 0$
$y = x^{1/3}$ 27. Cube root characteristic		$n_p = 1.16\, A^{-2/3}$ $n_q = 0$
$y = x^b \quad (x \geq 0)$ $= -(-x)^b \quad (x < 0)$ 28.	$b > -2$ Γ(arg.) is the gamma function. See Sec. 2.3	$n_p = \dfrac{2}{\sqrt{\pi}} \dfrac{\Gamma\left(\dfrac{b+2}{2}\right)}{\Gamma\left(\dfrac{b+3}{2}\right)} A^{b-1}$ $n_q = 0$

$y = M \sin mx$ 29. Harmonic nonlinearity	$J_1(mA)$ is the Bessel function of order 1 for real arguments. See Fig. B.4 and Sec. 2.3	$n_p = 2M \dfrac{J_1(mA)}{A}$ $n_q = 0$
$y = M \sinh mx$ 30.	$I_1(mA)$ is the modified Bessel function of order 1.	$n_p = 2M \dfrac{I_1(mA)}{A}$ $n_q = 0$
$y = 1 - e^{-cx} \quad (x \geq 0)$ $= -(1 - e^{cx}) \quad (x < 0)$ 31. Exponential saturation	$I_1(cA)$ is the modified Bessel function of order 1 and $S_1(cA)$ is the modified Struve function of order 1. See Ref. 25 of Chap. 2	$n_p = \dfrac{2}{A} [I_1(cA) - S_1(cA)]$ $n_q = 0$
$y = \dfrac{cx}{\sqrt{1 + (cx)^2}}$ 32. Algebraic saturation	$K(k)$ and $E(k)$ are the elliptic integrals of first and second kind, respectively.	$n_p = \dfrac{4}{\pi c A^2}\left[-\dfrac{1}{\sqrt{1+(cA)^2}} K\left(\dfrac{cA}{\sqrt{1+(cA)^2}}\right) + \sqrt{1+(cA)^2}\, E\left(\dfrac{cA}{\sqrt{1+(cA)^2}}\right)\right]$ $n_q = 0$
$y(t) = x(t - T_d)$ 33. Time delay	This phenomenon is linear, hence the *DF* is exactly the transfer function $\exp(-j\omega T_d)$.	$n_p = \cos \omega T_d$ $n_q = -\sin \omega T_d$

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
 34.	$m_1 > m_2$ $A \geq h + \frac{D}{m_1} + \frac{\delta m_1}{m_1 - m_2}$	$n_p = m_2 - \frac{m_1}{2}\left[f\left(\frac{h+\delta}{A}\right) + f\left(\frac{h-\delta}{A}\right)\right] + \left(\frac{m_1 - m_2}{2}\right)\left[f\left(\frac{h + D/m_1 + \delta m_1/(m_1 - m_2)}{A}\right) + f\left(\frac{h + D/m_1 - \delta m_1/(m_1 - m_2)}{A}\right)\right]$ $n_q = -\frac{4D\delta}{\pi A^2}$
 35.	$A > h + \frac{D}{m} + \delta$	$n_p = \frac{m}{2}\left[-f\left(\frac{h+\delta}{A}\right) + f\left(\frac{h+\delta+D/m}{A}\right) + f\left(\frac{h-\delta+D/m}{A}\right) - f\left(\frac{h-\delta}{A}\right)\right]$ $n_q = -\frac{4D\delta}{\pi A^2}$
 36.	$A > h + \delta$	$n_p = \frac{m}{2}\left[1 + f\left(1 - \frac{2\delta}{A}\right) - f\left(\frac{h+\delta}{A}\right) - f\left(\frac{h-\delta}{A}\right)\right]$ $n_q = -\frac{4\delta m}{\pi A^2}(A - h - \delta)$

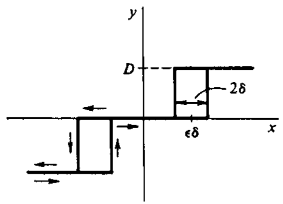
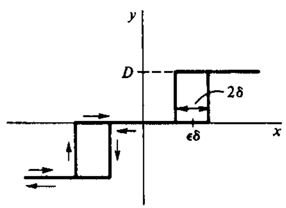
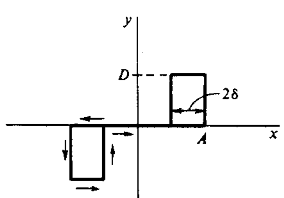
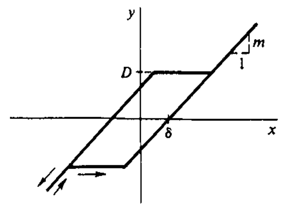
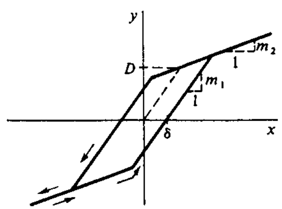
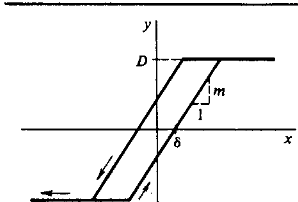
	$A < \delta(1+\epsilon)$	$n_p = 0$ $n_q = 0$
	$A > \delta(1+\epsilon)$	$n_p = \dfrac{2D}{\pi A}\left[\sqrt{1-\left(\dfrac{\delta}{A}\right)^2(1-\epsilon)^2} + \sqrt{1-\left(\dfrac{\delta}{A}\right)^2(1+\epsilon)^2}\right]$ $n_q = -\dfrac{4D\delta}{\pi A^2}$
37. (positive) Hysteresis	See Fig. B.5 and Sec. 2.3	
	$A < \delta(\epsilon-1)$	$n_p = 0$ $n_q = 0$
	$\delta(\epsilon-1) < A < \delta(\epsilon+1)$	$n_p = \dfrac{4D}{\pi A}\sqrt{1-\left(\dfrac{\delta}{A}\right)^2(1-\epsilon)^2}$ $n_q = 0$
38. (negative) Hysteresis	$A > \delta(\epsilon+1)$	$n_p = \dfrac{2D}{\pi A}\left[\sqrt{1-\left(\dfrac{\delta}{A}\right)^2(1-\epsilon)^2} + \sqrt{1-\left(\dfrac{\delta}{A}\right)^2(1+\epsilon)^2}\right]$ $n_q = \dfrac{4D\delta}{\pi A^2}$
	$A > \delta$	$n_p = \dfrac{2D}{\pi A}\sqrt{1-\left(1-\dfrac{2\delta}{A}\right)^2}$ $n_q = -\dfrac{4D\delta}{\pi A^2}$
39.		

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
 40.	$A > \delta + \frac{D}{m}$	$n_p = \frac{m}{2}\left[2 - f\left(\frac{\frac{D}{m} + \delta}{A}\right) + f\left(\frac{\frac{D}{m} - \delta}{A}\right)\right]$ $n_q = -\frac{4D\delta}{\pi A^2}$
 41.	$m_1 > m_2$ $A > \frac{D}{m_1} + \frac{\delta m_1}{m_1 - m_2}$	$n_p = \frac{m_1 - m_2}{2}\left[f\left(\frac{\frac{D}{m_1} + \frac{m_1\delta}{m_1 - m_2}}{A}\right) + f\left(\frac{\frac{D}{m_1} - \frac{m_1\delta}{m_1 - m_2}}{A}\right)\right] + m_2$ $n_q = -\frac{4D\delta}{\pi A^2}$
 42.	$A > \frac{D}{m} + \delta$	$n_p = \frac{m}{2}\left[f\left(\frac{\frac{D}{m} + \delta}{A}\right) + f\left(\frac{\frac{D}{m} - \delta}{A}\right)\right]$ $n_q = -\frac{4D\delta}{\pi A^2}$

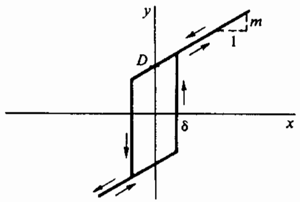
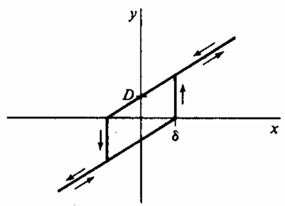
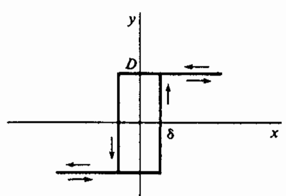
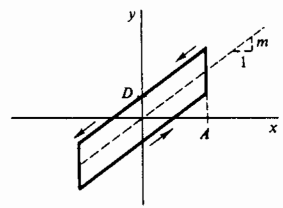
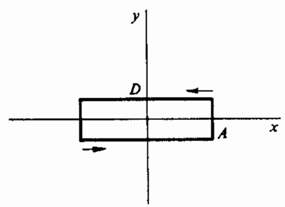
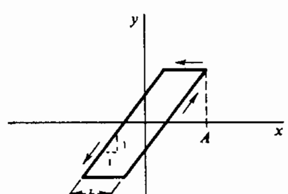
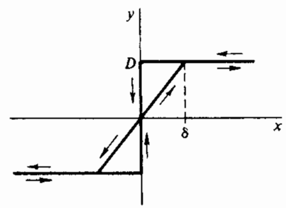
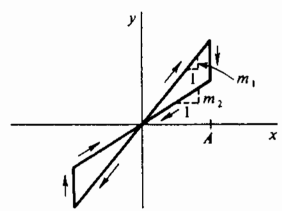
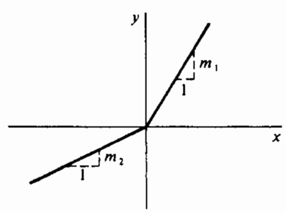
 43.	$A > \delta$ See Sec. 2.5	$n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2} + m$ $n_q = -\frac{4D\delta}{\pi A^2}$
 44. Negative deficiency	$A > \delta$ See Fig. B.6	$n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2} + \frac{D}{\delta}$ $n_q = -\frac{4D\delta}{\pi A^2}$
 45. Rectangular hysteresis or toggle	$A > \delta$ See Fig. B.6 and Sec. 2.3	$n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2}$ $n_q = -\frac{4D\delta}{\pi A^2}$

TABLE OF SINUSOIDAL-INPUT DESCRIBING FUNCTIONS (DFs) *(Continued)*

Nonlinearity	Comments	$n_p(A,\omega)$ and $n_q(A,\omega)$
 46.		$n_p = m$ $n_q = -\dfrac{4D}{\pi A}$
 47.		$n_p = 0$ $n_q = -\dfrac{4D}{\pi A}$
 48. Friction-controlled backlash	$A > \dfrac{b}{2}$ See Fig. B.7 and Sec. 2.3	$n_p = \dfrac{1}{2}\left[1 + f\left(1 - \dfrac{b}{A}\right)\right]$ $n_q = -\dfrac{1}{\pi}\left[2\dfrac{b}{A} - \left(\dfrac{b}{A}\right)^2\right]$

 49.	$A > \delta$ Multivalued nonlinearity for which $n_q(A,\omega) = 0$.	$n_p = \frac{D}{2\delta} f\left(\frac{\delta}{A}\right) + \frac{2D}{\pi A}$ $n_q = 0$
 50.	Multivalued nonlinearity for which the DF is independent of A.	$n_p = \frac{m_1 + m_2}{2}$ $n_q = \frac{m_1 - m_2}{\pi}$
 51.	Asymmetric characteristic equivalent to the parallel combination of a linear gain $(m_1 + m_2)/2$ and an absolute value characteristic $(m_1 - m_2)/2$. The even part does not contribute to the DF.	$n_p = \frac{m_1 + m_2}{2}$ $n_q = 0$

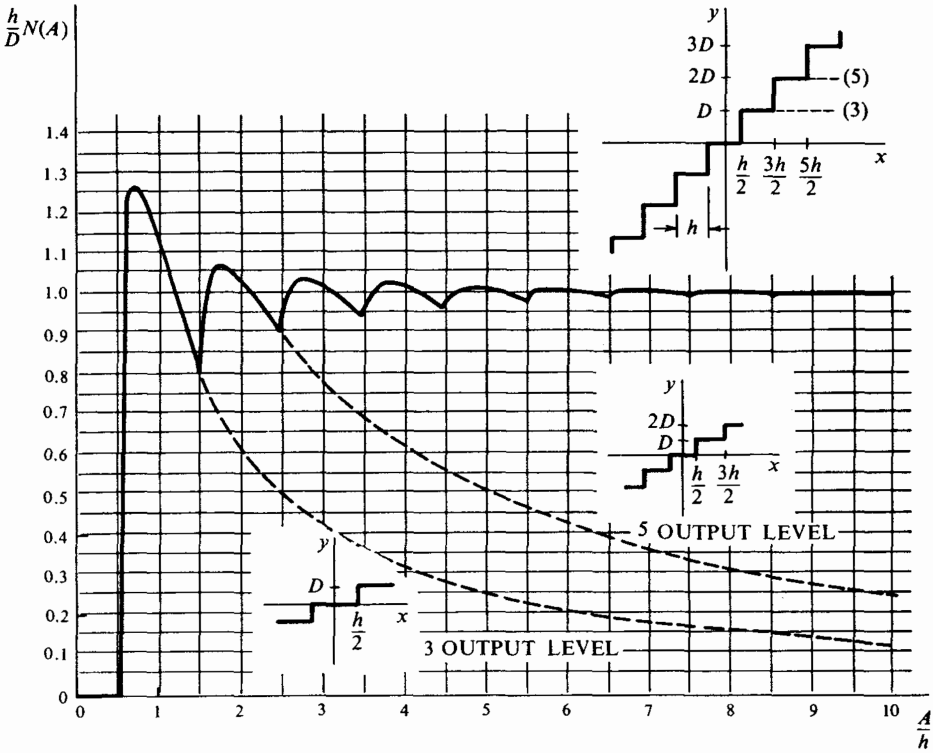


Figure B.1 Quantizer DF.

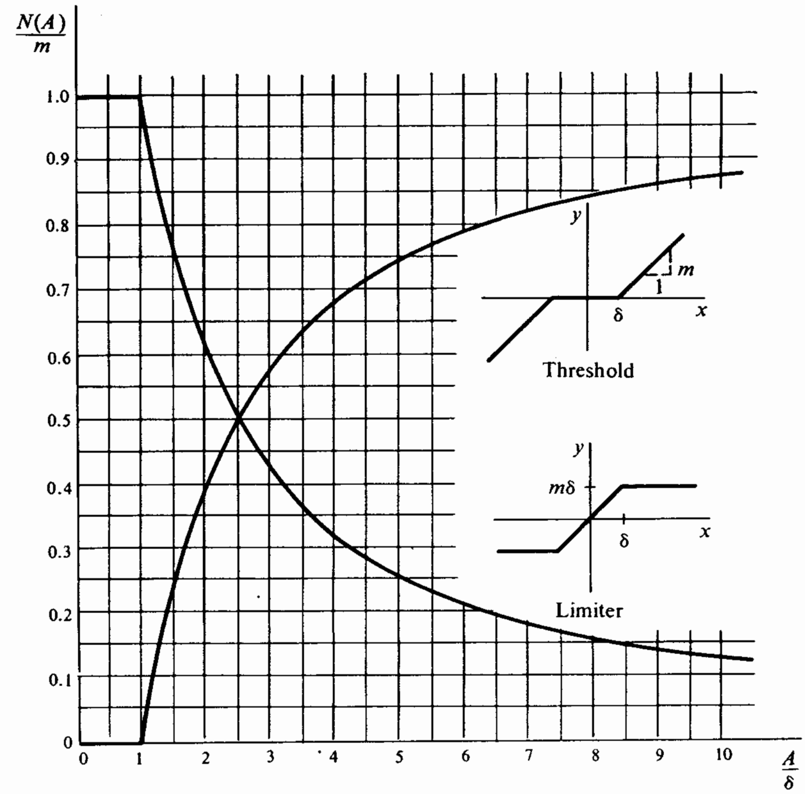


Figure B.2 DFs for limiter and threshold characteristics.

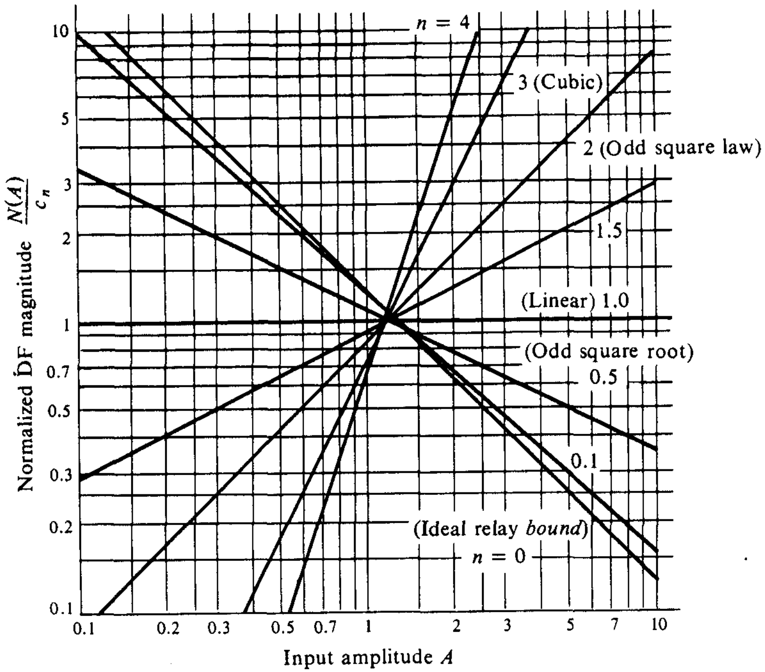


Figure B.3 *DF for the simple polynomial nonlinearity* $y = c_n x^n$ *(n odd) or* $y = c_n x^{n-1}|x|$ *(n even).*

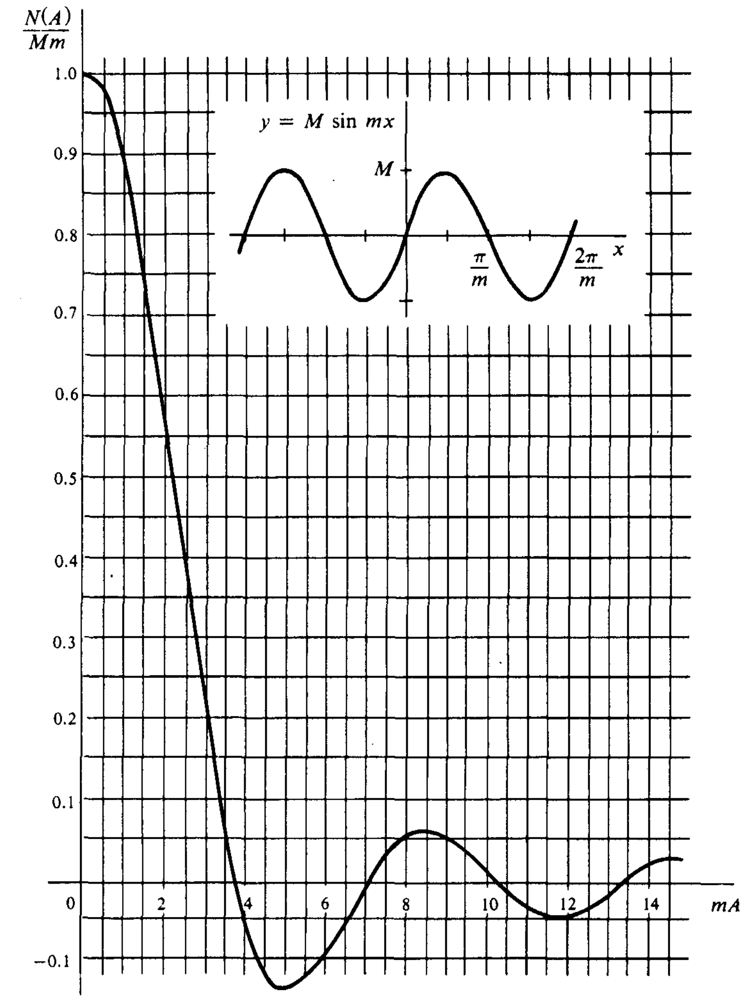


Figure B.4 *Normalized harmonic nonlinearity DF.*

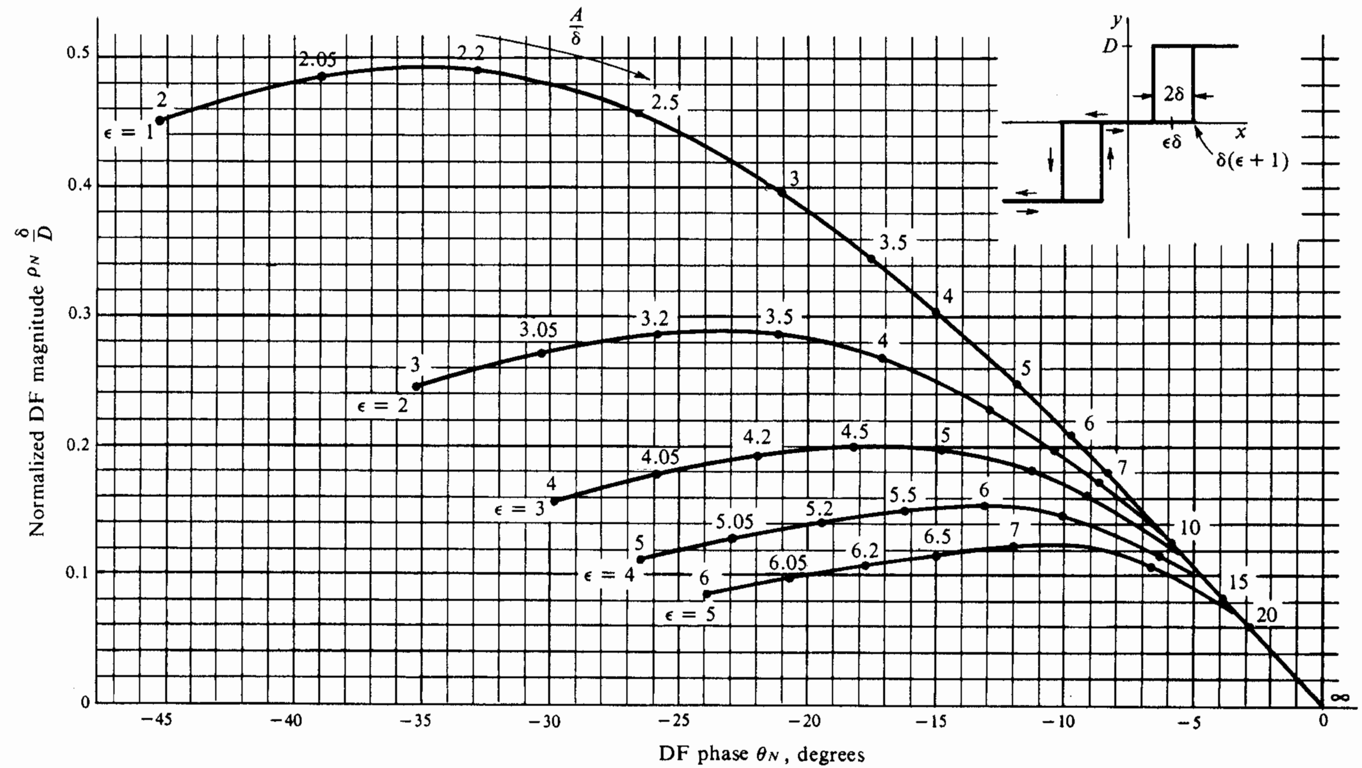


Figure B.5 *DF magnitude vs. phase for hysteresis characteristics.*

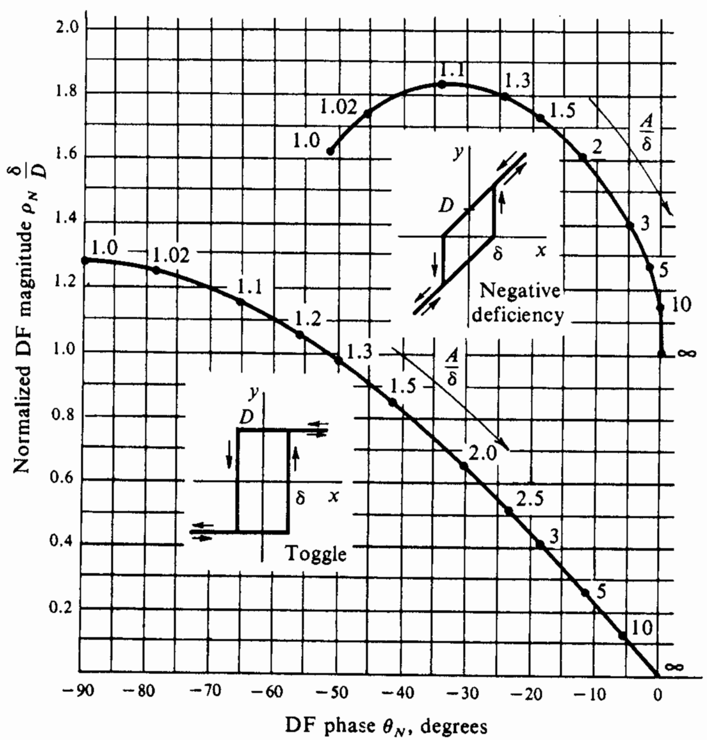


Figure B.6 DF magnitude vs. phase for toggle and negative deficiency characteristics.

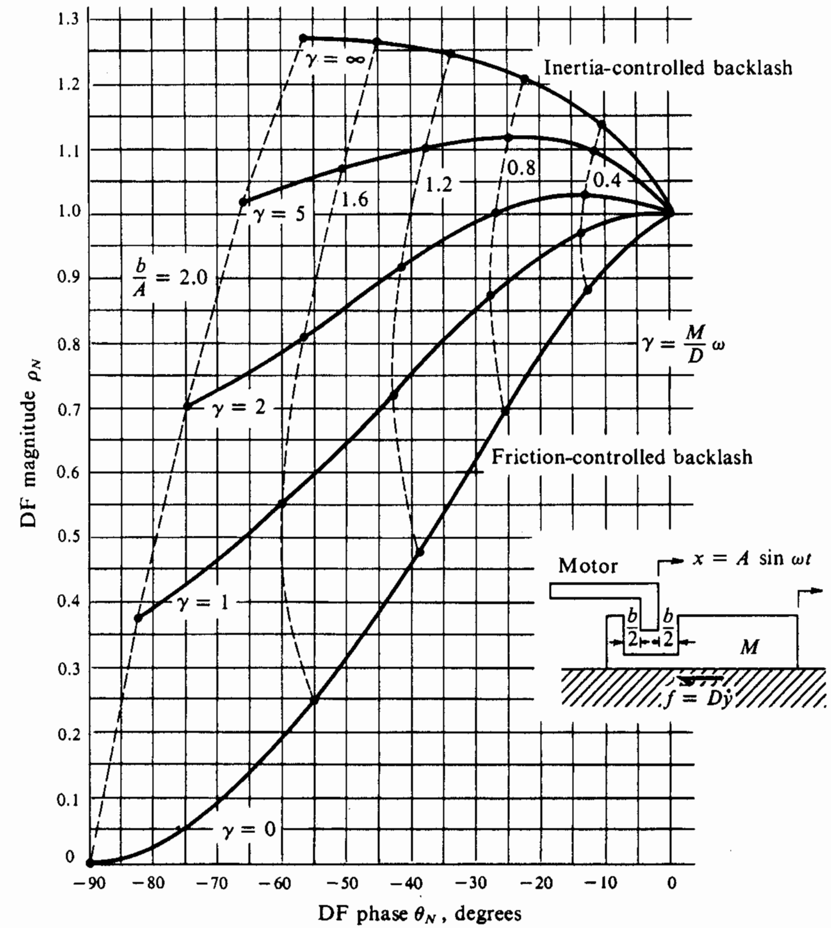


Figure B.7 DF for backlash with inertia and viscous-friction loading.