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Teorema de Kharitonov
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A Simple Proof of Kharitonov's Theorem

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Abstract—**An alternative proof of Kharitonov's theorem on the stability of linear time-invariant continuous systems under parameter variations is presented.**

I. INTRODUCTION

In 1978, Russian researcher Kharitonov presented a theorem on the stability of linear continuous systems under parameter variations [1]. Containing a simple, necessary and sufficient condition, the theorem is by far the most useful and powerful check on linear continuous system stability [1]–[3]. Nevertheless, the theorem is little well-known to the English reader [2], and its proof is not simple. The purpose of this correspondence is to provide an alternative proof of the theorem, which is much simpler and more insightful.

II. KHARITONOV'S THEOREM

Given the family of characteristic polynomials

$$P(s;\, a_0,\, a_2,\, \cdots;\, a_1,\, a_3,\, \cdots) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 \tag{1}$$

where a_i are the perturbed values which are bounded by

$$\underline{a}_i \le a_i \le \bar{a}_i,\; 0 < \underline{a}_i, \qquad i = 0,\, 1,\, 2,\, \cdots,\, n. \tag{2}$$

The entire family of polynomials in (1) will be stable if and only if the following four polynomials are stable:

$$P(s,\, E_1,\, O_1) = \bar{a}_0 + \bar{a}_1 s + \underline{a}_2 s^2 + \underline{a}_3 s^3 + \bar{a}_4 s^4 + \bar{a}_5 s^5 + \cdots$$

$$P(s,\, E_1,\, O_2) = \bar{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2 + \bar{a}_3 s^3 + \bar{a}_4 s^4 + \underline{a}_5 s^5 + \cdots$$

$$P(s,\, E_2,\, O_1) = \underline{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \underline{a}_3 s^3 + \underline{a}_4 s^4 + \bar{a}_5 s^5 + \cdots$$

$$P(s,\, E_2,\, O_2) = \underline{a}_0 + \underline{a}_1 s + \bar{a}_2 s^2 + \bar{a}_3 s^3 + \underline{a}_4 s^4 + \underline{a}_5 s^5 + \cdots \tag{3}$$

where E_1, E_2, O_1, and O_2 denote the sets $\{\bar{a}_0,\, \underline{a}_2,\, \bar{a}_4,\, \underline{a}_6,\, \cdots\}$, $\{\underline{a}_0,\, \bar{a}_2,\, \underline{a}_4,\, \bar{a}_6,\, \cdots\}$, $\{\bar{a}_1,\, \underline{a}_3,\, \bar{a}_5,\, \underline{a}_7,\, \cdots\}\{\underline{a}_1,\, \bar{a}_3,\, \underline{a}_5,\, \bar{a}_7,\, \cdots\}$, respectively.

III. PROOF

The proof for necessity is obvious, since the coefficient values in (3) are chosen from the given bounds (2).

The proof for sufficiency is based on three lemmas.

Lemma 1: With reference to (1) and (2), let E^* denote a set of arbitrary but fixed even coefficients $\{a_0^*,\, a_2^*,\, a_4^*,\, \cdots\}$ which are taken from the bounds (2). If the two polynomials $P(s,\, E^*,\, O_1)$ and $P(s,\, E^*,\, O_2)$ are stable, then the family of polynomials $P(s,\, E^*,\, O)$ will be stable [O denotes all possible sets of odd coefficients bounded by (2)].

To prove the above, we invoke the separation property of a Hurwitz (stable) polynomial [4], which states that for a polynomial $P(s)$ with all positive coefficients to be Hurwitz, it is necessary and sufficient that its even and odd parts (i.e., $P(s) + P(-s)$ and $P(s) - P(-s)$, respectively) have simple zeros restricted to the imaginary axis where they mutually separate each other. Equivalently, the separation property requires that the polar plot of $P(jw)$, $0 \le w < \infty$, will cut the real and imaginary axes alternately a total of n times in the finite plane, where n is the degree of $P(s)$. Hence, the two polynomials $P(jw,\, E^*,\, O_1)$ and $P(jw,\, E^*,\, O_2)$ will cut the real and imaginary axes of the polar plot alternately a total of n times. Variation of odd coefficients means shifting the plots vertically by different amounts for different w from one polynomial to the other (see Fig. 1), since

$$\operatorname{Re} P(jw,\, E^*,\, O) = a_0^* - a_2^* w^2 + a_4^* w^4 - \cdots$$

$$= \text{fixed for a given } w \tag{4}$$

$$\operatorname{Im} P(jw,\, E^*,\, O) = a_1 w - a_3 w^3 + a_5 w^5 - \cdots \tag{5}$$

$$\operatorname{Im} P(jw,\, E^*,\, O_2) \le \operatorname{Im} P(jw,\, E^*,\, O) \le \operatorname{Im} P(jw,\, E^*,\, O_1). \tag{6}$$

Hence, by continuity, any member from the family $P(jw,\, E^*,\, O)$ will cut the axes at least n times. Since it cannot cut the axes more than n times [because of the degree constraints in (4) and (5)], it must cut them in an *alternate* fashion. Therefore, according to the separation property, all members of $P(jw,\, E^*,\, O)$ will be stable.

Lemma 2: Let O^* denote an arbitrary set of fixed odd coefficients taken from the bounds (2). If the two polynomials $P(s,\, E_1,\, O^*)$ and $P(s,\, E_2,\, O^*)$ are stable, then the entire family of polynomials $P(s,\, E,\, O^*)$ will be stable [E denotes all sets of even coefficients from the bounds (2)].

The proof is similar to that of Lemma 1.

Lemma 3: If $P(s,\, E_1,\, O_1)$, $P(s,\, E_1,\, O_2)$, $P(s,\, E_2,\, O_1)$, and $P(s,\, E_2,\, O_2)$ are stable, then the family of polynomials $P(s,\, E,\, O)$ in (1) will be stable.

The proof is based on applications of Lemmas 1 and 2. First, let $E^* = E_1$ in Lemma 1, then the family of polynomials $P(s,\, E_1,\, O)$ will be stable. Similarly, if we let $E^* = E_2$, we see that $P(s,\, E_2,\, O)$ will be stable.

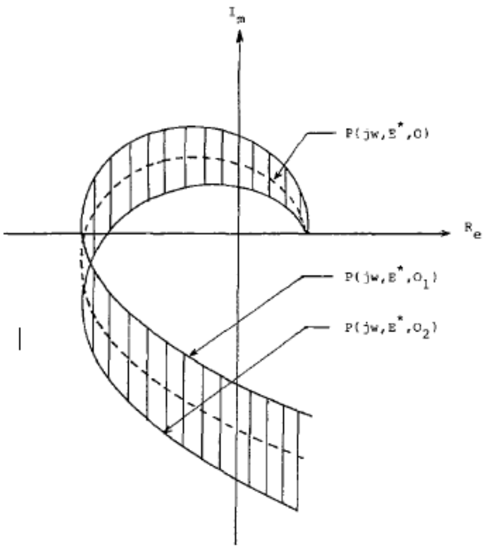

Fig. 1. Variation of odd coefficients in $P(jw,\, E^*,\, O)$ amounts to shifting each point of the polar plot vertically by different amounts between $P(jw,\, E^*,\, O_1)$ and $P(jw,\, E^*,\, O_2)$.

Using Lemma 2, the stability of $P(s,\, E_1,\, O)$ and $P(s,\, E_2,\, O)$ will imply stability of the family $P(s,\, E,\, O)$.

Lemma 3 constitutes the sufficiency proof of Kharitonov's theorem.

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