7 Stability of Nonlinear Systems

Key points

- Stability, asymptotic stability, uniform stability, global stability
- The Second Method of Lyapunov allows one to determine (global, asymptotic, uniform) stability of an equilibrium point without explicitly solving for system solutions.
- The hardest condition to meet in Lyapunov's conditions is finding a Lyapunov function candidate such that $\dot{V} < 0$. Oftentimes $\dot{V} \leq 0$. LaSalle's invariance principle can be used if this is the case for an autonomous or periodic system. Barbalat's lemma and a Lyapunov-like lemma from Slotine and Li can be used if $\dot{V} \leq 0$ for non-autonomous systems (such as adaptive systems).
- Domains of attraction can be calculated or approximated for some equilibrium points.
- The Aizerman and Kalman conjectures deal with global asymptotic stability of a nonlinear system for a whole class of nonlinearities. Both are false, but two different criteria, the circle criterion and the Popov criterion, are applicable.
- References: Khalil and Slotine and Li

We will consider the case of unforced, autonomous systems, as represented by the equation:

$$\dot{x} = f(x)$$

We will not consider disturbances, and we will restrict the analysis to systems that do not have an explicit time dependence (in a first time).

So far, we have started by looking for the **equilibrium points**:

$$f(x_e) = 0$$

We have then considered **perturbations about the equilibrium points**:

$$x = x_e + \delta x$$
$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_e .\delta x + HOT$$
$$J = \left.\frac{\partial f}{\partial x}\right|_e$$

and if $\mathrm{Re}(\lambda_i) \neq 0$, then <u>local stability can be determined from the eigenvalues of J</u>.

If $\mathrm{Re}(\lambda_i) = 0$, one can use the <u>center manifold theorem</u> to determine local stability.

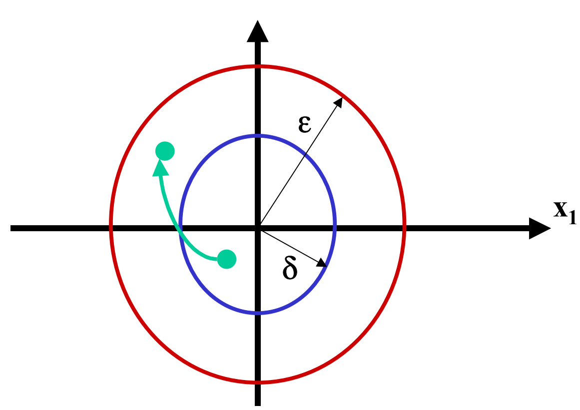
What about **global stability**?

Definition: Stability in the sense of Lyapunov

Assume $x_e = 0$.

<u>Stable:</u>

The equilibrium x = 0 is stable iff $\forall \varepsilon > 0, \forall t_0 \geq 0, \exists \delta > 0 \; \|x(t_0)\|_2 < \delta \Rightarrow \|x(t)\|_2 < \varepsilon, \forall t \geq t_0$.



"That is, if I start within δ, I stay within ε. In general, I give you an ε, you give me the corresponding δ. Things remain bounded."

Asymptotically stable:

The equilibrium x = 0 is asymptotically stable iff:
(i) x = 0 is a stable equilibrium
(ii) $\forall t_0 \geq 0, \exists \delta(t_0) \left\| x(t_0) \right\|_2 < \delta \Rightarrow \lim_{t\to+\infty} \left\| x(t) \right\| = 0$

Uniformly stable:

The equilibrium x = 0 is uniformly stable iff:
(i) x = 0 is a stable equilibrium
(ii) $\delta(\varepsilon, t_0) = \delta(\varepsilon)$

These conditions refer to **stability in the sense of Lyapunov**.

The Second Method of Lyapunov

- Originally proposed by Lyapunov (around 1890) to investigate stability in the small (local stability)
- Later extended to cover global stability
- Stability can be determined without explicitly solving for the system solutions.
- Generalization of an "energy" argument for non-energetic systems (e.g. forecasting the stock market, etc…)
- Difficulty is finding the Lyapunov function

Positive Definite Functions

A function is positive definite if $V(x) > 0 \quad \forall x \neq 0$ and $V(0) = 0$

Examples

For example, suppose $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Note that x is a vector, while V(x) is a scalar function.

Suppose $V_1(x) \equiv x_1^2 + x_2^2$. Is V_1 positive definite? YES.

To check, verify both conditions:

- Does $V_1 = 0 \Rightarrow x_1 = 0 \quad and \quad x_2 = 0$? Yes.
- Do we have: $V_1(x) > 0 \quad \forall (x_1, x_2) \in \Re^*$ Yes.

Suppose $V_2(x) \equiv x_1^2$. Is V_2 positive definite? NO.

V_2 is positive but not definite. For example, $x_1 = 0 \quad and \quad x_2 = 10 \Rightarrow V_2 = 0$

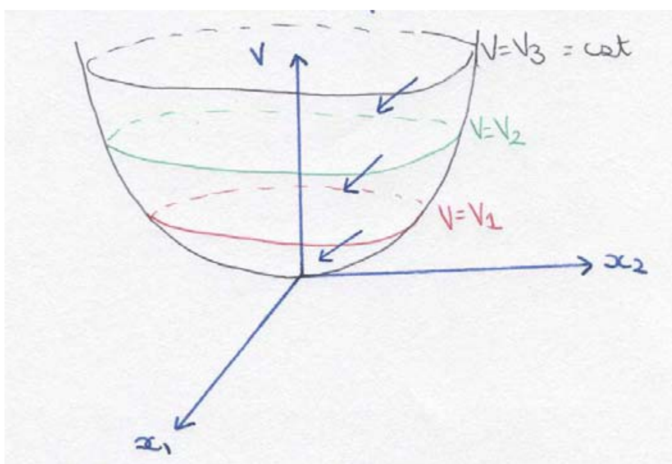
This property is called positive semi-definiteness.

Intuition for Lyapunov's theorem

Consider a second order system:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $V(x_1,x_2)$ be a positive definite function.



If $V(x_1,x_2)$ always decreases, then it must reach zero eventually. That is, for a stable system, all trajectories must move so that the values of V are decreasing. This is similar to the energy argument for stability of mechanical systems.

To relate V to the system dynamics, we compute $\dot{V}$.

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x_i} . \dot{x}_i$$
$$= \frac{\partial V}{\partial t} + \nabla V^T . f$$

The second term of this expression relates V to the vehicle dynamics. In our notation we have assumed:

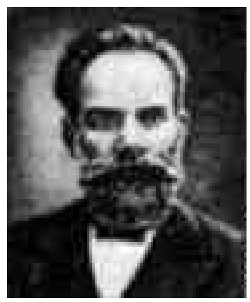
$$f(x) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \text{ and } \nabla V^T = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]$$

We need $\dot{V}$ to be negative definite for our intuitive condition to be true.

$\dot{V}$ is the rate of change of the scalar field V along the flow of the vector field f.

$\dot{V} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial t} = L_f V$ Lie derivative of V (if V does not depend on t)

<u>Aleksandr Mikhailovich Lyapunov</u>



- Born: 6 June 1857 in Yaroslavl, Russia
- Died: 3 Nov 1918 in Odessa, Russia

- Aleksandr Lyapunov was a school friend of Markov and later a student of Chebyshev. He did important work on differential equations, potential theory, stability of systems and probability theory. His work concentrated on the stability of equilibrium and motion of a mechanical system and the stability of a uniformly rotating fluid. He devised important methods of approximation. Lyapunov's methods, introduced by him in 1899, provide ways of determining the stability of sets of ordinary differential equations.
- http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lyapunov.html

<u>Theorem: Lyapunov's Second Method</u>

VERY IMPORTANT NOTE: This theorem provides a SUFFICIENT condition, not a NECESSARY condition.

Consider the system: $\dot{x} = f(x,t)$, $f(0,t) = 0 \forall t$

If a scalar function is defined such that:

(i) V(0,t) = 0

(ii) V(x,t) is positive definite, i.e. there exists a continuous, non-decreasing scalar function α(x) such that α(0) = 0 and $\forall x \neq 0, 0 < \alpha(\|x\|) < V(x,t)$

(iii) $\dot{V}(x,t)$ is negative definite, that is $\dot{V}(x,t) \leq -\gamma(\|x\|) < 0$ where γ is a continuous non-decreasing scalar function such that γ(0) = 0

(iv) $V \leq \beta(\|x\|)$ where β is a continuous non-decreasing function and β(0) = 0, i.e. V is decrescent, i.e. the Lyapunov function is upper bounded

(v) V is radially unbounded, that is $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$

Then the equilibrium point is uniformly asymptotically stable in the large and V(x,t) is called a Lyapunov function.

Relaxed conditions:
- Asymptotic stability requires $\dot{V}$ negative definite.
- Stability requires $\dot{V}$ negative semi-definite.
- Condition (iv) yields uniformity for the time-varying system.
- Global stability is given by condition (v).

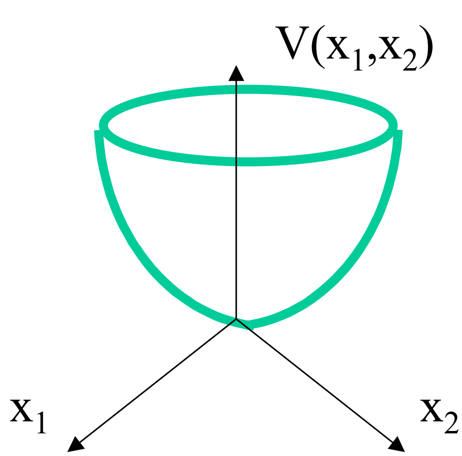
Note:
The most difficult condition is to show that $\dot{V}$ is negative semi-definite. Quite often, $\dot{V}$ will be negative semi-definite (→ globally stable, but not asymptotically stable).

In "English":
- Uniform: it does not matter when I start.
- Asymptotic: no matter how large the perturbation, we go back to the origin.
- In the large (also sometimes referred to as "globally"): true everywhere.

<u>Lyapunov pictures:</u>

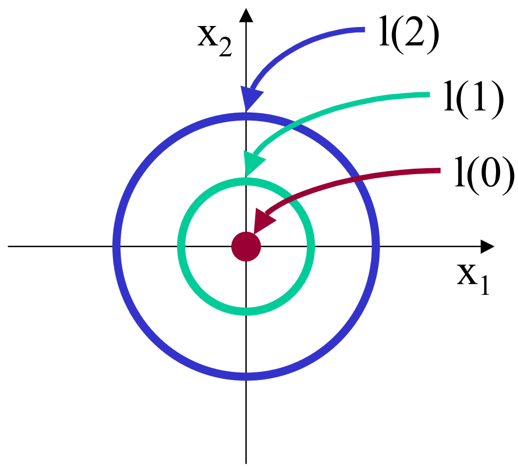
We limit ourselves to autonomous systems, that is $\dot{x} = f(x)$, not $\dot{x} = f(x,t)$.

For a 2D system, conditions 1 and 3 give something like this:



A more convenient way of visualizing this is through level sets. A level set can be defined as:

$l_C = \{x \in \Re^2 | V(x) = C\} = l(C)$

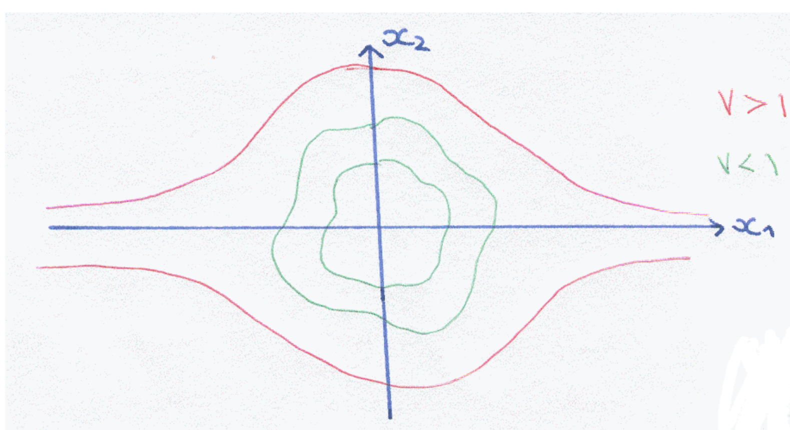


Why do we need the "radially unbounded" argument?

For example, consider the following Lyapunov function:

$$V(x) \equiv \frac{x_1^2}{1+x_1^2} + x_2^2$$

The level sets of Lyapunov functions are not closed contours in general. For example, for V as given above, the contours are shown below.



For V>1, we have tails of the contours that trail off to infinity. These can get you in trouble from a stability perspective.

An Aside on Lyapunov Theory for Linear Systems

Consider the autonomous linear system $\dot{x} = Ax$. Suppose we would like to investigate the stability of this system using Lyapunov theory (this is not the easiest way to do it, but let's consider it for the sake of argument).

Let's select $V = x^T Px$ where P is a positive definite matrix.

Then, $\dot{V} = \dot{x}^T Px + x^T P\dot{x} = x^T [A^T P + PA]x$

Let $A^T P + PA = -Q$, where Q is positive definite.

Then if Q is positive definite then the linear system is globally asymptotically stable.

$A^T P + PA = -Q$ is called the Lyapunov equation.

How does one obtain a (P,Q) pair?

- Option 1: Choose a P matrix that is positive definite, compute Q, and check if it is positive definite. This is not a smart approach, since if A is stable, then not every P>0 will yield a Q>0.
- Option 2: If A is stable, any Q>0 will yield a P>0. The usual approach is to set Q=I, then solve for P.

LaSalle's Invariance Theorem

Fact: in general the hardest condition to meet in Lyapunov's second method is to find a V function that yields $\dot{V} < 0$.What happens quite often is that a given V yields $\dot{V} \le 0$ (the system is stable, instead of asymptotically stable). A very useful theorem, due to LaSalle, called the invariance principle, can be used in this case for autonomous or periodic systems.

Joseph P. LaSalle
American Mathematician.
Ph.D.: 1941. California Institute of Technology.
Professor of Mathematics at Brown University.
J.P. LaSalle, "*Stability Theory for Ordinary Differential Equations*", J. Differential Equations **4** (1968), 57-65.

Theorem

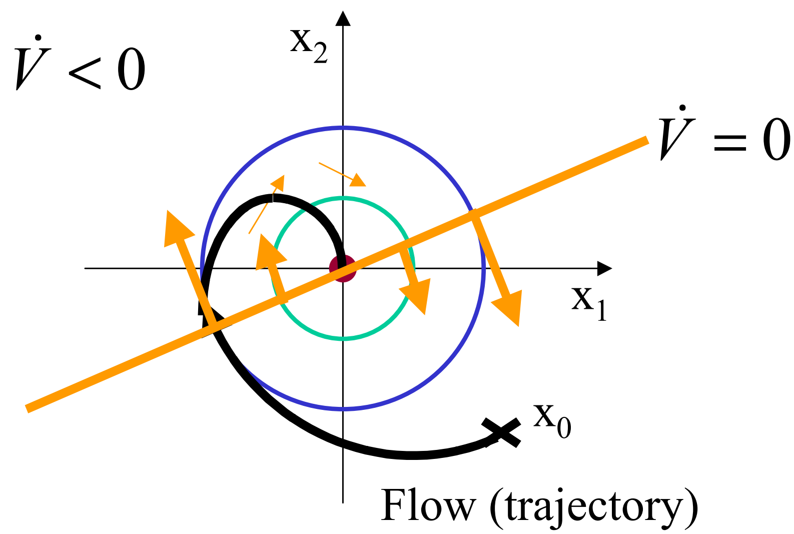
Consider a system described by either $\dot{x} = f(x)$ or $\dot{x} = f(x, t+T)$ where T is the period of the system.

If $V(x,t) = V(x,t+T)$, and V>0, and V is radially unbounded, if $\dot{V} \le 0$ (negative semidefinite) and $\dot{V} \neq 0$ along any solution of the differential equation except the origin, then the origin is globally uniformally asymptotically stable in the sense of Lyapunov.

"General idea": the trajectories cannot get "hung up" on $\dot{V} = 0$, that is V = cst.

<u>What does $\dot{V} = 0$ look like in a picture?</u>

Whenever one of the vectors is tangent to a level set, then V will not decrease at that point in the phase plane.



The $\dot{V} = 0$ line is NOT an invariant set.

<u>Example (linear)</u>

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 - ay_2 \end{cases} \qquad (a>0)$$

Let's pick $V(y) \equiv y_1^2 + y_2^2$. (V>0, V(0)=0).

Compute $\dot{V}$.

$$\dot{V} = 2y_1\dot{y}_1 + 2y_2\dot{y}_2 = -2ay_2^2 \Rightarrow \dot{V} \le 0$$

The system (linear system, so no need for "equilibrium point") is globally stable (conditions i, ii and v are met).

Apply LaSalle's invariance theorem:

$$\dot{V} = 0 \Rightarrow y_2 = 0 \Rightarrow \dot{y}_2 = 0 \Rightarrow y_1 = 0$$

Then the equilibrium point (0, 0) (the origin) is globally asymptotically stable.

A More General Statement for Autonomous Systems

Consider an autonomous system $\dot{x} = f(x)$ with f continuous and let V be a scalar with continuous first partial derivatives. Assume that:

- For some l>0, the region Ω_l defined by V(x)<l is bounded.
- $\dot{V} \leq 0$ for all x in Ω_l.

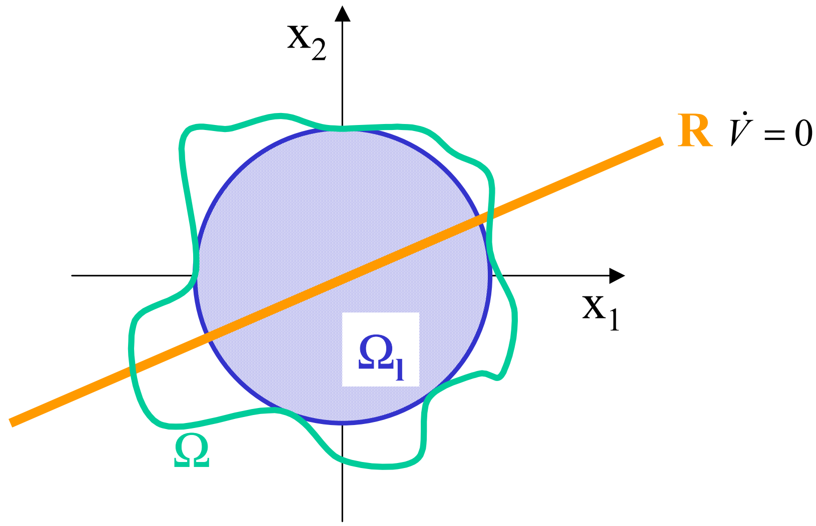
Then let R be the set of all points within Ω_l where $\dot{V} = 0$. Let M be the largest invariant set in R. Then every solution x(t) originating in Ω_l tends to M as t tends to infinity.

(Reminder: for an invariant set, if I start in the set I stay in the set).

Corollary (Slotine and Li):
Consider the system: $\dot{x} = f(x)$, f(0) = 0
Assume that within a neighborhood Ω of the origin,

i) V(x) is locally positive definite
ii) $\dot{V}(x)$ is negative semi-definite
iii) The set $R = \{x \in \Omega | \dot{V}(x) = 0\}$ contains no solutions to $\dot{x} = f(x)$ apart for the origin

Then: the origin is asymptotically stable and the largest connected region of the form $\Omega_l = \{x \in \Omega | V(x) < l\}$ and $\Omega_l \subset \Omega$ is a domain of attraction

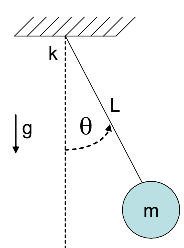


Note: R is NOT an invariant set, i.e. if you start on R you don't necessarily stay on R.

The Global Invariant Set Theorem

Same conditions/assumptions as the corollary, plus V(x) is radially unbounded → global asymptotic stability.

Example: The Pendulum



Consider first the case where there is no friction (not mechanically realizable):

$$mL^2\ddot{\theta} + mgL\sin\theta = 0$$

This can be re-written as:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

By convention and without lack of generality, we select $\frac{g}{L} = 1$.

That is, our dynamic equation reduces to:

$$\ddot{\theta} + \sin\theta = 0$$

Start by finding the equilibrium points: $\dot{\theta} = 0, \quad \theta = \pm n\pi$

Investigate the stability of (0, 0):

One possible choice would be: $V(\theta, \dot{\theta}) \equiv \quad \theta^2 + \dot{\theta}^2$. However, this choice is not interesting.

Let: $$V(\theta, \dot{\theta}) \equiv \quad energy \quad = (1 - \cos\theta) + \frac{1}{2}\dot{\theta}^2$$

Where the first term represents the potential energy and the second term represents the kinetic energy of the system.

The function is locally positive definite but not radially unbounded, which implies we will get a local, not a global result.

Compute $\dot{V}$:

$$\dot{V} = \sin\theta\dot{\theta} + \dot{\theta}\ddot{\theta} = 0$$

$$(\text{since } \ddot{\theta} + \sin\theta = 0)$$

Note that since there is no dissipation mechanism, the energy has to be conserved, and therefore $\dot{V} = 0$ everywhere was predictable.

LaSalle's will not help us here, as $\dot{V} = 0$ everywhere. Plus, from experience an engineering student ought to know that without damping, the pendulum will swing forever, and therefore it won't be asymptotically stable about its (0, 0) equilibrium point.

Let's add some viscous damping in the bivot/bearing. The dynamic equation is now:

$$\ddot{\theta} + \sin\theta + k\dot{\theta} = 0$$

We keep the same choice for V:

$$V(\theta, \dot{\theta}) \equiv \quad energy \quad = (1 - \cos\theta) + \frac{1}{2}\dot{\theta}^2$$

However, the expression for $\dot{V}$ has now changed:

$$\dot{V} = \sin\theta\dot{\theta} + \dot{\theta}\ddot{\theta} = -k\dot{\theta}^2$$

The expression for $\dot{V}$ is negative semi-definite, so the system is stable about its (0,0) equilibrium point (not asymptotically).

Let's try LaSalle's theorem:

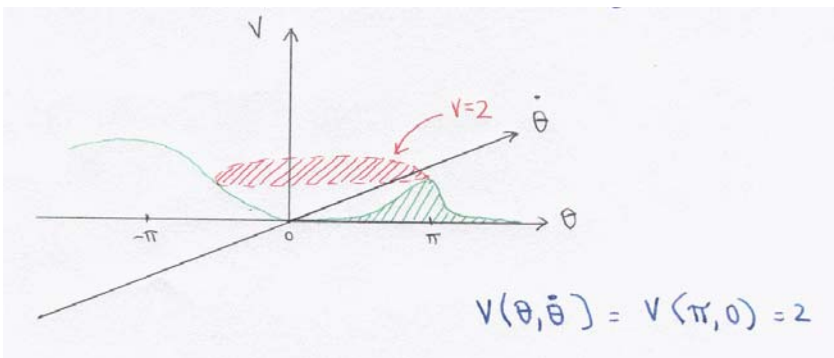
$$\dot{V} = 0 \Rightarrow \dot{\theta} = 0 \Rightarrow \ddot{\theta} = 0$$

And from the dynamics, $\dot{V} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \pm n\pi$

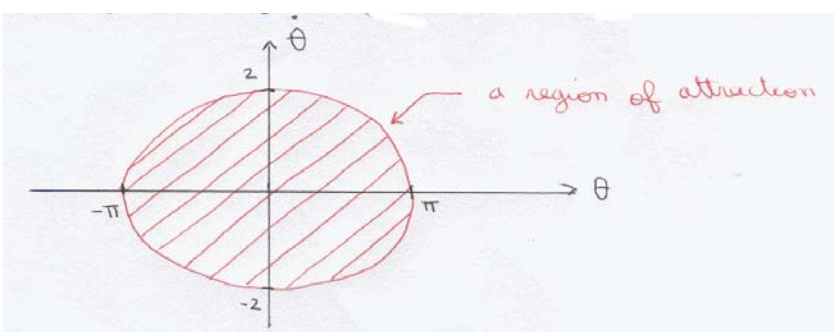
We want to find Ω_l, as defined by V<l, which is the largest set of points for which the pendulum will not "go over" and make a complete turn.

To define l, let $\theta = \pm\pi, \quad \dot{\theta} = 0$. Then, V=2.

$$\Omega_l = \left\{ (\theta, \dot{\theta}) \mid (1 - \cos\theta) + \frac{1}{2}\dot{\theta}^2 < 2 \right\}$$



If we plot it in the phase-plane instead:



The region of attraction is "ellipsoid-looking" (it is not really an ellipse).

If the initial conditions are in this region, the pendulum will stay in this region (and not swing over the top).

The disturbing part of this result is that the region of attraction does not depend on k, that is, k could be 10^{-27} or 10^{27}.

In fact, some points outside the "ellipse" might still work depending on k.

This is an incredibly conservative result.

Domains of Attraction

Reference: Khalil, chapter 8.

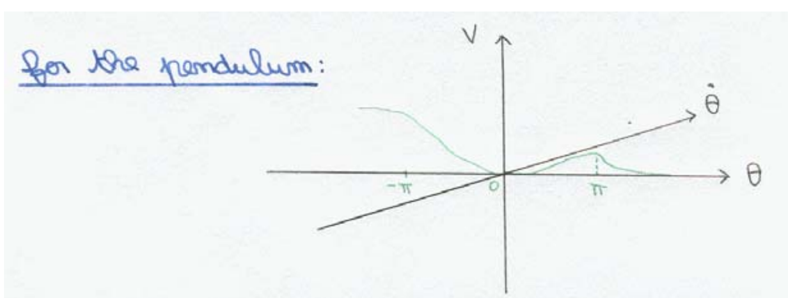
In many applications, it is not enough to determine that a given system has an asymptotically stable equilibrium point. Rather, it is important to find the region of attraction of that point, or at least an estimate of it.

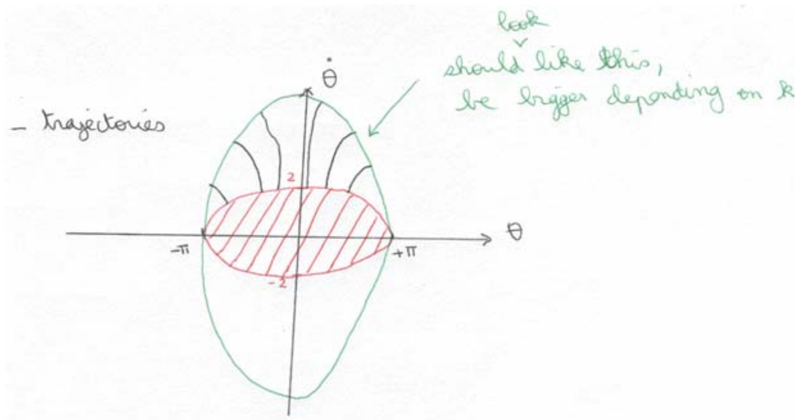
Lemma: If x = 0 is an asymptotically stable equilibrium point for $\dot{x} = f(x)$, then its (exact) region of attraction Ω_a is an invariant set whose boundaries are formed by trajectories.

Trajectory Reversing Method

One method to obtain the domain of attraction, called the trajectory reversing method, uses backward integration.

1. Step1: Find an approximate region Ω_c, say, by V(x)<c
2. Starting from the boundary of Ω_c, integrate $\dot{x} = f(x)$ backward in time (which is the equivalent of integrating $\dot{x} = -f(x)$ forward in time.





The "Alternative" Method (due to Khalil)

The "alternative" method uses a quadratic Lyapunov function.

Consider the system described by: $\dot{x} = f(x)$ with $f(0) = 0$.

Let $f(x) = Ax + f_1(x)$.

Let's assume that x = 0 is an asymptotically stable equilibrium point of the system. Then A is a stable matrix and has eigenvalues with negative real parts.

Let: $V(x) = x^T Px$, where P is positive definite. Then:

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T (A^T P + PA)x + f_1^T Px + x^T Pf_1 \\ &= x^T (A^T P + PA)x + 2x^T Pf_1 \\ &= -x^T Qx + 2x^T Pf_1\end{aligned}$$

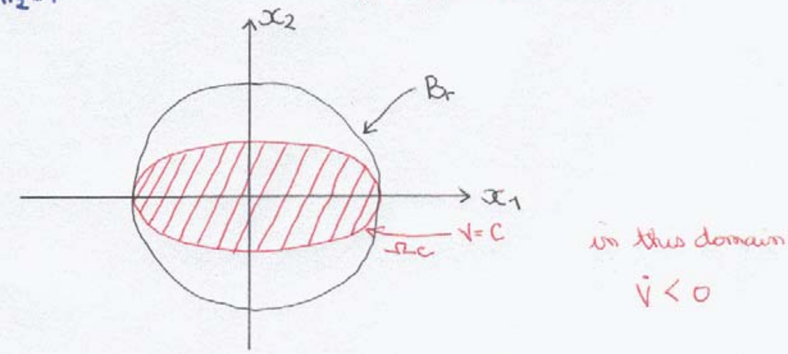
The $x^T Pf_1$ *and* $f_1^T Px$ terms are scalars, so their transposes are equal. $\dot{V}$ is a scalar expression. Also, Q is positive definite.

We are looking for the largest domain, Ω_c, defined by V(x)<c such that $\dot{V}$ is negative definite (or $\dot{V} \le 0$ + LaSalle's invariance principle).

We know that there is a ball $B_r = \|x\|_2 \le r$ such that $\dot{V} < 0$ in B_r. Let Ω_c be contained in B_r by choosing $c = \min_{\|x\|_2 = r} V(x)$.

Let $\Omega_c \in B_r$.

Choose $C = \min\limits_{\|x\|_2 = r} V(x)$ ← the min. value of $V(x)$ that touches the ball



We have selected: $V(x) = x^T P x$

We know that: $\lambda_{\max}(P)\|x\|_2^2 \geq V \geq \lambda_{\min}(P)\|x\|_2^2$

Let $c = \lambda_{\min}(P) r^2$

Assume that f_1 is Lipschitz or at least locally Lipschitz), that is, $\|f_1\|_2 < \gamma \|x\|_2$, where γ is the Lipschitz constant.

$$
\begin{aligned}
&\dot{V}(x) = -x^T Q x + 2x^T P f_1 \\
&\Rightarrow \dot{V}(x) \leq -\lambda_{\min}(Q)\|x\|_2^2 + 2\lambda_{\max}(P)\gamma\|x\|_2^2 \quad (\textit{conservative}) \\
&\Rightarrow \dot{V}(x) \leq -[\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P)]\|x\|_2^2
\end{aligned}
$$

The quantity in brackets in the last equations must be greater than zero.

We have not explained how to choose Q yet. We want:

- $\lambda_{\min}(Q)$ to be as large as possible
- $\lambda_{\max}(P)$ to be as small as possible

In essence, we want to maximize the ratio: $\dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$.

There is a (nice) known solution to this problem (referenced (not proven) in textbook by Khalil), and that choice of Q is:

$$Q = I$$

That is, we set

$$A^T P + PA = -I$$

Cookbook procedure:

1) $\dot{x} = f(x) = Ax + f_1(x)$
2) Solve $A^T P + PA = -I$ for P.
3) Set $V(x) = x^T Px$, $\dot{V}(x) = -x^T Qx + 2x^T Pf_1$. We need to find the largest ball on which $\dot{V} < 0$ (is negative definite), and find the radius r of that ball.
4) $c = \lambda_{\min}(P)r^2$
5) $\Omega_c = \{x \mid V(x) < c\} \Rightarrow x^T Px < \lambda_{\min}(P)r^2$

Example

Consider the following system:

$$\begin{cases} \dot{x}_1 = -2x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 + x_1 x_2 \end{cases}$$

The system has equilibrium points at both (0, 0) and (1, 2).

We will consider the equilibrium point at (0, 0) here.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda_{1,2} = -2, -1$$

We have an asymptotically stable equilibrium point.

We want to calculate the region of attraction of that equilibrium point.

$$A^T P + PA = -I \Rightarrow P = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$V(x) = x^T Px$

$\dot{V}(x) = -(x_1^2 + x_2^2) + \left(\frac{1}{2} x_1^2 x_2 + x_1 x_2^2 \right)$

We transform the system into polar coordinates, that is, we let $x_1 = \rho\cos\theta, x_2 = \rho\sin\theta$.

$$\dot{V} = -\rho^2 + \rho^3\cos\theta\sin\theta\left(\sin\theta + \frac{1}{2}\cos\theta\right)$$

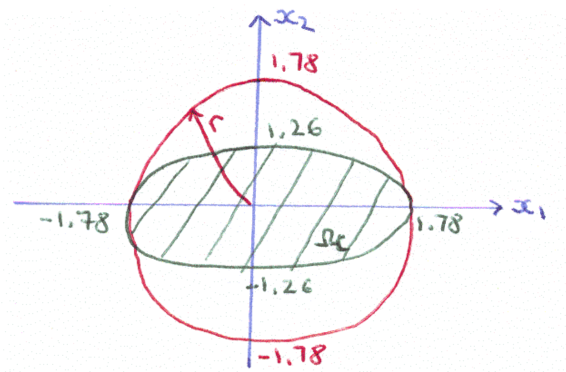
$$\Rightarrow \dot{V} \le -\rho^2 + \frac{1}{2}\rho^3\,|\sin 2\theta|.\left|\sin\theta + \frac{1}{2}\cos\theta\right|$$

$$\Rightarrow \dot{V} \le -\rho^2 + \frac{\sqrt{5}}{4}\rho^3$$

$$\Rightarrow \dot{V} \le 0 \;\; for\;\rho < \frac{4}{\sqrt{5}} \approx 1.8$$

Let $c = \lambda_{\min}(P)r^2 = \frac{1}{4}\left(\frac{4}{\sqrt{5}}\right)^2 = 0.8$

Then $\Omega_c = \{x\,|\,V(x) < c\} \Rightarrow x^T P x < 0.8 \Rightarrow \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 < 0.8$



<u>Zubov's Method of Construction</u>

References: Khalil, Hahn, Zubov

This method yields an <u>exact solution for the domain of attraction</u> (very hard to get).

<u>Theorem:</u>

The necessary and sufficient conditions for Ω_1 to be the EXACT domain of attraction for an equilibrium point is the existence of two scalar functions V and Φ such that:

a. V is positive definite and continuous in Ω_1, Φ is positive definite and continuous for all x

b. Within Ω_1, $0 \le V < 1$ and $V(0)=0$
c. $\dot{V} = \nabla V^T . f = \Phi(V-1)$ where $\dot{x} = f(x)$

Notes:
i. $\Phi(x)$ is an arbitrary positive definite function and is chosen for convenience.
ii. Condition (c) is a partial differential equation for V: $\dot{V} = \dfrac{\partial V}{\partial x} . f = \Phi(V-1)$ *and* $V(0) = 0$

Example (a bit contrived, due to Hahn)

Consider the following system:

$$\begin{cases} \dot{x} = -x + 2x^2 y & \leftarrow \textit{nonlinear} \\ \dot{y} = -y & \leftarrow \textit{stable} \end{cases}$$

The exact domain of attraction for (0,0) can be obtained by solving the PDE:

$$\frac{\partial V}{\partial x}(2x^2 y - x) + \frac{\partial V}{\partial y}(-y) = \Phi(V-1) \textit{ and } V(0) = 0$$

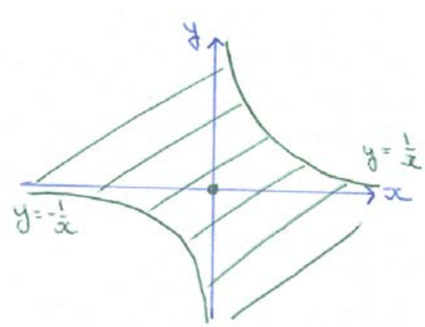
Now, we have to select a function $\Phi(x)$. Let us pick: $\Phi(x) = x^2 + y^2$

It can be shown that the exact solution for the domain of attraction is then:

$$V = 1 - \exp\left[-\frac{y^2}{2} - \frac{x^2}{2(1-xy)} \right]$$

$V<1$ defines the set Ω_1, $V=1$ defines the boundary of Ω_1

$\Rightarrow 1 - xy = 0$ *or* $xy = 1$ is the boundary of Ω_1



Stability of Nonlinear Time-Varying Systems

Up to now, we have only considered time-invariant (or autonomous) systems. We now turn our attention to time-varying systems.

Example from adaptive control

$$\dot{x} = U + a\omega(t)$$

In this example, a is an unknown constant and ω(t) is a known, bounded function.

We want: $x \to x_d(t)$

Let $U = \dot{x}_d - (x - x_d) - \hat{a}(t)\omega(t)$ where we define $\dot{\hat{a}}(t) = (x - x_d)\omega(t)$.

Claim: with this control law, $x \to x_d(t)\ as\ t \to \infty$ and $\hat{a}(t)$ is bounded.

Proof:

Start by defining the error coordinates:

$$\begin{cases} e = x - x_d \\ \tilde{a} = a - \hat{a} \end{cases}$$

The error dynamics are given by:

$$\begin{cases} \dot{e} = -e + \tilde{a}\omega(t) \\ \dot{\tilde{a}} = -e\omega(t) \end{cases}$$

We want to prove that: $x \to x_d(t)\ as\ t \to \infty$, that is, $e \to 0\ as\ t \to \infty$

Let's try a Lyapunov function:

$$V = e^2 + \tilde{a}^2$$

This function is positive definite and decrescent (OK, I can just use 2V to bound it).

$$\dot{V} = -2e^2$$

Hence we know that the origin: $(e(0), \tilde{a}(0))$ is uniformly stable.

That is, $e(t)\ and\ \tilde{a}(t)$ are bounded.

A reminder on the meaning of autonomous

Autonomous	**Non-Autonomous**
$\dot{x} = f(x)$ **or** $\dot{x} = f(x) + g(x).u$ where u(.) is some control which is chosen so that the system is independent of time For example, consider the linear system: $\dot{x} = Ax + Bu$ If u is chosen to be –kx, then $\dot{x} = (A - Bk)x$ becomes autonomous.	$\dot{x} = f(x,t)$

Example of Autonomous System (taken from homework problems)

$$\dot{x} = f(x)\theta + u$$

We want to obtain an estimation algorithm for θ, the unknown parameter.

$$\begin{cases} \dot{\hat{x}} = f(x)\hat{\theta} + u + (x - \hat{x}) \\ \dot{\hat{\theta}} = ef(x) \end{cases}$$

where $\begin{cases} e = x - \hat{x} \\ \tilde{\theta} = \theta - \hat{\theta} \end{cases}$

So the error dynamics are:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = f(x)\theta + u - f(x)\hat{\theta} - u - e$$

$$\text{and } \begin{cases} \dot{e} = \tilde{\theta} - e \\ \dot{\tilde{\theta}} = -ef(x) \end{cases}$$

The error dynamics have no explicit mention of time in them ⇒ the system $\begin{pmatrix} e \\ \tilde{\theta} \end{pmatrix}$ is **autonomous**.

⇒ Use LaSalle's invariant theorem.

<u>Example of Non-Autonomous System (taken from homework problems)</u>

$$\begin{cases} \dot{e} = -e\tilde{\theta} + w(t) \\ \dot{\tilde{\theta}} = -ew(t) \end{cases}$$

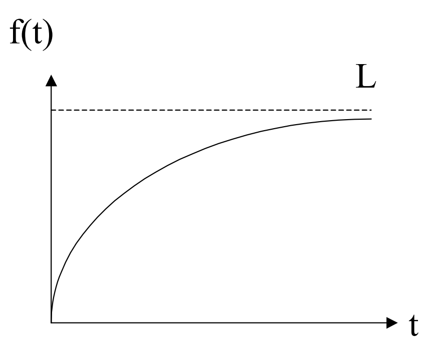
Depends on some function of time from the outside
$\Rightarrow$ the system $\begin{pmatrix} e \\ \tilde{\theta} \end{pmatrix}$ is **non-autonomous.**
$\Rightarrow$ LaSalle's can't be used
$\Rightarrow$ Need a new tool called **Barbalat's lemma**.

Note: In general adaptive systems are non-autonomous.

<u>Lyapunov's second method</u> provides the "decrescent" condition, $V(x) < \beta(\|x\|)$, to guarantee uniform stability of an equilibrium point. For asymptotic stability, we need $\dot{V} < 0$ (negative definite), if we have $\dot{V} \le 0$ (negative semi-definite), LaSalle's invariance principle cannot be used. We use a result presented in the next section, Barbalat's lemma, instead.

<u>Barbalat's Lemma</u>

If we have f(t) which is differentiable and has a limit, that is f$\rightarrow$L and $\dot{f}$ is uniformly continuous, then $\dot{f} \rightarrow 0$.



Why do we need the uniform continuity condition?

f has a limit $\not\longrightarrow$ $\dot{f} \rightarrow 0$

For example: f(t) = $e^{-t}\sin(e^{2t})$ sinusoidal decrescence to zero in exponential envelope.
$\dot{f}$ is infinite (faster and faster oscillations).

$$\boxed{\dot{f} \text{ has a bound derivative} \Rightarrow \dot{f} \text{ is uniformly continuous}}$$

Note that $\dot{f}$ is uniformly continuous if $\ddot{f}$ exists and is bounded.

<u>Lyapunov-Like Lemma</u> (Slotine and Li)

If a scalar function V(x,t) has the following properties:

i. V(x,t) is lower-bounded
ii. $\dot{V}$ is negative semi-definite
iii. $\dot{V}$ is uniformly continuous ($\ddot{V}$ is bounded)

Then $\dot{V} \to 0 \ as \ t \to \infty$

Note: This is a VERY USEFUL lemma, especially for adaptive systems.

<u>Return to a previous example (adaptive control):</u>

The error dynamics for an adaptive control problem were given by:

$$\begin{cases} \dot{e} = -e + \tilde{a}\,\omega(t) \\ \dot{\tilde{a}} = -e\omega(t) \end{cases}$$

We want to prove that: $x \to x_d(t) \ as \ t \to \infty$, that is, $e \to 0 \ as \ t \to \infty$

Let's try a Lyapunov function:

$$V = e^2 + \tilde{a}^2$$

This function is positive definite and decrescent (OK, I can just use 2V to bound it).

$$\dot{V} = -2e^2$$

Hence we know that the origin: $(e(0), \tilde{a}(0))$ is uniformly stable.

That is, $e(t) \ and \ \tilde{a}(t)$ are bounded.

Then, (this is a "sloppy" argument), if $e(t) \to 0 \ and \ \dot{e}(t) \to 0 \ as \ t \to \infty$, then $\tilde{a}\,\omega(t) \to 0$, and if $\omega(t) \neq 0$, then $\tilde{a} \to 0$

Example (Slotine and Li)
Stability of Invariant Sets other than Equilibrium Points

Consider the system given by the following equations:

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^4 + 2x_2^2 - 10) \\ \dot{x}_2 = -x_1^3 - 3x_2^5(x_1^4 + 2x_2^2 - 10) \end{cases}$$

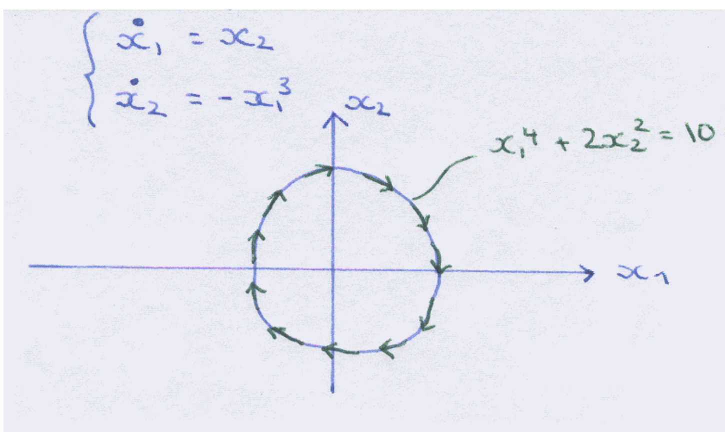
We can prove that the set $x_1^4 + 2x_2^2 = 10$ is invariant.

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -(4x_1^2 + 12x_2^6)(x_1^4 + 2x_2^2 - 10) = 0$$

That is, if I start on this set, I stay on this set.

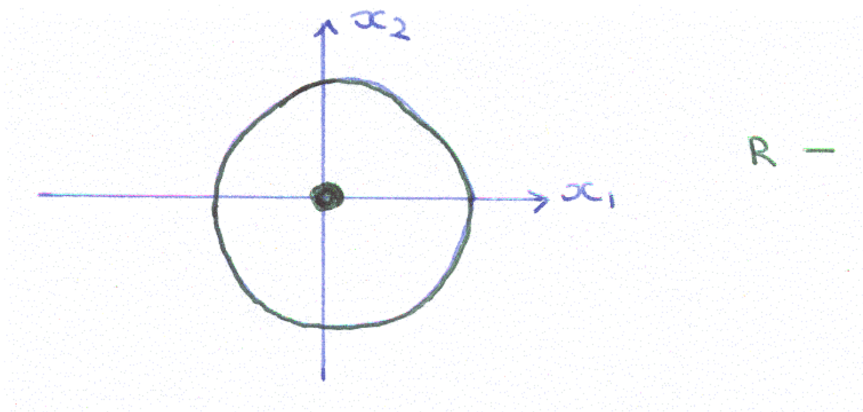
We pick a Lyapunov function candidate: $V(x_1, x_2) = (x_1^4 + 2x_2^2 - 10)^2$

Then $\dot{V} = -8(x_1^4 + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$



Note that the first parenthesis in the expression for $\dot{V}$ is zero at the origin. We can compute the set:

$$R = \{(0,0) \cup (x_1, x_2) \in \Re^2 \mid x_1^4 + 2x_2^2 = 10\}$$



Consider Ω_{100}, that is V(x)<100.

All trajectories in Ω_{100} go to $x_1^4 + 2x_2^2 = 10$, since V>0, $\dot{V}$<0, and the only invariant set is $x_1^4 + 2x_2^2 = 10$. That is, the set $x_1^4 + 2x_2^2 = 10$ is a stable limit cycle.

$\dot{V}$ is always <0, that is, we have a globally stable limit cycle if you exclude the origin.

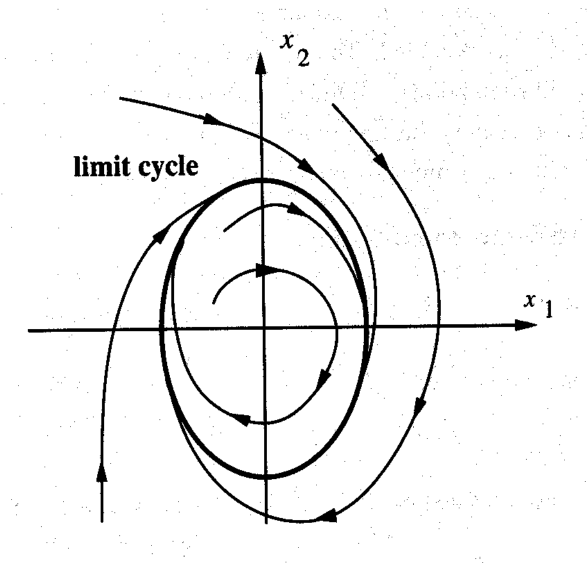
Girard-Hedrick conjecture: The origin of the above system is unstable. Prove it!

Answer:

$\dot{V}$ is strictly negative, except if:

$$x_1^4 + 2x_2^2 - 10 = 0 \text{ or if } x_1^{10} + 3x_2^6 = 0 \text{ in which case } \dot{V} = 0.$$

The first equation is simply that defining the limit cycle, while the second equation is verified only at the origin. Since both the limit circle and the origin are invariant sets, the set M simply consists of their union. Thus, all system trajectories starting in the region Ω_l converge either to the limit cycle, or the origin.



Convergence to a limit cycle

Moreover, the equilibrium point at the origin can actually be shown to be unstable. However, this result cannot be obtained from linearization, since the linearized system $(\dot{x}_1 = x_2, \dot{x}_2 = 0)$ is only marginally stable. Instead, and more astutely, consider the region Ω_{100}, and note that while the origin 0 does not belong to Ω_{100}, every other point in the region enclosed by the limit cycle is in Ω_{100} (in other words, the origin corresponds to a local *maximum* of V). Thus, while the expression for $\dot{V}$ is the same as before, now the set M is just the limit cycle. Therefore, re-application of the invariant set theorem shows that any state trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular, this implies that the equilibrium point at the origin is unstable.

Example: Asymptotic Stability with Time-Varying Damping (Slotine and Li)

Consider the following second-order dynamics:

$$\ddot{x} + c(t)\dot{x} + k_0 x = 0 \qquad (*)$$

which can represent a mass-spring damper system (with mass = 1), where $c(t) \geq 0$ is a time-varying damping coefficient and k_0 is a spring constant. Physical intuition may suggest that the equilibrium point (0,0) is stable as long as the damping coefficient c(t) remains larger that a strictly positive constant (implying constant dissipation of energy), as in the case for autonomous nonlinear mass-spring-damper systems. However, this is not necessarily true. Indeed, consider the system:

$$\ddot{x} + (2 + e^t)\dot{x} + x = 0$$

One easily verifies that, for instance, with the initial condition x(0)=2, $\dot{x}(0) = 1$, the solution is $x(t) = 1 + e^{-t}$ which tends to x=1 instead! Here the damping increases so fast that the system gets "stuck" at x=1.

Let us study the asymptotic stability of this class of systems using a Lyapunov analysis.

Lyapunov stability of the system (though not its asymptotic stability) can be easily established using the mechanical energy of the system as a Lyapunov function. Let us now use a different Lyapunov function to determine sufficient conditions for the asymptotic stability of the origin of the system (*). Consider the following positive definite function:

$$V(x,t) = \frac{[\dot{x} + \alpha x]^2}{2} + \frac{b(t)}{2} x^2$$

where α is any positive constant smaller than $\sqrt{k_0}$, and

$$b(t) = k_0 - \alpha^2 + \alpha c(t)$$

$\dot{V}$ can be easily computed as:

$$\dot{V} = [\alpha - c(t)]\dot{x}^2 + \frac{\alpha}{2}[\dot{c}(t) - 2k_0]x^2$$

Thus, if there exists positive numbers α and β such that:

$$c(t) > \alpha > 0 \text{ and } \dot{c}(t) \le \beta < 2k_0$$

then $\dot{V}$ is negative definite. Assuming in addition that c is upper-bounded (guaranteeing the decrescence of V), the above conditions imply the asymptotic convergence of the system.

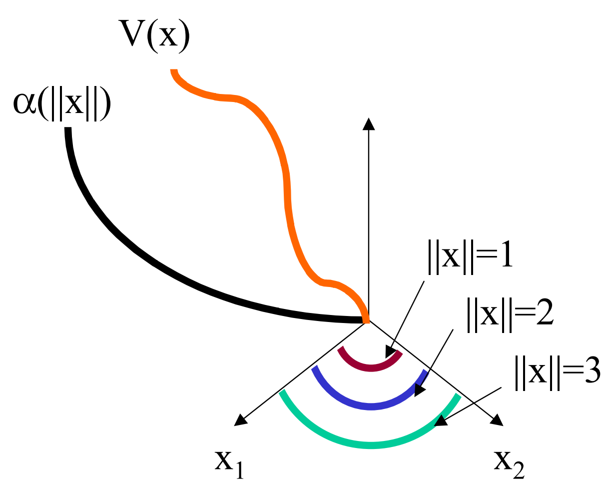
It can be shown that, actually, the technical assumption that c(t) is upper bounded is not necessary. Thus, for instance, the system:

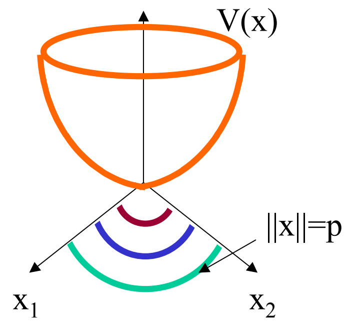
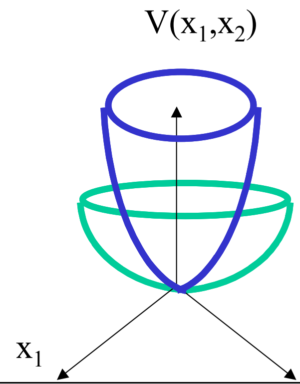
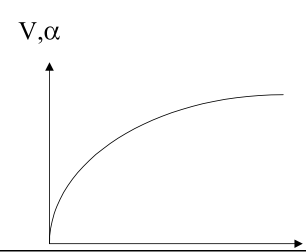
$$\ddot{x} + (2 + 8t)\dot{x} + 5x = 0$$

is asymptotically stable.

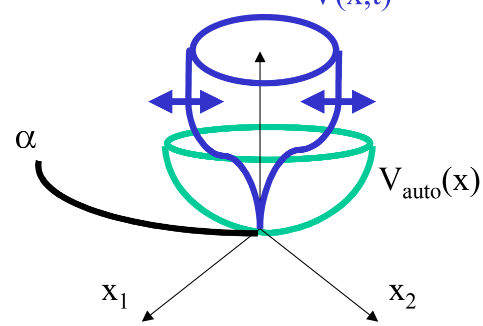
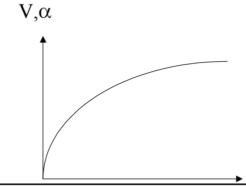
Lyapunov Stability in Pictures

Pictures and Comments for the Time-Invariant Case

	Condition	Picture/Comments
i	**V(0,t) = 0**	Need to check this condition V(x) = 0 at x = 0. We need x = 0 to coincide with V = 0 because the method relies on making V $\to$ 0.
ii	**V(x,t) is positive definite, i.e. there exists a continuous, non-decreasing scalar function α(x) such that α(0) = 0 and** $\forall x \neq 0, 0 < \alpha(\lVert x \rVert) < V(x,t)$	Need to check this. Simplifies to $\alpha(\lVert x \rVert) \leq V(x)$.  The level sets of α are circles. We are used to dealing with the definition of positive definite to mean V(x)≥0 everywhere and V(x) = 0 ⇔ x = 0. Does this coincide with the definition out of the Sastry book? (it better!)

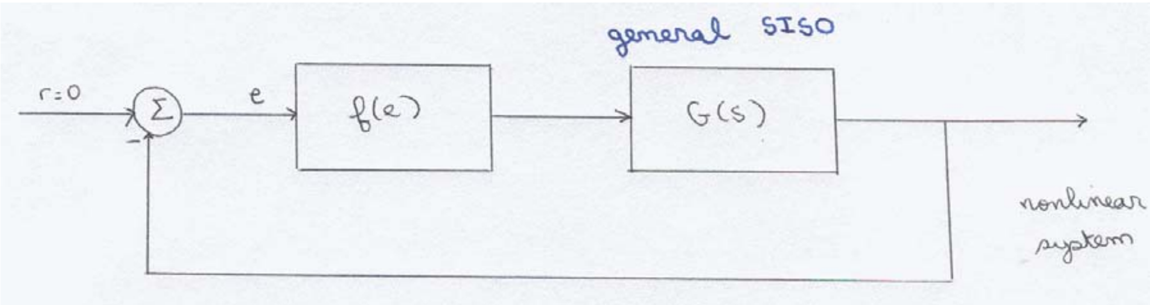
		 $\alpha(p) = \min_{\lVert x\rVert = p\,circle} V(x)$
iii	$\dot{V}(x,t)$ **is negative definite, that is** $\dot{V}(x,t) \le -\gamma(\lVert x\rVert) < 0$ **where γ is a continuous non-decreasing scalar function such that γ(0) = 0**	Similar interpretation to that of (ii), only looking at $-\dot{V}(x)$ If the condition does not hold, we can use LaSalle's (time-invariant cases).
iv	$V(x,t) \le \beta(\lVert x\rVert)$ **where β is a continuous non-decreasing function and β(0) = 0, i.e. V is decrescent, i.e. the Lyapunov function is upper bounded**	**Doesn't matter for time-invariant systems** Simplifies to $V(x) \le \beta(\lVert x\rVert)$. 
v	**V is radially unbounded, that is** $\alpha(\lVert x\rVert) \to \infty$ **as** $\lVert x\rVert \to \infty$	Need to check to get a global result. 

Pictures and Comments for the Time-Variant Case

	Condition	**Picture/Comments**
i	**V(0,t) = 0**	Need to check this condition V(x) = 0 at x = 0. We need x = 0 to coincide with V = 0 because the method relies on making V $\rightarrow$ 0.
ii	**V(x,t) is positive definite, i.e. there exists a continuous, non-decreasing scalar function α(x) such that α(0) = 0 and** $\forall x \neq 0, 0 < \alpha(\lVert x \rVert) < V(x,t)$	Finding $\alpha(\lVert x \rVert), \alpha \in K$ could be hard. If $\exists$ $V_{autonomous}(x)$ positive definite such that $V_{autonomous}(x) \leq V(x,t)$ then $\alpha \in K$ exists. 
iii	$\dot{V}(x,t)$ **is negative definite, that is** $\dot{V}(x,t) \leq -\gamma(\lVert x \rVert) < 0$ **where γ is a continuous non-decreasing scalar function such that γ(0) = 0**	Similar interpretation to that of (ii), only looking at $\dot{V}(x,t)$ and bounding it by $\dot{V}_{auto}(x)$ If we don't get this, we can use **Barbalat's lemma**.
iv	$V(x,t) \leq \beta(\lVert x \rVert)$ **where β is a continuous non-decreasing function and β(0) = 0, i.e. V is decrescent, i.e. the Lyapunov function is upper bounded**	Need this for uniformity of stability. Find $V_{autonomous}(x) \geq V(x,t) \Rightarrow \exists \beta \in K$ such that $V(x,t) \leq V_{auto}(x) \leq \beta(\lVert x \rVert)$
v	**V is radially unbounded, that is** $\alpha(\lVert x \rVert) \rightarrow \infty$ **as** $\lVert x \rVert \rightarrow \infty$	Need to check to get a global result. 

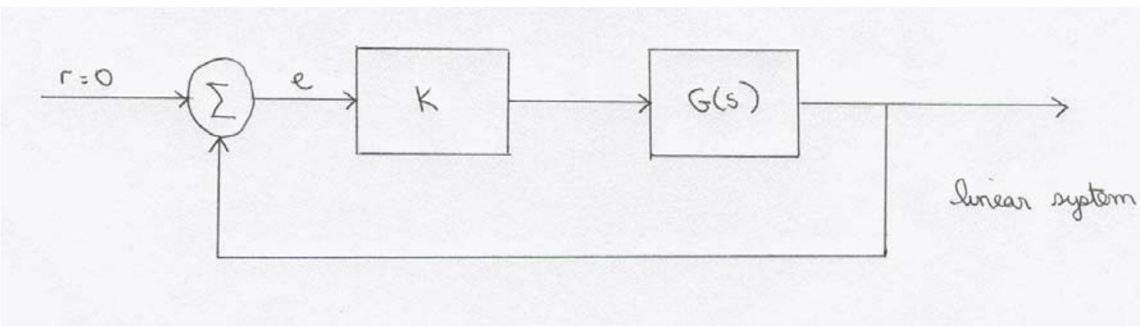
Frequency Domain Stability Criteria

We now consider a special kind of nonlinear system, where a general SISO linear system is connected in series with a nonlinearity (for example, actuator saturation) as shown below. This class of systems was studied in detail in the 1950s.



The conjectures of Aizerman and Kalman

We start by considering the conjectures of Aizerman and Kalman. Both relate the stability of the nonlinear system above to that of a system of the following type:



Both conjectures were proven wrong by counter-example.

Aizerman's conjecture

If the nonlinearity is bounded by $k_1 \leq \frac{f(e)}{e} \leq k_2$, and if the linear system is stable for $k_1 \leq K \leq k_2$, then the nonlinear system is stable.

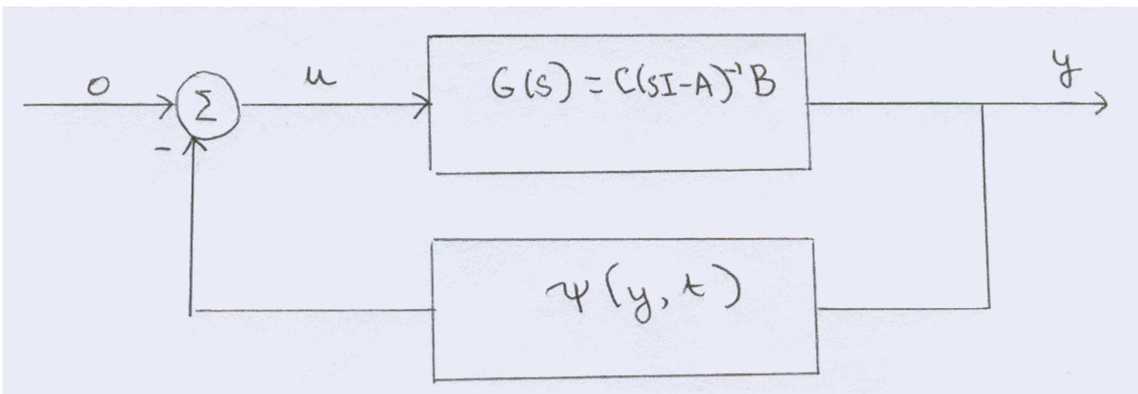
Kalman's conjecture

If the nonlinearity is bounded by $k_1 \leq f'(e) \leq k_2$, and if the linear system is stable for $k_1 \leq K \leq k_2$, then the nonlinear system is stable.

Absolute Stability

We would like to prove global asymptotic stability of the nonlinear system for a whole class of nonlinearities (hence the name "absolute stability"). Even though the two conjectures above are false, two different criteria, the circle criterion and the Popov criterion, are applicable. Both provide sufficient, but not necessary, conditions. They both derive from the Kalman-Yakubovitch lemma, which relies on the concept of positive real functions. We will consider these different concepts in turn.

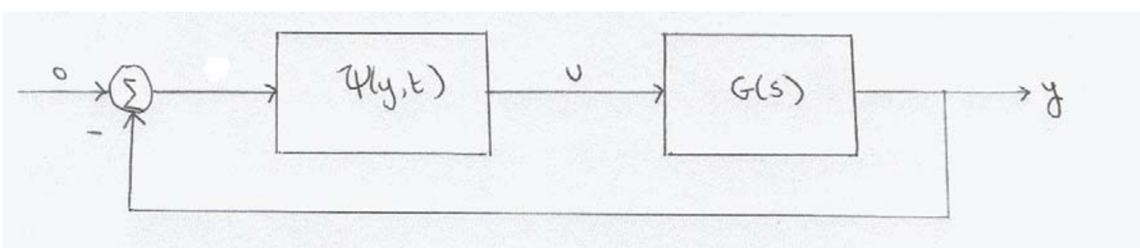
Consider a system of the following form:



The systems is described by the following equations:

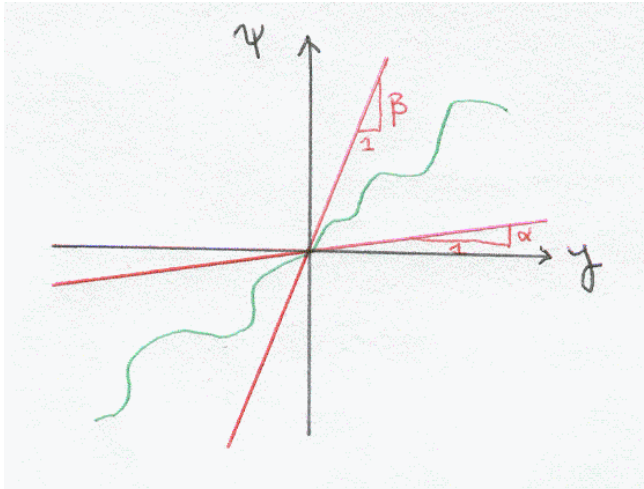
$$\begin{cases} \dot{x} = Ax - B\psi(y,t) \\ y = Cx \end{cases} \qquad (1)$$

The system can be redrawn as follows (since I have no input):



We assume that the linear system G(s) is minimal, that is:
- A, B is controllable
- A, C is observable

In addition, let us assume that ψ(y,t) satisfies sector constraints, that is:



$$\alpha y^2 \le y\psi \le \beta y^2$$

We consider two different Lyapunov function candidates, which yield the two different techniques to evaluate absolute stability, namely the circle criterion and the Popov criterion.

$$V_1 = x^T P x$$

$$V_2 = x^T P x + \eta \int_0^y \psi d\tau$$, where P is positive definite, and η>0 (the integral is also >0).

V_1 yields the circle criterion and V_2 yields the Popov criterion.

Luré was the first to state this problem, sometimes called Luré's problem.

Before we start, we need to set some definitions.

Definition (positive real, strictly positive real)
A transfer function Z(s) is called positive real if Z(s) is positive semi-definite for Re(s)>0. It is called strictly positive real (SPR) if Z(s-ε) is positive real for some ε>0.

Kalman-Yakubovitch Lemma (K-Y Lemma)

Importance: relates the frequency domain to the state space

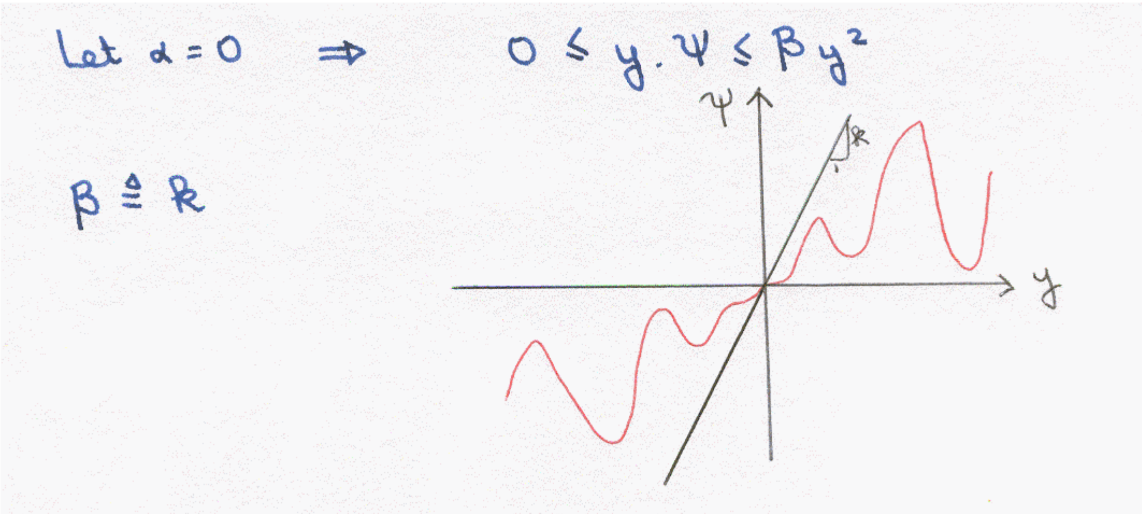
Let $Z(s) = C(sI - A)^{-1} B + D$ be a scalar (matrix case is treated in Khalil) where the matrix A is Hurwitz, (A, B) is a controllable pair, (A, C) is an observable pair.

Then Z(s) is strictly positive real if and only if there exists a symmetric positive definite matrix P, matrices W and L and a positive constant ε such that:

$$\begin{cases} PA + A^T P = -L^T L - \varepsilon P \\ PB = C^T - L^T W \\ W^T W = D + D^T \end{cases}$$

The proof of this result is provided in Khalil.

<u>The Circle Criterion</u>



As discussed above, we pick $V = x^T P x$, where P is symmetric and positive definite.

Note: If P is not symmetric, it can be separated into a symmetric and an asymmetric component:

$$P = P_S + P_{AS} \Rightarrow V = x^T P_S x + x^T P_{AS} x = x^T P_S x \; as \; x^T P_{AS} x = 0$$

The system equations are given by:

$$\begin{cases} \dot{x} = Ax - B\psi(y,t) \\ y = Cx \end{cases}$$

We compute the expression for $\dot{V}$:

$$\dot{V} = x^T (PA + A^T P)x - 2x^T PB\psi(y,t)$$

Now, from the sector constraints that we imposed above, we can derive the following equation:

$$\psi(\psi - ky) < 0$$

This can be rewritten as:

$$-\psi(\psi - ky) > 0$$

We then add a positive quantity to the RHS of the equation for $\dot{V}$ and use the fact that y=Cx:

$$\dot{V} \leq x^T(PA + A^T P)x - 2x^T PB\psi(y,t) - 2\psi(\psi - ky)$$

Or alternatively:

$$\dot{V} \leq x^T(PA + A^T P)x + 2x^T(C^T k - PB)\psi(y,t) - 2\psi^2$$

This result was first derived by Luré.

Suppose that we have:

$$PA + A^T P = L^T L - \varepsilon P$$
$$PB = C^T k - \sqrt{2}L^T$$

If this is true, then:

$$\dot{V} \leq -\varepsilon x^T Px - (Lx - \sqrt{2}\psi)^2$$

Then $\dot{V}$ will be negative definite if one can find P, L and ε, and this will guarantee asymptotic stability of the origin.

At this point we can use the K-Y Lemma if we make the following substitutions:

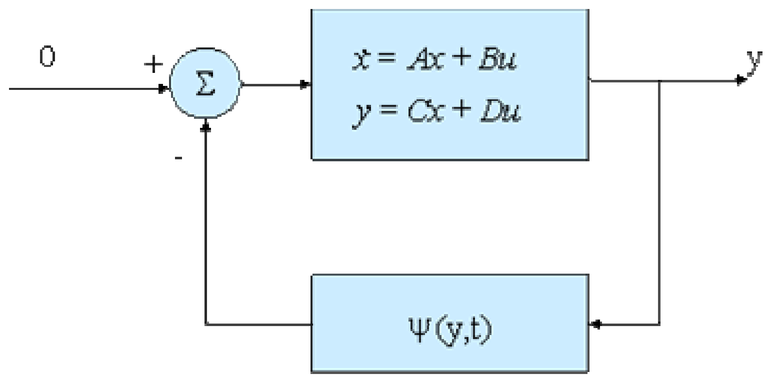
$$C^T \Rightarrow C^T k$$
$$W \Rightarrow \sqrt{2}I$$
$$D \Rightarrow I$$

Then the K-Y lemma implies that P, L and ε exist if and only if:

$$Z(s) \equiv 1 + KC(sI - A)^{-1}B = 1 + KG(s) \text{ is SPR}$$

Lemma (The Circle Criterion)

The origin of the system:



with $G(s) = C(sI - A)^{-1}B + D$

is <u>globally asymptotically stable</u> if:

$$\psi(\psi - ky) < 0$$

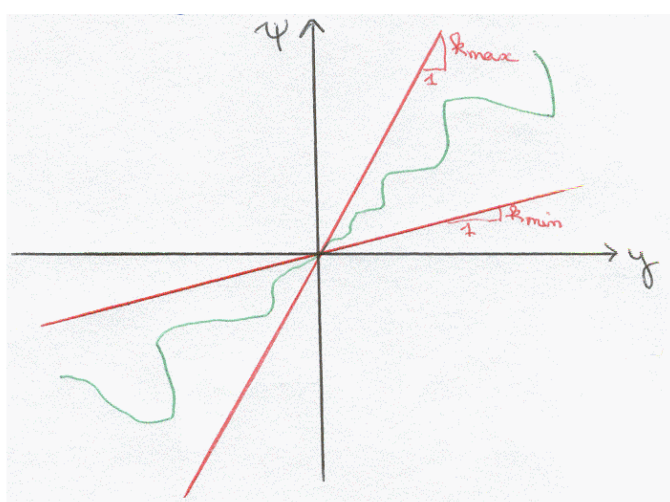
and

$$Z(s) \equiv 1 + KG(s) \; is \; SPR$$

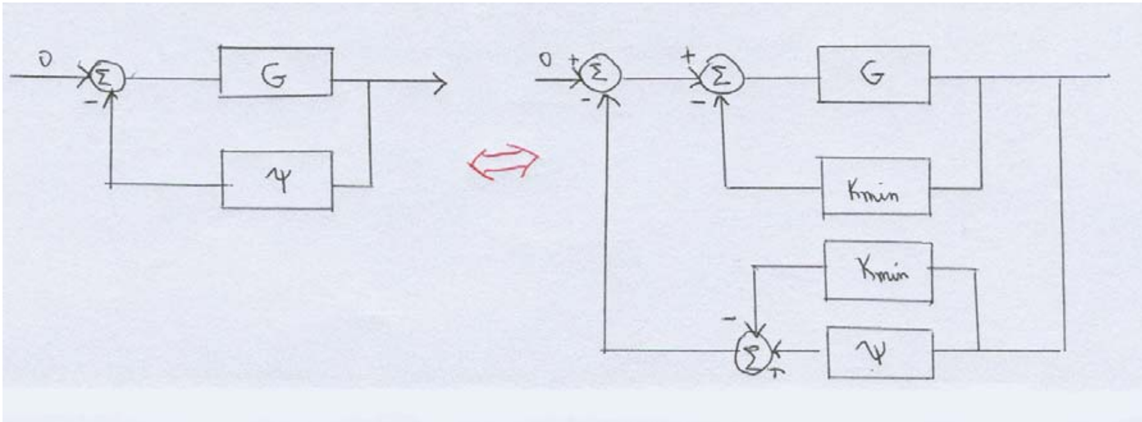
<u>Note:</u> The A matrix has to be Hurwitz.

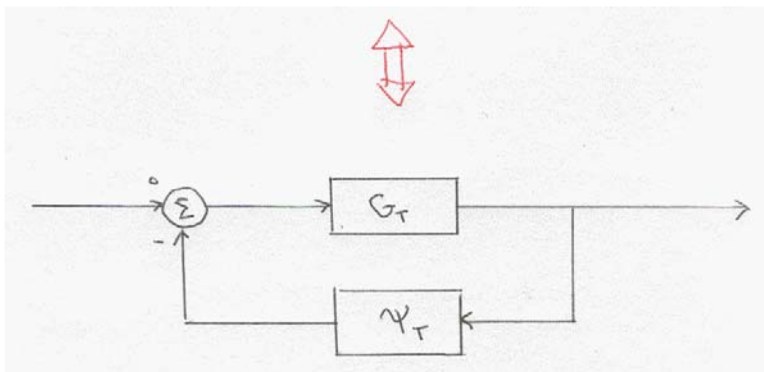
The sector restriction can be relaxed by a transformation. Suppose the nonlinearity lives in a sector defined by:

$$(\psi - k_{\min} y)(\psi - k_{\max} y) \leq 0$$

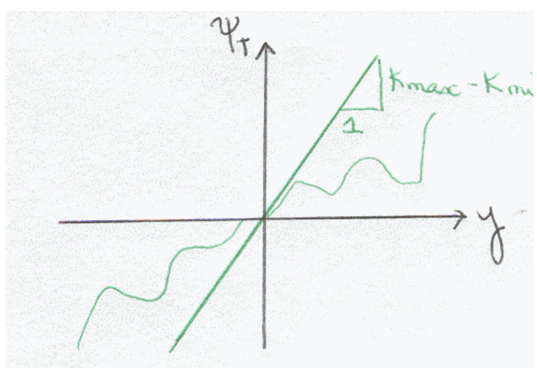


Then we can apply the following transformation:





With $G_T = \dfrac{G}{1 + k_{\min} G}$ and $\psi_T = \psi - k_{\min} y$, $k = k_{\max} - k_{\min}$



If we would like to rewrite this condition in the state-space (as for example in Sastry):

$$\begin{cases} \dot{x} = (A - k_{\min} BC) - B\psi_T \\ y = Cx \\ \psi_T = \psi - k_{\min} y \end{cases}$$

Then if A-Bk$_{\min}$C is Hurwitz, the system is <u>globally asymptotically stable</u> if:

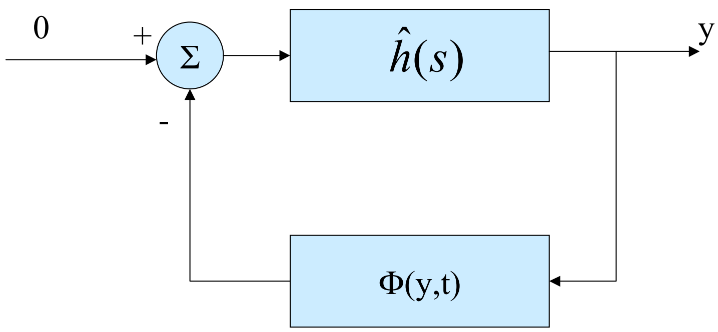
$$Z_T(s) \equiv 1 + KG_T(s) \text{ is } SPR$$

where $G_T = \dfrac{G}{1 + k_{\min} G}$, and $Z_T = \dfrac{1 + k_{\max} G}{1 + k_{\min} G}$

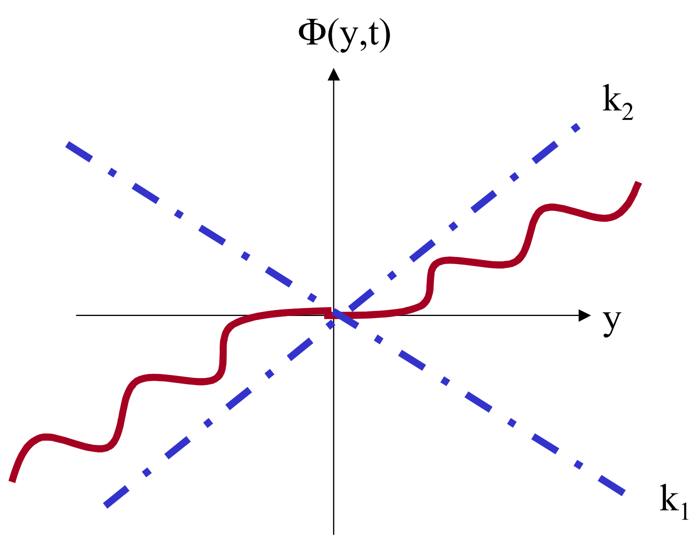
<u>The Graphical Circle Criterion</u>

1. For details (lots available), see Khalil
2. This is where the name "circle criterion" comes from

The closed loop system in standard form:

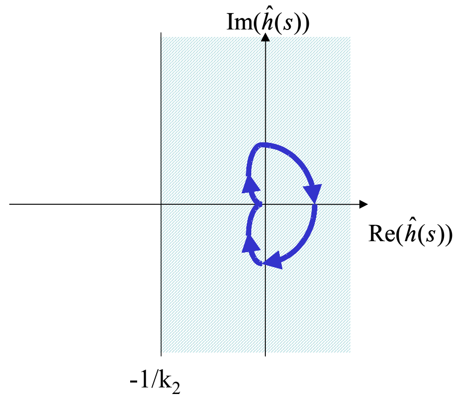


with Φ bounded like:

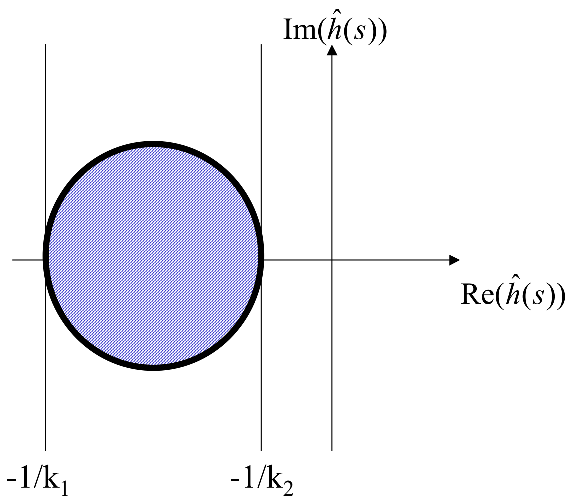


is globally asymptotically stable if any of the following hold:

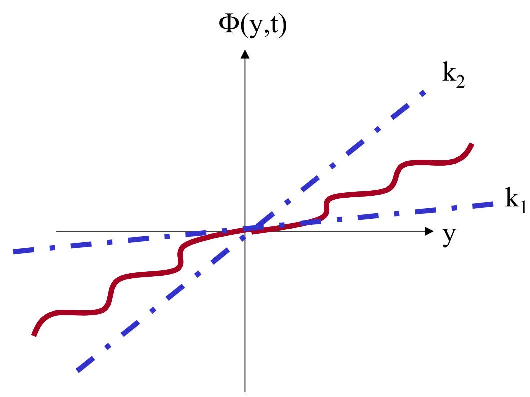
1) If $0=k_1<k_2$, the Nyquist plot lies to the right of the vertical line Re(s)=-1/K and $\hat{h}(s)$ is stable.



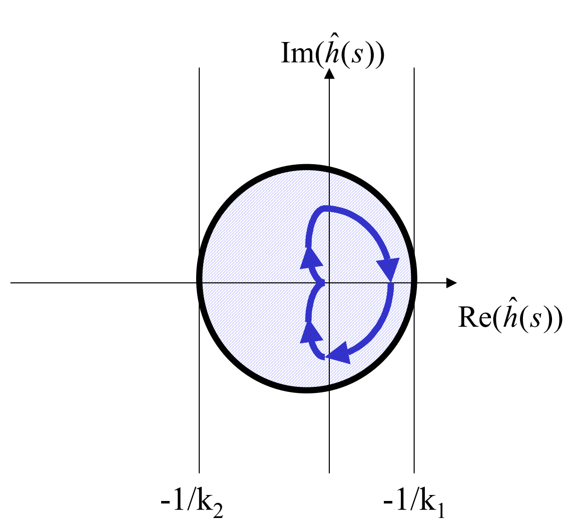
2) If $0<k_1<k_2$, the Nyquist plot of $\hat{h}(s)$ encircles this circle:

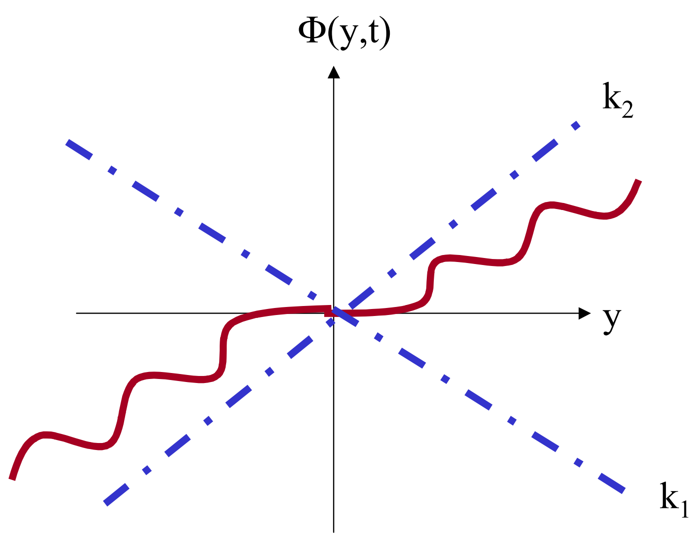


n times counterclockwise where n is the number of poles of $\hat{h}(s)$



3) If $k_1<0<k_2$, if $\hat{h}(s)$ is stable and the Nyquist plot of $\hat{h}(s)$ lies inside the following circle:



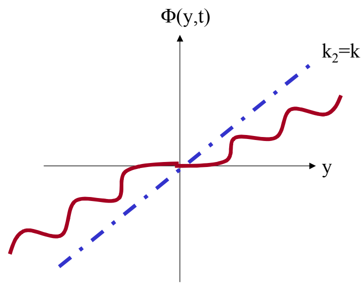


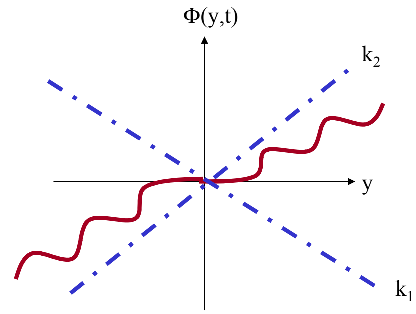
<u>Note:</u> MIMO versions of the circle criterion do exist.

<u>To use this criterion:</u>

Find out if $\hat{h}(s)$ is stable or unstable.

(i) If stable: try both tests 1 and 3

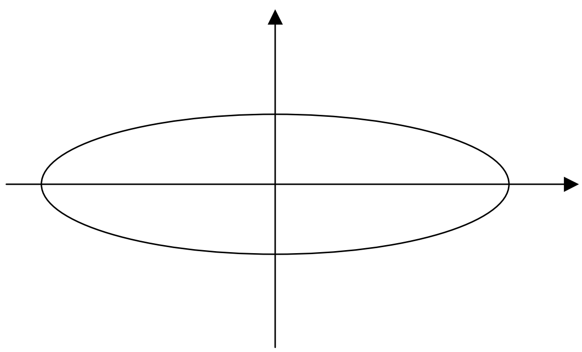




(ii) If unstable: use condition 2, plot the Nyquist contour of $\hat{h}(s)$ and try to fit discs inside it so that they encircle the disc n times.

<u>Practicalities:</u>

Matlab typically doesn't plot a circle as a circle. For example, $x^2 + y^2 = 1$



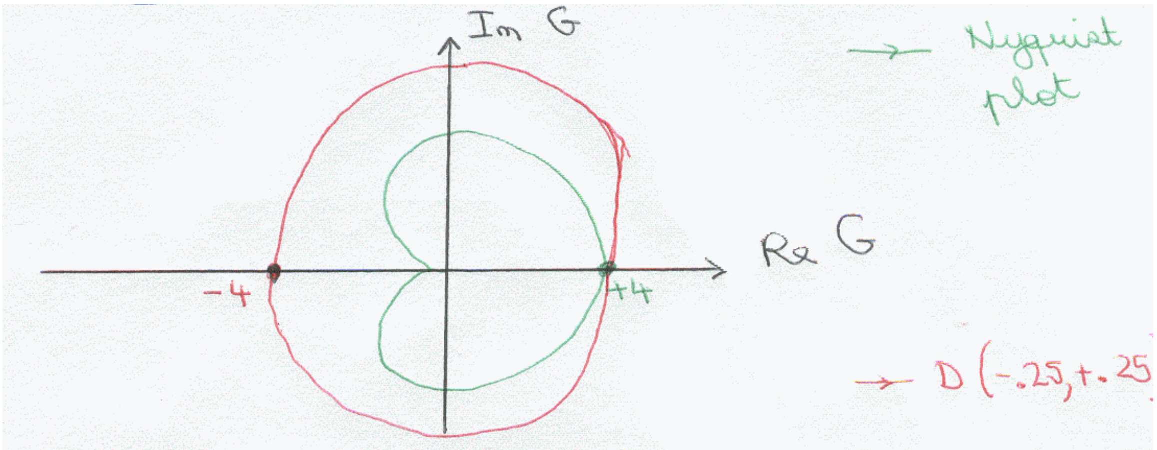
This means that Nyquist plots provided by Matlab typically do not have "even" axis.

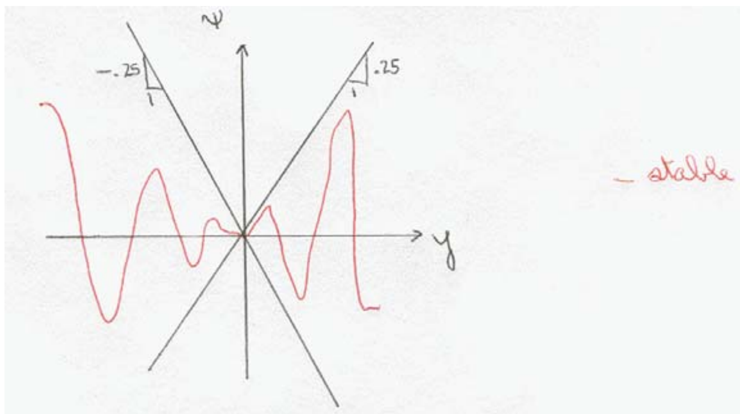
Make sure to equalize the axes of the Nyquist plots by using the command "`axis equal`" before trying any graphical tests.

<u>Example: (Khalil)</u>

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

We use the third condition:





<u>Note:</u> the circle criterion provides a sufficient condition

<u>Note:</u> The circle criterion gives a powerful result (can be any nonlinearity in the sector). As a result, it tends to be <u>very conservative</u>.

Another method, due to Popov, is less conservative as it considers only time-invariant nonlinearities.

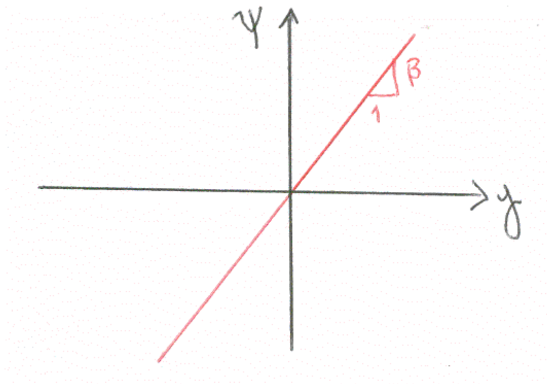
<u>The Popov criterion</u>

Popov: Roumanian mathematician (in his 90s). Refused the ASME trophy.

Consider the system described by the following equations:

$$\begin{cases} \dot{x} = Ax - B\psi(y) \\ y = Cx \end{cases}$$

A is Hurwitz, ψ(y) is time-invariant and $0 \le y\psi \le \beta\psi^2$



<u>Note:</u> The original derivation does not use Lyapunov techniques!

As discussed above, we will consider a Lyapunov function candidate given by:

$$V = V_2 = x^T Px + 2\eta \int_0^y \beta\psi d\tau$$

The derivation follows that presented above for the circle criterion:

We compute the expression for $\dot{V}$:

$$\dot{V} = x^T(PA + A^T P)x - 2x^T PB\psi(y,t) + 2\eta\psi\beta C(Ax - B\psi)$$

$$\ldots$$

$$\dot{V} \le -\varepsilon x^T Px - (Lx - W\psi)^2$$

$$\ldots$$

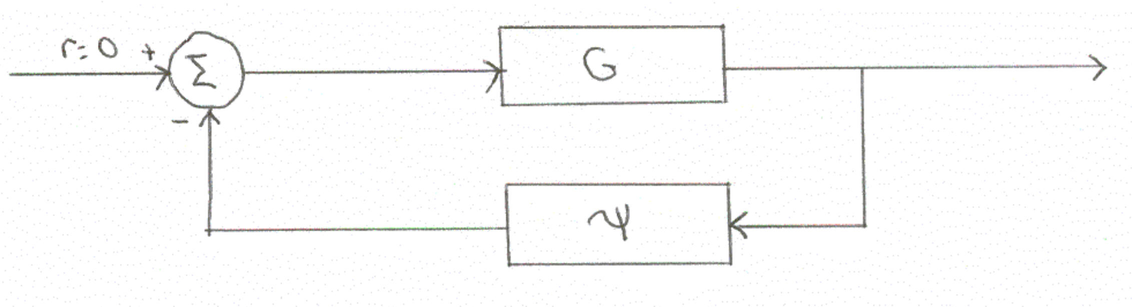
$$\begin{aligned} &2(1+\eta\beta CB) = W^2 \\ &PA + A^T P = L^T L - \varepsilon P \\ &PB = C^T\beta + \eta\beta A^T C^T - L^T W \end{aligned}$$

Using the K-Y Lemma, one can show that W, P, L and ε, exist if:

$$Z(s) \equiv 1 + (1+\eta s)\beta G(s) \text{ is SPR}$$

<u>The Popov Theorem</u>

The following system:



is globally asymptotically stable if: A is Hurwitz, (A, B) is a controllable pair, (A, C) is an observable pair, ψ is time-invariant, $0 \le y\psi \le \beta\psi^2$ and if η≥0 (and -1/η cannot be an eigenvalue of A) and

$$Z(s) \equiv 1 + (1 + \eta s)\beta G(s) \textit{ is SPR}$$

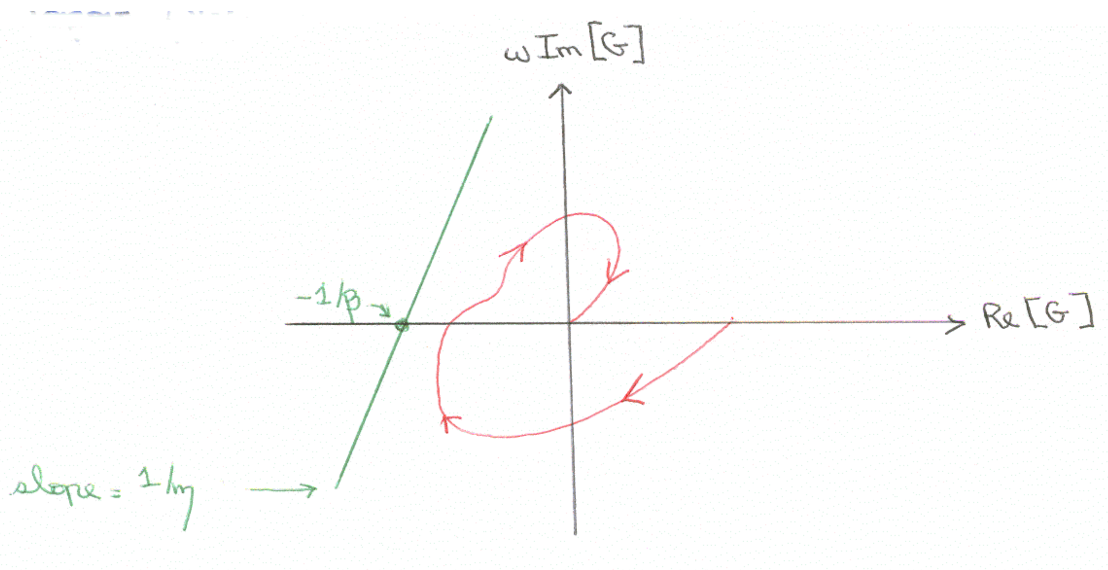
This last condition is equivalent to:

$$\mathrm{Re}\left[1 + (1 + \eta j\omega)\beta G(j\omega)\right] > 0 \quad \textit{for all } \omega$$

Or:

$$\frac{1}{\beta} + \mathrm{Re}\left[G(j\omega)\right] - \eta\omega\,\mathrm{Im}\left[G(j\omega)\right] > 0 \quad \textit{for all } \omega$$

The "ω" in the third term prevents us from using Nyquist plots for this criterion. To display this condition graphically, we need to define a "Popov plot".



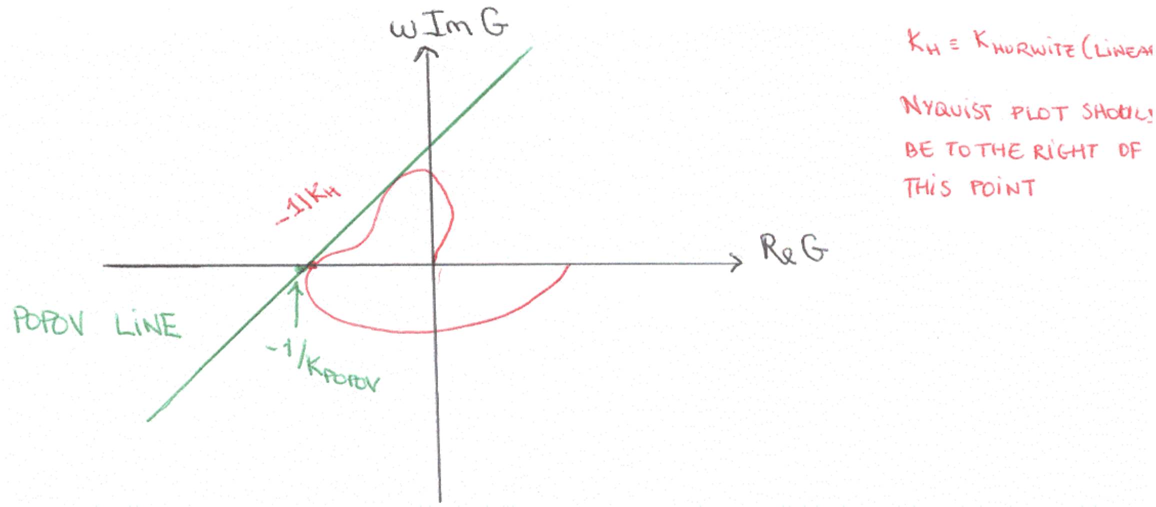
<u>The Popov plot has to be to the right of the Popov line for global asymptotic stability.</u>

The <u>Popov line</u> is defined by:

$$\frac{1}{\beta}+\operatorname{Re}[P(\omega)]-\eta\operatorname{Im}[P(\omega)]=0 \quad \text{where } P(\omega)=\operatorname{Re}[G]+j\omega\operatorname{Im}[G]$$

<u>Example:</u>

Suppose $\psi = ky$ (linear, for a test of how conservative this is)



The origin will be globally asymptotically stable if $K_H \geq K_{POPOV}$

How conservative the results will be depends on ψ.

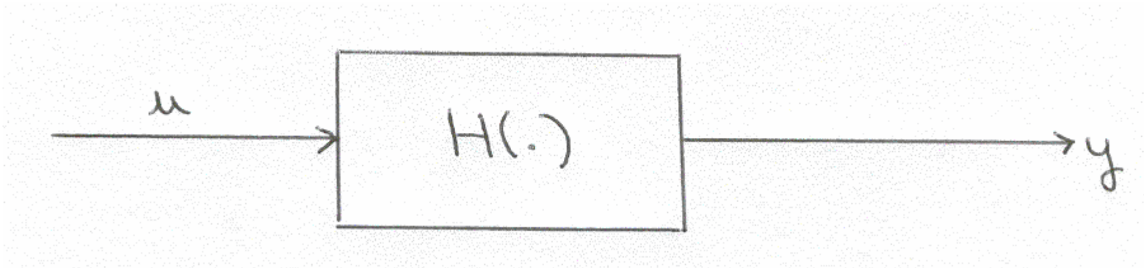
Input-Output Stability (also called L-stability)

Reference: Khalil

Consider a system of the form:

$$\begin{cases} \dot{x} = f(x,t,u) \\ y = h(x,t,u) \end{cases}$$

We can rewrite the second equation as $y = Hu$ where H is a nonlinear, time varying operator.



In this case, we consider that u belongs to a class of signals; for example:

$$\|u\|_\infty \equiv \sup_{t\geq 0}\|u(t)\| < \infty$$

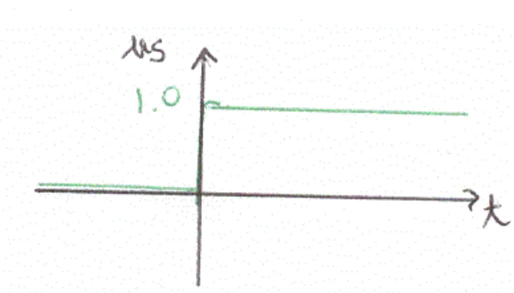
or

$$\|u\|_2 \equiv \sqrt{\int_0^\infty \|u(t)\|^2\, dt} < \infty$$

or more generally, $L_p^m \; for \; 1 \leq p < \infty$ is defined as the set of all piecewise continuous functions in $u \in \Re^m$ such that:

$$\|u\|_{L_p} \equiv \left(\int_0^\infty \|u(t)\|^p\, dt\right)^{1/p} < \infty$$

Note: The step function $u_S(t)$ does not satisfy those norms (integral goes to infinity)

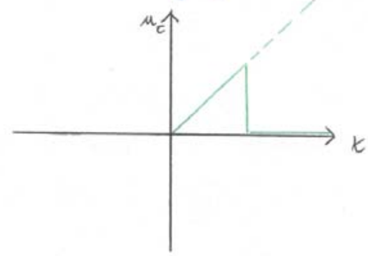


In order to admit step functions, ramps etc…, we can define an extended space:

$$L_e^m = \{u \mid u_\tau \in L^m, \forall \tau > 0\} \text{ where } u_\tau(t) \equiv \begin{cases} u(t) & 0 \leq t \leq \tau \\ 0 & \tau > t \end{cases}$$

$$\mathcal{L}_e^m = \{ u \mid u_\tau \in \mathcal{L}^m , \forall \tau > 0 \}$$

where $u_\tau(t) \triangleq \begin{cases} u(t) & 0 \le t \le \tau \\ 0 & \tau > t \end{cases}$



Definition:

A mapping $H : L_e^m \to L_e^q$ is called "L-stable" if there exists non-negative constants, γ and β, such that:

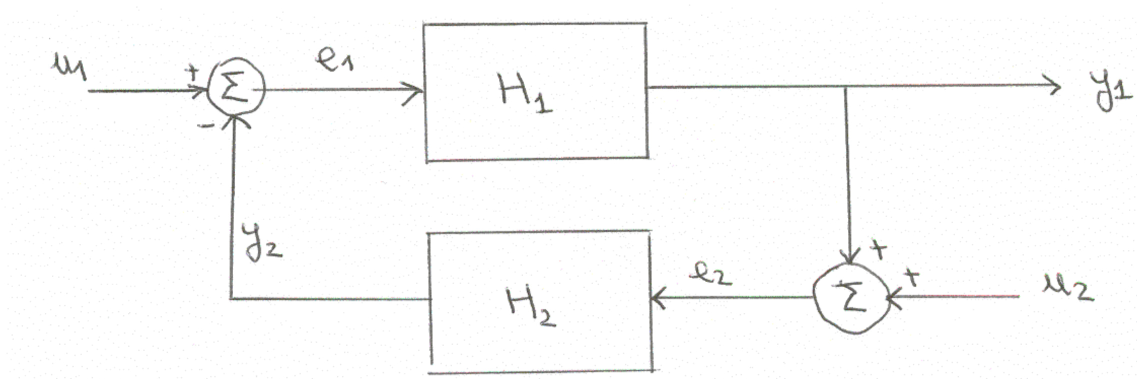
$$\|(Hu)_\tau\| \le \gamma \|u_\tau\| + \beta \text{ for all } u \in L_e^m \text{ and all } \tau \in [0, \infty[$$

(The LH term represents the output, which is bounded, by the input (RH term), which is in some class, so it is bounded).

We define the smallest gain γ for which a β exists to be the gain of the nonlinear mapping.

The Small Gain Theorem

Consider a feedback interconnection:



Assume that both H_1 and H_2 are L-stable with finite γ_1, β_1, γ_2 and β_2.

What are the conditions under which the interconnection is L-stable?

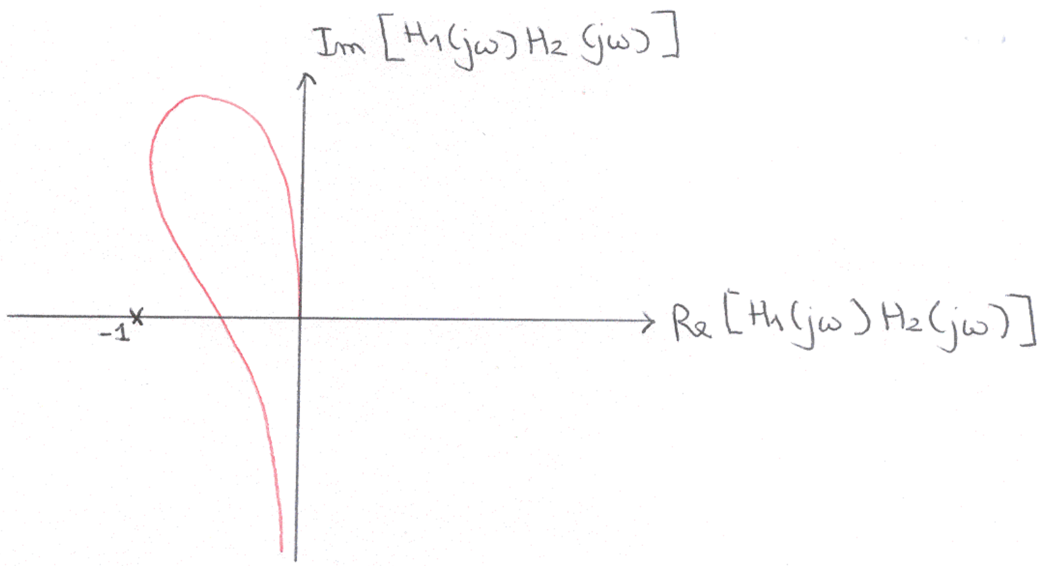
Answer:

$$\boxed{\gamma_1\gamma_2<1}$$

(This answer is provided by the small gain theorem. We will justify this later.)

What if $H_1(s)$ and $H_2(s)$ are linear operators, both of them open-loop stable?

$$|H_1(j\omega)|<\gamma_1$$
$$|H_2(j\omega)|<\gamma_2$$



For a linear system: need to miss the (-1,0) point.
The nonlinear condition requires that the system stay in the unit cycle (this is a lot more conservative).

Sketch of a proof for the small gain theorem:

Result: the interconnection (as shown above) is L-stable if:

$$\gamma_1\gamma_2<1$$

Let
$$e_{1\tau}=u_{1\tau}-(H_2e_2)_\tau$$
$$e_{2\tau}=u_{2\tau}-(H_1e_1)_\tau$$

Then, using the triangular inequality,

$$\begin{aligned} \| e_{1\tau} \| &\le \| u_{1\tau} \| + \| (H_2 e_2)_\tau \| \\ &\le \| u_{1\tau} \| + \gamma_2 \| e_{2\tau} \| + \beta_2 \end{aligned}$$

That is,

$$\| e_{1\tau} \| \le \gamma_1 \gamma_2 \| e_{1\tau} \| + \| u_{1\tau} \| + \gamma_2 \| u_{2\tau} \| + \gamma_2 \beta_1 + \beta_2$$

Now, if $\gamma_1\gamma_2<1$,

$$\| e_{1\tau} \| \le \frac{1}{1-\gamma_1\gamma_2} [\| u_{1\tau} \| + \gamma_2 \| u_{2\tau} \| + \gamma_2 \beta_1 + \beta_2]$$

That is, the error is bounded. The result is identical for $\|e_{2\tau}\|$.

<u>Note:</u> in later studies you may find this condition useful to assess <u>stability robustness</u> (of linear and/or nonlinear systems):