# Lyapunov Stability Theory: Nonautonomous Systems

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## Outline

- Equilibrium of nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$.
- Stability definitions.
- Stability theorems.



## Equilibrium Point

- Consider nonautonomous systems.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \boldsymbol{f}: D \times \mathcal{R}^+ \to \mathcal{R}^n$$

$D$ = open connected subset of $\mathcal{R}^n$

$\boldsymbol{f}$ locally Lipschitz in $\boldsymbol{x}$, continuous in $t$ on $D \times \mathcal{R}^+$

Equilibrium point at $t_0$

$$\boldsymbol{f}(\mathbf{0}, t) = \mathbf{0}, \forall t \geq t_0$$



## Equilibrium at the Origin

- W.l.og., assume $\boldsymbol{x}_e = \mathbf{0}$ equilibrium point at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$
- If $\boldsymbol{x}_e \neq \mathbf{0}$ is an equilibrium point at $t_0$, translate the axes.
- Translation of a nonzero trajectory $\overline{\boldsymbol{x}}(t)$

$$\boldsymbol{z} = \boldsymbol{x}(t) - \overline{\boldsymbol{x}}(t)$$

$$\dot{\boldsymbol{z}} = \dot{\boldsymbol{x}} - \dot{\overline{\boldsymbol{x}}}$$

$$= \boldsymbol{f}(\boldsymbol{z} + \overline{\boldsymbol{x}}(t), t) - \boldsymbol{f}(\overline{\boldsymbol{x}}, t) = \mathbf{0}, \boldsymbol{z} = \mathbf{0}, \forall t \geq t_0$$

$$\dot{\boldsymbol{z}} = \overline{\boldsymbol{f}}(\boldsymbol{z}, t), \boldsymbol{z}_e = \mathbf{0}$$

## Stability Definitions

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is

- Stable if $\forall \epsilon > 0, \exists\ \delta(\epsilon, t_0) > 0$ such that

  $\|\boldsymbol{x}(t_0)\| < \delta(\epsilon, t_0) \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon, \forall t \geq t_0 > 0$

  Otherwise, $\boldsymbol{x}_e$ is unstable.

- Convergent at $t_0$ if $\exists\ \delta_1(t_0) > 0$ s.t.

  $\|\boldsymbol{x}(t_0)\| < \delta_1(t_0) \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{0}$ *(or)*

  $\forall \epsilon > 0, \exists T(\epsilon, t_0)$ *s.t.* $\|\boldsymbol{x}(t_0)\| < \delta_1 \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon,$
  $\forall t \geq t_0 + T(\epsilon, t_0)$

  Asymptotically stable if it is both stable & convergent.



## Uniform Stability Definitions

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is

- Uniformly stable if $\forall \epsilon > 0, \exists\ \delta(\epsilon) > 0$ s.t

  $\|\boldsymbol{x}(t_0)\| < \delta(\epsilon) \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon, \forall t \geq t_0 > 0$

- Uniformly convergent if $\exists\ \delta_1 > 0$ s.t.

  $\|\boldsymbol{x}(t_0)\| < \delta_1 \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{0}$ *(or)*

$\forall \epsilon > 0, \exists T(\epsilon)$ *s.t.* $\|\boldsymbol{x}(t_0)\| < \delta_1 \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon,$
$\forall t \geq t_0 + T(\epsilon)$



## Uniform Asymptotic Stability

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is

- Uniformly asymptotically stable if it is both uniformly stable & uniformly convergent.
- Globally uniformly asymptotically stable if it is uniformly asymptotically stable and every motion converges to the origin.



## Exponential Stability

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ is exponentially stable if $\exists \alpha, \lambda > 0$ s.t.

$$\|\boldsymbol{x}(t)\| \leq \alpha \|\boldsymbol{x}(t_0)\| e^{-\lambda t}$$

$$\forall t \geq t_0, \forall \|\boldsymbol{x}(t_0)\| < \delta$$

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ is globally exponentially stable if the condition holds $\forall x \in \mathcal{R}^n$

## Positive Semidefinite Functions

$$W(\boldsymbol{x},t), \qquad W: D \times \mathcal{R}^+ \to \mathcal{R}, \mathbf{0} \in D$$

- $W(\boldsymbol{x},t)$ continuously differentiable w.r.t. all its arguments.
- $W(\boldsymbol{x},t)$ Positive semidefinite in $D$

*i.* $W(\mathbf{0},t) = 0, \forall t \in \mathcal{R}^+$

*ii.* $W(\boldsymbol{x},t) \geq 0, \forall \boldsymbol{x} \in D, \boldsymbol{x} \neq \mathbf{0}, \forall t \in \mathcal{R}^+$



## Positive Definite Functions

$$W(\boldsymbol{x},t), \qquad W: D \times \mathcal{R}^+ \to \mathcal{R}, \mathbf{0} \in D$$

- $W(\boldsymbol{x},t)$ continuously differentiable w.r.t. all its arguments.
- $W(\boldsymbol{x},t)$ Positive definite in $D$:

*i.* $W(\mathbf{0},t) = 0, \forall t \in \mathcal{R}^+$

*ii.* $V(\boldsymbol{x}) \leq W(\boldsymbol{x},t), \forall \boldsymbol{x} \in D, \boldsymbol{x} \neq \mathbf{0}, \ \forall t \in \mathcal{R}^+$, $V(\boldsymbol{x})$ positive definite.



## Decrescent Functions

$$W(\boldsymbol{x},t), \qquad W: D \times \mathcal{R}^+ \to \mathcal{R}, \mathbf{0} \in D$$

- $W(\boldsymbol{x},t)$ continuously differentiable w.r.t. all its arguments.
- $W(\boldsymbol{x},t)$ decrescent in $D$ if

$$|W(\boldsymbol{x},t)| \leq V(\boldsymbol{x}), \forall \boldsymbol{x} \in D, \forall t \in \mathcal{R}^+$$

$V(\boldsymbol{x})$ positive definite

- Decay of $W(\boldsymbol{x},t)$ a function of $\boldsymbol{x}$ only, not $t$



## Radially Unbounded Functions

$$W(\boldsymbol{x},t), \qquad W: \mathcal{R}^n \times \mathcal{R}^+ \to \mathcal{R}$$

Radially Unbounded if

$$W(\boldsymbol{x},t) \to \infty \ as \ \|\boldsymbol{x}\| \to \infty$$

uniformly on $t$

Or

$\forall M > 0, \ \ \exists N > 0$ s.t for all $t$

$$\|\boldsymbol{x}\| > N \Rightarrow W(\boldsymbol{x},t) > M$$

## Functions of Class $K, KL$

Class $K$: continuous function $\alpha: [0, a] \to \mathcal{R}^+$ with (i) $\alpha(0) = 0$, (ii) $\alpha(.)$ strictly increasing.

Class $KL$: Continuous function $\beta: [0, a] \times \mathcal{R}^+ \to \mathcal{R}^+$ s. t.

i. For fixed $s, \beta(r, s)$ is in class $K$ w.r.t. $r$.
ii. For fixed $r, \beta(r, s)$ is strictly decreasing w.r.t. $s$.
iii. $\beta(r, s) \to 0$ as $s \to \infty$



## In Terms of Class $K$ Functions

- Positive Definite

$$\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}) \leq W(\boldsymbol{x}, t), \forall \boldsymbol{x} \in B_r \subset D, \quad \forall t \in \mathcal{R}^+$$

- Decrescent

$$|W(\boldsymbol{x}, t)| \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|), \forall \boldsymbol{x} \in B_r \subset D, \quad \forall t \in \mathcal{R}^+$$

- Positive Definite & Decrescent

$$\alpha_1(\|\boldsymbol{x}\|) \leq W(\boldsymbol{x}, t) \leq \alpha_2(\|\boldsymbol{x}\|)$$
$$\forall \boldsymbol{x} \in B_r \subset D, \qquad \forall t \in \mathcal{R}^+$$



## Lemmas: Functions of Class $K$

$\boldsymbol{x}_e = \boldsymbol{0}$ equilibrium point at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is

- Uniformly stable if and only if $\exists$ a class $K$ function $\alpha(.)$ and a constant $c$ s.t.

$$\|\boldsymbol{x}(t_0)\| < c \Rightarrow \|\boldsymbol{x}(t)\| < \alpha(\|\boldsymbol{x}(t_0)\|), \forall t \geq t_0$$

- Uniformly asymptotically stable if and only if $\exists$ a class $KL$ function $\beta(.,.)$ & a constant $c$ s.t.

$$\|\boldsymbol{x}(t_0)\| < c \Rightarrow \|\boldsymbol{x}(t)\| < \beta(\|\boldsymbol{x}(t_0)\|, t - t_0)$$
$$\forall t \geq t_0$$



## Examples

$$W(\boldsymbol{x}, t) = (x_1^2 + x_2^2)e^{-\alpha t}, \alpha > 0$$
$$W(\boldsymbol{0}, t) = 0. e^{-\alpha t} = 0, \forall t \in \mathcal{R}^+$$
$$W(\boldsymbol{x}, t) \geq 0, \forall \boldsymbol{x} \neq \boldsymbol{0}, \forall t \in \mathcal{R}^+$$
$$\lim_{t \to \infty} W(\boldsymbol{x}, t) = 0, \forall \boldsymbol{x}$$

- $W(\boldsymbol{x}, t)$ does not satisfy $\alpha_1(\|\boldsymbol{x}\|) \leq W(\boldsymbol{x}, t)$, $\forall t \in \mathcal{R}^+$
- $W(\boldsymbol{x}, t)$ is positive semidefinite but not positive definite.

## Example

$$W(\boldsymbol{x},t) = (t^2+1)(x_1^2+x_2^2)/(x_1^2+2)$$

$$V_2(\boldsymbol{x}) = (x_1^2+x_2^2)/(x_1^2+2) \le W(\boldsymbol{x},t), \forall \boldsymbol{x} \in \mathcal{R}^n, \quad \forall \boldsymbol{x} \ne \boldsymbol{0}, \quad \forall t \in \mathcal{R}^+$$

- No $V(\boldsymbol{x})$ s.t.

$$|W(\boldsymbol{x},t)| \le V(\boldsymbol{x}), \forall \boldsymbol{x} \in D, \forall t \in \mathcal{R}^+$$

$$W(\boldsymbol{x},t) \to (t^2+1)\ as\ x_1 \to \infty, not\ \infty\ \forall t$$

- $W(\boldsymbol{x},t)$ positive definite, not decrescent, not radially unbounded.



## Example

$$W(\boldsymbol{x},t) = \boldsymbol{x}^T\boldsymbol{x}(t^2+1)$$

$$V_3(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 \le W(\boldsymbol{x},t), \forall \boldsymbol{x} \in \mathcal{R}^n, \boldsymbol{x} \ne \boldsymbol{0}, \quad \forall t \in \mathcal{R}^+$$

- $W(\boldsymbol{x},t)$ positive definite and radially unbounded.
- We cannot find $V(\boldsymbol{x})$ s.t.

$$|W(\boldsymbol{x},t)| \le V(\boldsymbol{x}), \forall \boldsymbol{x} \in D, \forall t \in \mathcal{R}^+$$

$W(\boldsymbol{x},t)$ is not decrescent



## Example

$$W(\boldsymbol{x},t) = (x_1^2+x_2^2)/(x_1^2+2)$$

$$V_2(\boldsymbol{x}) = (x_1^2+x_2^2)/(x_1^2+2) \le W(\boldsymbol{x},t), \forall \boldsymbol{x} \in \mathcal{R}^2, \quad \forall \boldsymbol{x} \ne \boldsymbol{0}, \quad \forall t \in \mathcal{R}^+$$

$$|W(\boldsymbol{x},t)| \le V_3(\boldsymbol{x}) = \|\boldsymbol{x}\|^2, \forall \boldsymbol{x} \in \mathcal{R}^2, \forall t \in \mathcal{R}^+$$

$$W(\boldsymbol{x},t) \to 1\ as\ x_1 \to \infty, not\ \infty\ \forall t$$

- $W(\boldsymbol{x},t)$ positive definite, decrescent, not radially unbounded.



## Example

$$W(\boldsymbol{x},t) = \|\boldsymbol{x}\|^2(t^2+1)/(t^2+2)$$

$$\alpha V_3(\boldsymbol{x}) = \alpha\|\boldsymbol{x}\|^2 \le W(\boldsymbol{x},t),$$

$$\forall \boldsymbol{x} \in \mathcal{R}^n, \boldsymbol{x} \ne \boldsymbol{0}, \quad \forall t \in \mathcal{R}^+, \alpha < 1/2$$

- $W(\boldsymbol{x},t)$ positive definite and radially unbounded.

$$|W(\boldsymbol{x},t)| \le V_3(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{R}^n, \forall t \in \mathcal{R}^+$$

$W(\boldsymbol{x},t)$ is decrescent

## Derivative

- Along the trajectories of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$$

$$\dot{W}(\boldsymbol{x}, t) = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \boldsymbol{x}} . \dot{\boldsymbol{x}}$$

$$\dot{W}(\boldsymbol{x}, t) = \frac{\partial W}{\partial t} + \nabla W . \boldsymbol{f}(\boldsymbol{x}, t)$$



## Theorem: Stability

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is

- Stable if $\exists$ a continuously differentiable positive definite $W(\boldsymbol{x}, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(\boldsymbol{x}, t)$ is negative semidefinite in $D$
- Uniformly stable if $\exists$ a continuously differentiable positive definite decrescent $W(\boldsymbol{x}, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(\boldsymbol{x}, t)$ is negative semidefinite in $D$



## Theorem: Uniform Asymptotic Stability

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is uniformly asymptotically stable if $\exists$ a continuously differentiable positive definite decrescent $W(\boldsymbol{x}, t)$, $W: D \times \mathcal{R}^+ \rightarrow \mathcal{R}$, $\mathbf{0} \in D$, s. t. $\dot{W}(\boldsymbol{x}, t)$ is negative definite in $D$



## Theorem: Global Uniform Asymptotic Stability

The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ at $t_0$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is globally uniformly asymptotically stable if $\exists$ a continuously differentiable positive definite, decrescent, & radially unbounded $W(\boldsymbol{x}, t)$, $W: \mathcal{R}^n \times \mathcal{R}^+ \rightarrow \mathcal{R}$, s. t. $\dot{W}(\boldsymbol{x}, t)$ is negative definite $\forall \boldsymbol{x} \in \mathcal{R}^n$

## Summary of Stability Theorems

| $W(\boldsymbol{x},t)$ | $\dot{W}(\boldsymbol{x},t)$ | **Conclusion** |
|---|---|---|
| Pos. definite | Neg. semi-definite | Stable |
| Pos. definite decrescent | Neg. semi-definite | Uniformly stable |
| Pos. definite decrescent | Neg. definite | Uniformly asymptot. stable |
| Globally pos. definite decrescent Rad.unbounded | Globally neg. definite | Globally, uniformly asymptot. stable |



## Theorem: Exponential Stability

The equilibrium at $t_0$ $\boldsymbol{x}_e = \mathbf{0}$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},t)$ is exponentially stable if $\exists K_1, K_2, K_3 > 0$ s. t. $\forall t \geq t_0$

$$K_1 \|\boldsymbol{x}(t)\|^p \leq W(\boldsymbol{x},t) \leq K_2 \|\boldsymbol{x}(t)\|^p$$
$$\dot{W}(\boldsymbol{x},t) \leq -K_3 \|\boldsymbol{x}(t)\|^p$$

- If the conditions hold globally, $\boldsymbol{x}_e = \mathbf{0}$ is globally exponentially stable.
- Proof: similar to autonomous case.



## Example

$$\dot{x}_1 = -x_1 - x_2 e^{-2t}$$
$$\dot{x}_2 = x_1 - x_2$$

Equilibrium: $\boldsymbol{x}_e = \mathbf{0}$ $(\boldsymbol{f}(\mathbf{0},t) = \mathbf{0}, \forall t \geq t_0, \forall t_0$

$$W(\boldsymbol{x},t) = x_1^2 + (1+e^{-2t})x_2^2$$

$$V_1(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 \leq W(\boldsymbol{x},t) \leq x_1^2 + 2x_2^2 = V_2(\boldsymbol{x})$$

$$\forall \boldsymbol{x} \in \mathcal{R}^2, \qquad \forall t \in \mathcal{R}^+$$

- Positive definite, decrescent, radially unbounded



## Derivative

$$\dot{W}(\boldsymbol{x},t) = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \boldsymbol{x}} . \dot{\boldsymbol{x}}$$
$$= -2x_2^2 e^{-2t}$$
$$+ 2\begin{bmatrix} x_1 & x_2(1+e^{-2t}) \end{bmatrix} \begin{bmatrix} -x_1 - x_2 e^{-2t} \\ x_1 - x_2 \end{bmatrix}$$
$$= -2[x_1^2 - x_1 x_2 + x_2^2(1+2e^{-2t})]$$
$$\leq -2[x_1^2 - x_1 x_2 + 3x_2^2] = -\boldsymbol{x}^T Q \boldsymbol{x} < 0$$
$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix} > 0$$

$\dot{W}(\boldsymbol{x},t)$ globally negative definite

$\boldsymbol{x}_e = \mathbf{0}$ is globally exponentially stable.

## Linear Time-varying Systems

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$$

$$A(t) = [a_{ij}(t)] \in \mathcal{R}^n \times \mathcal{R}^n$$

$$a_{ij}(.): \mathcal{R}^+ \to \mathcal{R}, i, j = 1, \dots, n$$

- Continuous functions $\forall t \in \mathcal{R}^+$
- State-transition matrix: $\Phi(.,.)$

$$\boldsymbol{x}(t) = \Phi(t, t_0)\boldsymbol{x}(t_0)$$

$$\|\Phi(t, t_0)\| = \max_{\|\boldsymbol{x}\|=1} \|\Phi(t, t_0)\boldsymbol{x}\|$$



## Theorem: Exponential Stability

The equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$ of $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$

is exponentially stable if and only if $\exists K_1, K_2 > 0$ s. t.

$$\|\Phi(t, t_0)\| \leq K_1 e^{-K_2(t-t_0)}$$

$$\forall t \geq t_0, \forall t_0 \geq 0$$



## Proof (Sufficiency)

- Assume $\exists K_1, K_2 > 0$ s. t.

$$\|\Phi(t, t_0)\| \leq K_1 e^{-K_2(t-t_0)}$$

$$\forall t \geq t_0, \forall t_0 \geq 0$$

$$\|\boldsymbol{x}(t)\| = \|\Phi(t, t_0)\boldsymbol{x}(t_0)\ \|$$

$$\|\boldsymbol{x}(t)\| \leq \|\Phi(t, t_0)\| \|\boldsymbol{x}(t_0)\|$$

$$\leq \|\boldsymbol{x}(t_0)\| K_1 e^{-K_2(t-t_0)}$$

$$\forall t \geq t_0, \forall t_0 \geq 0$$

i. e. exponential stability



## Proof (Necessity)

- Assume exponential stability

$$\|\boldsymbol{x}(t)\| = \|\Phi(t, t_0)\boldsymbol{x}(t_0)\ \| \leq c e^{-\lambda(t-t_0)}$$

$$\forall \boldsymbol{x}(t_0) \in \mathcal{R}^n, \forall t \geq t_0, \forall t_0 \geq 0$$

- Assume w.l.o.g. $\|\boldsymbol{x}(t_0)\| = 1$

$$\|\Phi(t, t_0)\| = \max_{\|\boldsymbol{x}\|=1} \|\Phi(t, t_0)\boldsymbol{x}\|$$

$$\leq K_1 e^{-K_2(t-t_0)}, \forall t \geq t_0, \forall t_0 \geq 0$$

## Lyapunov Function

$$W(\boldsymbol{x},t) = \boldsymbol{x}^T P(t)\boldsymbol{x}$$

$P(t)$ continuously differentiable, symmetric, bounded, and positive definite.

$\exists K_1, K_2 > 0$ s. t.

$$K_1\|\boldsymbol{x}\|^2 \le W(\boldsymbol{x},t) \le K_2\|\boldsymbol{x}\|^2$$
$$\forall t \ge t_0, \forall \boldsymbol{x} \in \mathcal{R}^n$$

$W(\boldsymbol{x},t)$ positive definite, decrescent, radially unbounded



## Derivative

$$\begin{aligned}\dot W(\boldsymbol{x},t) &= \dot{\boldsymbol{x}}^T P(t)\boldsymbol{x} + \boldsymbol{x}^T P(t)\dot{\boldsymbol{x}} + \boldsymbol{x}^T \dot P(t)\boldsymbol{x}\\ &= \boldsymbol{x}^T\left[A^T(t)P(t) + P(t)A(t) + \dot P(t)\right]\boldsymbol{x}\\ &= -\boldsymbol{x}^T Q(t)\boldsymbol{x}\end{aligned}$$

$Q(t)$ symmetric by construction

- If $Q(t)$ uniformly positive definite, $\dot W(\boldsymbol{x},t) < 0$

$\boldsymbol{x}_e = \boldsymbol{0}$ is uniformly asymptotically stable

C.T. Chen, 1984, p.404: Equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$ of $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$ is uniformly asymptotically stable iff it is exponentially stable.



## Theorem: Exponential Stability

The equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$ of $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$

is exponentially stable iff $\forall Q(t)$ continuous, symmetric, bounded, and positive definite

$$A^T(t)P(t) + P(t)A(t) + \dot P(t) = -Q(t)$$

$\exists P(t)$ continuously differentiable, symmetric, bounded, and positive definite.



## Asymptotic Stability of A(t)

- The LTV system $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$ is exponentially stable if there exists $\lambda > 0$ s.t. $\forall i$ and $\forall t \ge 0$

$$\lambda_i\{A(t) + A^T(t)\} \le -\lambda$$

- Proof: $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x}$

$$\begin{aligned}\dot V(\boldsymbol{x}) &= \boldsymbol{x}^T\{A(t) + A^{T(t)}\}\boldsymbol{x} \le -\lambda\, \boldsymbol{x}^T\boldsymbol{x}\\ &= -\lambda\, V\end{aligned}$$

$$V(t) \le V(0)e^{-\lambda t}$$

## Asymptotic Stability of A(t)

- The LTV system $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$ is exponentially stable if there exists $\lambda > 0$ s.t. $\forall i$ and $\forall t \geq 0$

*i.* $Re\{\lambda_i[A(t) + A^T(t)]\} \leq -\lambda$

*ii.* $\int_0^\infty \{A^T(t)A(t)\}dt < \infty$





## Linearization Principle

- Linearize nonlinear $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ system in vicinity of equilibrium $\boldsymbol{x}_e = \mathbf{0}, \boldsymbol{f}(\mathbf{0}, t) = \mathbf{0}$

$$\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]_{\boldsymbol{x}_e=\mathbf{0}} \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t)$$

$$\approx A(t)\boldsymbol{x}$$

$$A(t) = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]_{\boldsymbol{x}_e=\mathbf{0}}$$

$$\boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t) - A(t)\boldsymbol{x}$$



## Theorem: Linearization

- The equilibrium $\boldsymbol{x}_e = \mathbf{0}$ of

$$\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]_{\boldsymbol{x}_e=\mathbf{0}} \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}, t)$$

is uniformly asymptotically stable if

*i.* $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$ is exponentially stable.

*ii.* $\lim_{\|\boldsymbol{x}\| \to 0} \frac{\|\boldsymbol{g}(\boldsymbol{x},t)\|}{\|\boldsymbol{x}\|} = \mathbf{0}$

*uniformly w.r.t.* $t$.

# Converse Theorems

- If the equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ is uniformly stable, then a Lyapunov function that satisfies the conditions for the uniform stability theorem exists.
- Similarly for uniform asymptotic stability, global uniform asymptotic stability.
- The theorems show that the search for $W(\boldsymbol{x}, t)$ may be worth it but do not tell us how to find the Lyapunov function: not useful in practice.