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A. M. Lyapunov's stability theory—100 years on*

P. C. PARKS
Royal Military College of Science (Cranfield), Shrivenham, Swindon, Wilts SN6 8LA, UK

On 12 October 1892 (by the modern calendar) Alexandr Mikhailovich Lyapunov defended his doctoral thesis *The general problem of the stability of motion* at Moscow University. A brief history of Lyapunov's life and tragic death is given, and followed by a section highlighting the important ideas in his thesis of 1892. Subsequent applications of these ideas in the control field are then reviewed, from which the great importance of Lyapunov's thesis in present day technology may be appreciated.

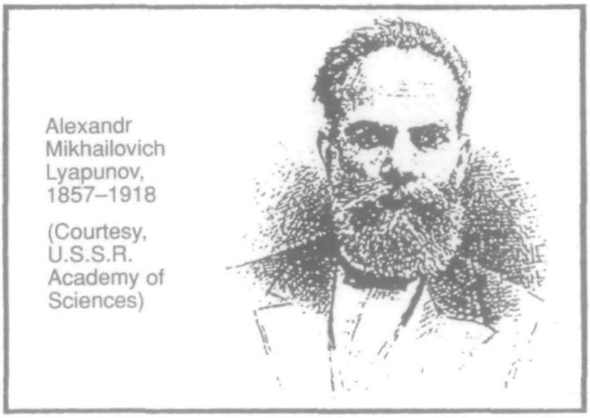

FIG. 1.

1. A brief history of the life of A. M. Lyapunov

Aleksandr Mikhailovich Lyapunov was born on 6 June 1857† at Yaroslavl' where his father was director of the Demidovsk Lycée, a general high school. In 1870 his mother, who had been widowed in 1868, moved with her three sons to Nizhny-Novgorod (known for many years as Gorki), and it was here that Lyapunov received his secondary school education, graduating from the gymnasium in 1876 with a gold medal. In the same year he became a student in mathematics at St. Petersburg University, where he was particularly influenced by the lectures of

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† All dates in this paper accord to the modern (Gregorian) calendar.

Professor P. L. Chebyshev. Lyapunov graduated in 1880, again winning a gold medal for an essay on hydrostatics. He stayed on at the university in the Mechanics Faculty, developing his Master's thesis *On the stability of ellipsoidal forms of equilibrium of rotating fluids* which he defended in 1884 (Lyapunov 1884). This work, which was translated into French in 1904, established his name in Europe.

In the autumn of 1885, Lyapunov moved to Kharkov where he became a 'privat-dozent' in the university's Department of Mechanics. In the winter break of 1885/6 he returned to St. Petersburg to marry his first cousin, Natalia Rafailovna Sechenova. In his work in Kharkov, Lyapunov first had to spend much time preparing lectures in mechanics and notes for his students, but in 1888 he started to publish papers devoted to the stability of motion. This work led to his famous doctoral thesis of 1892, *The general problem of the stability of motion* (Lyapunov 1892). He defended this thesis in Moscow on 12 October 1892, one of his 'opponents' being N. E. Zhukovski (better known here as Joukowski—'the father of Russian aviation'); this thesis was republished in French in 1908 and 1949, Russian in 1935, and English (Lyapunov 1992). Lyapunov was promoted in the following year 1893 to an ordinary professorship at Kharkov.

Although Lyapunov continued his work on stability problems until 1902, he also turned his attention to other topics, including potential theory and probability theory, particularly the central limit theorem of Laplace. He was also an active participant in activities of the Kharkov Mathematical Society, becoming its president and editor of its *Communications* from 1899 to 1902.

In 1902, Lyapunov moved back to St. Petersburg where he had been appointed Academician and the successor to his former teacher P. L. Chebyshev, who had died in 1894. He was now able to devote himself entirely to research, returning to the problem of forms of equilibrium of bodies of uniformly rotating fluid particles under mutual Newtonian gravitational attraction. In particular, Lyapunov showed that pear-shaped forms of rotating fluid were unstable. This result had important consequences in astronomy, since it disproved the then current theories of formation of satellites from bodies of rotating fluid e.g. the Earth–Moon system. G. H. Darwin (son of the famous naturalist Charles Darwin) at Cambridge had proposed such a mechanism at this time which naturally required pear-shaped bodies to be stable. In 1917, J. H. Jeans, in an Adams Prize Essay at Cambridge, was to confirm that Lyapunov's result was correct and that therefore Darwin's theory was incorrect.

In June 1917, Lyapunov left St. Petersburg for Odessa on doctor's orders, since his wife was suffering badly from tuberculosis. However, in Odessa her condition deteriorated, and on 31 October 1918 she died. Lyapunov himself was in poor health and his eyesight was failing. The tragic events of the revolution, the burning of the family estate on the River Volga including a great library built up by his father and grandfather, and the illness of his wife had greatly depressed Lyapunov. On the day his wife died, Lyapunov shot himself in the head with a pistol. He died on 3 November 1918, the day of his wife's funeral. In a suicide note he asked to be buried with her. The inscription on the gravestone reads:

> The creator of the theory of stability of motion, the doctrine of equilibrium figures of rotating liquid, methods of the qualitative theory of differential equations, author of the

central limit theorem and other deep investigations in areas of mechanics and mathematical analysis.

At this time, many of the later uses of Lyapunov's work could hardly have been foreseen. Some of these will be described in later sections of this paper.

2. Lyapunov's doctoral thesis *The general problem of the stability of motion*

The thesis was first published by the Kharkov Mathematical Society (Lyapunov 1892). A translation into French by É. Davaux was published by the University of Toulouse in 1908; this translation was reprinted as a book by Princeton University Press in 1949. In March 1992, the *International Journal of Control* devoted its entire issue to an English translation by A. T. Fuller of the French version (Lyapunov 1992). In the same issue, J. F. Barrett provided both a translation from Russian of a biography of Lyapunov by Academician V. I. Smirnov and a bibliography of all Lyapunov's works. Lyapunov's thesis runs to 242 printed pages in this English version. A number of important ideas are contained in the thesis, of which the following four topics may be highlighted.

2.1 *General definitions of stability of motion*

Lyapunov's thesis opens with some rather general definitions of the stability of motion. These include various measures of stability of a point following a particular time trajectory in an n-dimensional space. It is possible for a certain motion to be stable with respect to some measures but unstable with respect to others. Here Lyapunov gives as an example a particle in a circular orbit under an inverse-square law of attraction: this motion is stable with respect to the length of the radius vector and to the speed of the particle, but is unstable with respect to the rectangular coordinates of its position.

Stability, in the sense of points in phase-space which are initially close together subsequently staying close together, is lacking even in apparently well-behaved Hamiltonian systems where

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, \dots, n). \tag{1}$$

Consequently phase-space volumes v are preserved, because

$$v(t) = v(0) \exp \int_0^t \operatorname{trace} \boldsymbol{J} \, d\tau = v(0),$$

since

$$\operatorname{trace} \boldsymbol{J} = \sum_{i=1}^{n} \left(\frac{\partial^2 H}{\partial x_i \, \partial p_i} - \frac{\partial^2 H}{\partial p_i \, \partial x_i} \right) = 0,$$

where $\boldsymbol{J}$ is the Jacobian matrix formed from the right-hand sides of the Hamiltonian equations. This result was originally due to Liouville (1809–82). Figure 2, from

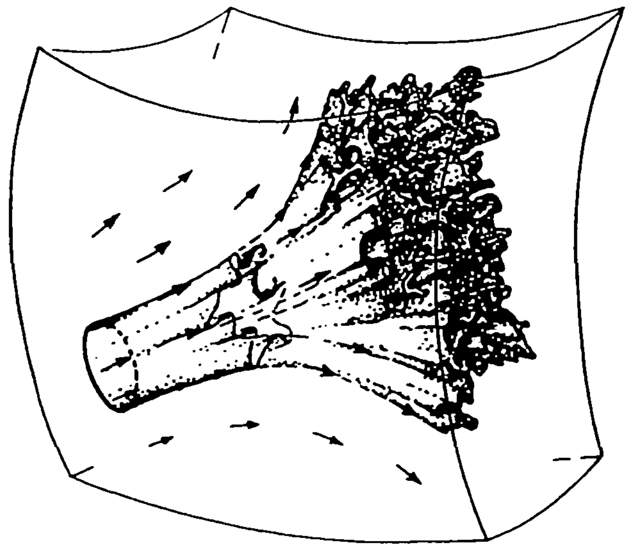

FIG. 2. Diffusion of trajectories.

Penrose (1990), shows how this volume preservation can nevertheless allow extensive diffusion of the trajectories in phase space (of dimension $2n$).

2.2 *Characteristic numbers of functions and 'Lyapunov exponents'*

This second concept, also developed quite early on in Lyapunov's thesis, is called by him the 'characteristic number' of a function of time. Given a function $x(t)$, then one can define λ_0 such that $x(t)\exp\lambda t$ is unbounded for $\lambda > \lambda_0$ and $x(t)\exp\lambda t \to 0$ as $t \to \infty$ for $\lambda < \lambda_0$. The critical (real) number λ_0 is the *characteristic number* of $x(t)$. For example, if $x(t) = t^2$, then $\lambda_0 = 0$; if $x(t) = \exp(-2t)$, then $\lambda_0 = 2$; and if $x(t) = \exp(t \sin t)$, then $\lambda_0 = -1$.

In recent years, the term 'characteristic exponent' or 'Lyapunov exponent' has been used in the context of chaotic motion. Here one is interested in whether or not two points $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ in the state space of a dynamical system which are initially very close together, stay close together in the subsequent motion. Here the function of time, $x(t)$, being considered is given by $x(t) = |\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)|$, where $|\boldsymbol{x}_1(0) - \boldsymbol{x}_2(0)|$ is 'small'. The largest *characteristic* or *Lyapunov* exponent can be defined as

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)|}{|\boldsymbol{x}_1(0) - \boldsymbol{x}_2(0)|}.$$

A chaotic motion is characterized by a *positive* Lyapunov exponent, which can be regarded as minus one times the characteristic number as originally defined by Lyapunov himself, and explained in the previous paragraph. Thus, in a chaotic motion, the two points move apart as time increases, making accurate prediction of the long term future impossible. More generally, for various attractors in state space motions in an n-dimensional space $\mathbb{R}^n$, a set of n Lyapunov exponents may be defined,

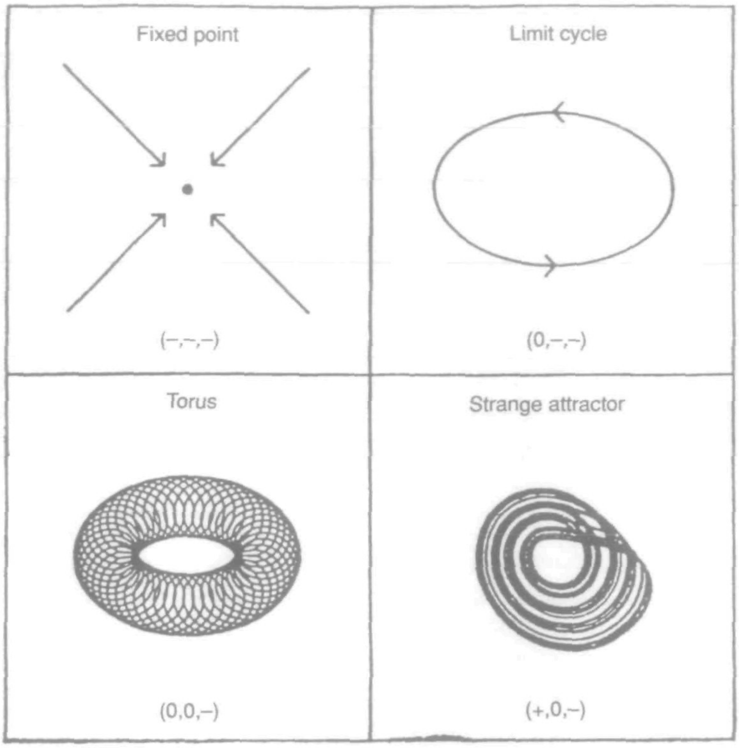

FIG. 3. Signs of Lyapunov exponents for various attractors.

and their signs characterize the attractor. This idea is illustrated for three-dimensional state space in Fig. 3, taken from Ruelle (1989).

2.3 *Lyapunov's first method*

Lyapunov's first method provides theoretical validity for the often-used technique of linearization. More precisely, given a set of nonlinear first-order differential equations

$$\dot{x}_i = f_i(x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{2}$$

where the f_i are analytic functions such that $f_i(0, \dots, 0) = 0$ $(i = 1, \dots, n)$, so that the origin $\boldsymbol{x} = \mathbf{0}$ is an equilibrium point, then a linearized set of equations

$$\dot{\boldsymbol{x}} = \boldsymbol{J}\boldsymbol{x} \tag{3}$$

may be found by forming the Jacobian matrix $\boldsymbol{J}$ with the (i, j) entry

$$J_{ij} = \left[\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right]_{\boldsymbol{x}=\mathbf{0}} \quad (i, j = 1, \dots, n),$$

or, equivalently, by expanding the f_i as a power series about $\boldsymbol{x} = \mathbf{0}$ and dropping all terms except those linear in the x_is.

If the matrix $\boldsymbol{J}$ has all its eigenvalues either negative or with negative real parts, so that the linearized equation (3) is asymptotically stable, *then so will be the original nonlinear equation* (2), *at least for initial conditions $\boldsymbol{x}$ lying in a finite region about the origin*, that is to say, initial displacements $\boldsymbol{x}$ from the origin within this region will decay to zero as time tends to infinity.

Lyapunov has an involved proof of this result involving a solution in series of the original nonlinear equation in which the f_i may also be functions of time. He also examines at some length various 'singular cases', that is, cases in which $\boldsymbol{J}$ has eigenvalues which are zero or have zero real parts. Here stability is determined by the nonlinear terms in the power-series expansion of the f_i. Specifically, the cases of one zero eigenvalue and of two purely imaginary eigenvalues, $\pm \mathrm{j}\lambda$, were considered in the thesis.

2.4 *Lyapunov's second method*

The fourth highlight of Lyapunov's thesis has proved subsequently to be the most important—this is his so-called 'second' or 'direct' method, making use of 'Lyapunov functions'. Once again, if we consider $\boldsymbol{x} = \boldsymbol{0}$ to be an equilibrium point of the nonlinear differential equation (2), then it may prove possible to investigate the stability of this equilibrium point by examining a positive-definite function $V = V(\boldsymbol{x})$, surrounding $\boldsymbol{x} = \boldsymbol{0}$ with a nest of closed surfaces defined by $V(\boldsymbol{x}) = c$ $(c > 0)$. The rate of change with respect to time of V following a trajectory of (2) is easily calculated as

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\boldsymbol{x}).$$

If this is always negative (except at $\boldsymbol{x} = \boldsymbol{0}$, where $V = 0$), then it follows that the trajectories must cross the surface $V =$ constant in an inwards directions, and consequently tend to the point $\boldsymbol{x} = \boldsymbol{0}$ at time t tends to infinity. A three-dimensional example is shown in Fig. 4. Thus asymptotic stability has been proved *without any need to find explicit solutions of the nonlinear differential equation* (2).

This powerful idea originated in Lyapunov's mind from his knowledge of astronomical problems and the earlier work of Lagrange, Laplace, Dirichlet, and Liouville (Fuller 1992). In many astronomical problems, stability—rather than asymptotic stability—is sought; in such cases, $\dot{V}$ is zero so that the disturbed motions persist as oscillations on some surface $V = c$ surrounding $\boldsymbol{x} = \boldsymbol{0}$. The greater interest in asymptotic stability shown in contemporary control theory did not develop until the 20th century.

3. Applications of Lyapunov's second method in control theory—developments in Russia before 1960

3.1 *Chetayev's work at Kazan (1930s)*

Lyapunov's (1892) thesis contains many mathematical examples of his theory not linked with physical stability problems (although Fuller's (1992) English translation adds an occasional control-engineering interpretation of Lyapunov's examples (e.g.

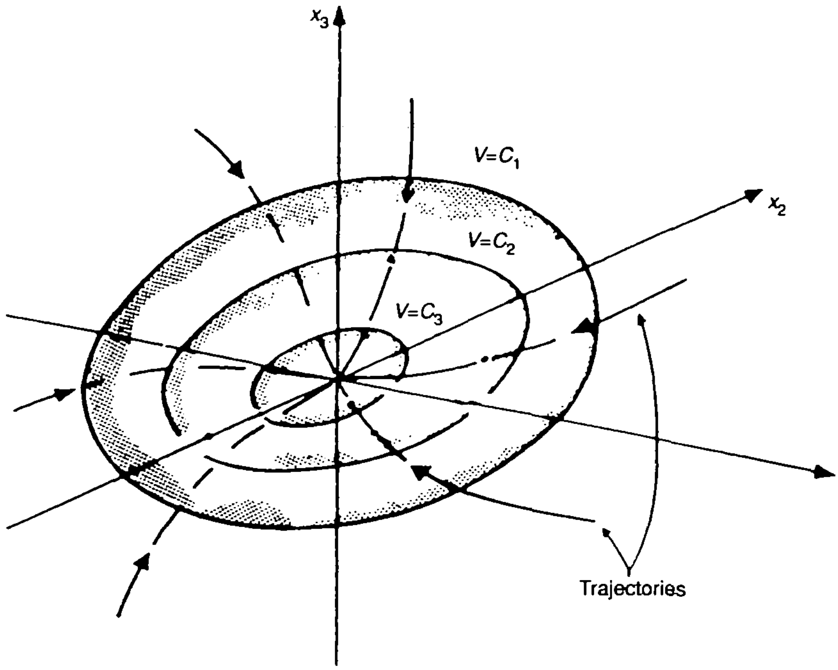

FIG. 4. Liapunov function contours and trajectories of $\dot{x} = f(x)$.

in Section 41, Examples V and VI)). Practical use of Lyapunov's stability theory in the control field came many years later, notably in the work of N. G. Chetayev and I. G. Malkin at the Kazan Aviation Institute in the 1930s (Chetayev 1961; Malkin 1952). A typical example of Chetayev's application of Lyapunov's second method to aeronautical stability problems is provided by the classical problem of spin stabilization of a rocket or artillery shell. The (simplified) model is shown in Fig. 5. The corresponding equations of motion, obtained from the moving-axes equation

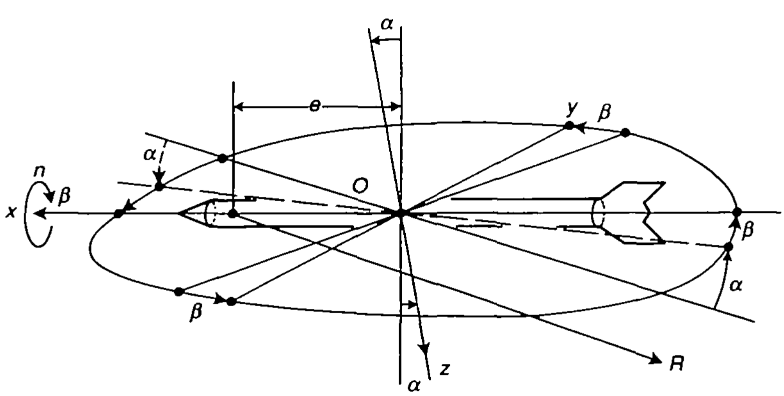

FIG. 5. Rotating rocket stability problem.

for angular momentum in conventional notation, are

$$A\ddot{\beta} + A\dot{\alpha}^2 \sin\beta\cos\beta - Cn\dot{\alpha}\cos\beta = eR\sin\beta\cos\alpha,$$
$$A\ddot{\alpha} - 2A\dot{\alpha}\dot{\beta}\sin\beta + Cn\dot{\beta} = eR\sin\alpha,$$

where A, C, n, e, and R are constants. Now two integrals of the motion exist, identified as energy F_1 and angular momentum F_2:

$$F_1 \equiv \tfrac{1}{2}A(\dot{\beta}^2 + \dot{\alpha}^2\cos^2\beta) + eR(\cos\alpha\cos\beta - 1)$$
$$F_2 \equiv A(\dot{\beta}\sin\alpha - \dot{\alpha}\cos\beta\sin\beta\cos\alpha) + Cn(\cos\alpha\cos\beta - 1).$$

A tentative Lyapunov function is formed as $V = F_1 - \lambda F_2$, with $\dot{V} \equiv 0$. Expanding the trigonometric terms in F_1 and F_2 gives

$$V = \tfrac{1}{2}[A\dot{\alpha}^2 + (Cn\lambda - eR)\beta^2 + 2A\lambda\dot{\alpha}\beta] + \tfrac{1}{2}[A\dot{\beta}^2 + (Cn\lambda - eR)\alpha^2 - 2A\lambda\alpha\dot{\beta}]$$
$$+ \text{higher-order terms}.$$

The two quadratic forms in $(\dot{\alpha}, \beta)$ and $(\dot{\beta}, \alpha)$ have matrices $\begin{bmatrix} A & \pm A\lambda \\ \pm A\lambda & Cn\lambda - eR \end{bmatrix}$. Each of these is positive definite if $A > 0$ and $-AeR + ACn\lambda - A^2\lambda^2 > 0$. A suitable constant λ can be found if $A\lambda^2 - Cn\lambda + eR = 0$ has two real roots, from which is obtained the (well known) criterion for spin stabilization:

$$C^2n^2 > 4AeR.$$

3.2 *Contributions of Lur'e and Letov*

A nonlinear control problem discussed extensively by A. I. Lur'e (1957) and A. M. Letov (1961) and dating back to the 1940s is illustrated in Fig. 6.

If we consider the demand n_D to be zero, then the stability of the feedback loop may be investigated by examining the nonlinear differential equations

$$\dot{y} = Ky + m\xi, \qquad \dot{\xi} = f(\sigma), \qquad \sigma = p^T y - r\xi, \tag{4}$$

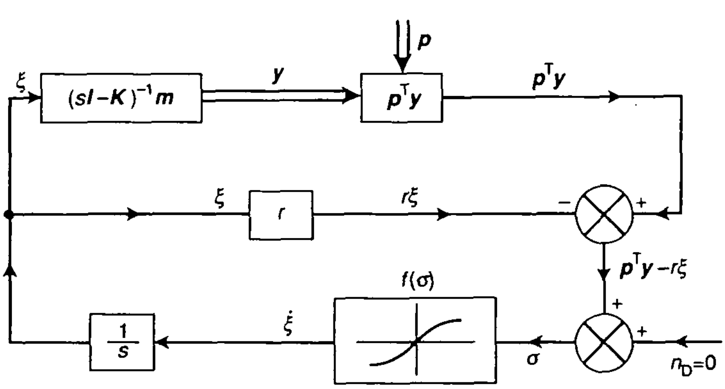

FIG. 6. The Lur'e–Letov nonlinear control problem.

where $\boldsymbol{K}$ is a constant matrix, $\boldsymbol{m}$ and $\boldsymbol{p}$ are constant vectors, and r is a non-negative real number. The equations are first transformed by the 'generalized Lur'e transformation'

$$\boldsymbol{y} = \boldsymbol{Dx} - \boldsymbol{Db}\xi,$$

where $\boldsymbol{D}$ is any non-singular matrix such that $\boldsymbol{Db} = \boldsymbol{K}^{-1}\boldsymbol{m}$, to yield

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{b}f(\sigma), \qquad \dot{\sigma} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - rf(\sigma), \tag{5}$$

where $\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{KD}$. The nonlinearity $f(\sigma)$ is assumed to be continuous in σ, with $f(0) = 0$, $\sigma f(\sigma) > 0$ $(\sigma \neq 0)$, and $\int_0^\infty f(\sigma)\,\mathrm{d}\sigma$ infinite. A Lyapunov function V is introduced as

$$V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Bx} + \int_0^\sigma f(\phi)\,\mathrm{d}\phi,$$

with $\dot{V} = -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Cx} + 2f(\sigma)\boldsymbol{g}^{\mathrm{T}}\boldsymbol{x} + f(\sigma)\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - rf^2(\sigma)$, where $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} + \boldsymbol{BA} = -\boldsymbol{C}$ and $\boldsymbol{g} = \boldsymbol{Bb}$. The matrix $\boldsymbol{C}$ is positive-definite, and $\boldsymbol{B}$ also, if we assume that $\boldsymbol{K}^{\mathrm{T}}$ (and $\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{KD}$) has all its eigenvalues with negative real parts. The Lyapunov derivative $\dot{V}$ may be made negative-definite, either (if $\boldsymbol{c}$ is given) by choosing the feedback constant r such that

$$r > (\boldsymbol{g}^{\mathrm{T}} + \tfrac{1}{2}\boldsymbol{c}^{\mathrm{T}})\boldsymbol{C}^{-1}(\boldsymbol{g} + \tfrac{1}{2}\boldsymbol{c}),$$

or (if r is given) by choosing the vector $\boldsymbol{c}$ as

$$\boldsymbol{c} = -2\boldsymbol{g} + 2\sqrt{r}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{\nu},$$

where $\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{\nu} < 1$ and $\boldsymbol{C} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{T}$, with $\boldsymbol{T}$ nonsingular.

3.3 *Aizerman's conjecture*

The problem considered by Lur'e and Letov led M. A. Aizerman (1949) to make a famous conjecture concerning the function $f(\sigma)$ appearing in equation (4) and Fig. 6. This conjecture, as applied to this problem, is as follows. 'If $f(\sigma)$ is replaced by a linear function $k\sigma$, and if the resulting linearized form of (4) is asymptotically stable for all constant k in the sector $0 < k < K$, then the nonlinear system (4) will also be asymptotically stable.' Here K is a constant such that $0 < \sigma f(\sigma) < K\sigma^2$ $(\sigma \neq 0)$; see Fig. 7.

Unfortunately the general Aizerman conjecture is untrue; the following counter-example was found by Krasovskii (1963), which is in a form different from equations (4) and (5). Consider

$$\dot{x}_1 = x_2 + f(x_1),$$
$$\dot{x}_2 = -x_1 - x_2.$$

If $f(x_1)$ is replaced by kx_1 then $0 < k < 1$ ensures asymptotic stability. However, if

$$f(x) = \begin{cases} x - [\mathrm{e}^{-2}/(1 + \mathrm{e}^{-1})]x & \text{for } |x| < 1, \\ x - [\mathrm{e}^{-2|x|}/(1 + \mathrm{e}^{-|x|})]\,\mathrm{sign}\,x & \text{for } |x| \geqslant 1, \end{cases}$$

then $0 < xf(x) < x^2$ $(x \neq 0)$, but the trajectory starting at $x_1 = 1$ and $x_2 = \mathrm{e}^{-1} - 1$ has the equation

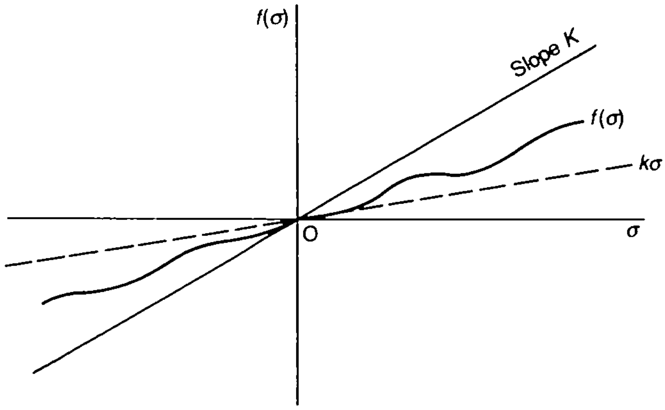

FIG. 7. $f(\sigma)$ for Aizerman's conjecture.

$$x_2 = e^{-x_1} - x_1,$$

and $x_1 \to \infty$ and $x_2 \to -\infty$ as $t \to \infty$.

Aizerman's conjecture stimulated later developments, in particular Popov's stability criterion (see Section 4.4).

4. Lyapunov's second method in control theory—developments since 1960 (mainly outside Russia)

4.1 *Introduction*

These developments started in the late 1950s and early 1960s and were greatly stimulated by challenging control problems associated with the so-called 'space race' between the USA and USSR, set off by the launch of the Soviet Sputnik 1 (4 October 1957). Of particular significance in making Lyapunov's second method better known outside Russia were the 1st World Congress of the International Federation of Automatic Control (IFAC), held in Moscow in June 1960, and a seminal article by R. E. Kalman and J. E. Bertram in the *Journal of the American Society of Mechanical Engineers*, published in the same month (Kalman & Bertram 1960). A group formed by Professor Solomon Lefschetz at the Research Institute for Advanced Study at Baltimore USA (of which R. E. Kalman and J. P. La Salle were members) was particularly active at this time in making Russian mathematical techniques better known. Cover-to-cover English translations of selected Russian journals also started in these years. A very readable book on Lyapunov's second method by Lefschetz and La Salle appeared in 1961.

4.2 *The Lyapunov Hermite–Routh–Hurwitz connection*

A question raised by Kalman & Bertram (1960) was how Lyapunov's second method could be employed to recover classical results for linear differential equations such as those derived by E. J. Routh (1877) and by A. Hurwitz (1895). The connection was firmly established by P. C. Parks (1964b) using a special solution of the Lyapunov matrix equation (see also Section 4.3).

If the nth-order linear differential equation with real constant coefficients is given by

$$p(\mathrm{D})y \equiv (\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \cdots + a_n)y = 0, \tag{6}$$

and its phase-space form by $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{7}$$

is the companion matrix of (6). Then a Lyapunov function is $V = x^{\mathrm{T}}Hx$, where

$$H = \begin{bmatrix} a_na_{n-1} & 0 & a_na_{n-3} & 0 & \cdots & \cdots & \cdots \\ 0 & -a_na_{n-3}+a_{n-1}a_{n-2} & 0 & \ddots & & & \cdots \\ a_na_{n-3} & 0 & \ddots & & & \cdots & \cdots \\ 0 & \ddots & & & & & \\ & & & & \ddots & \ddots & a_5 \\ \cdots & \cdots & & \ddots & 0 & -a_5+a_1a_4 & 0 \\ \cdots & \cdots & \ddots & 0 & a_5-a_1a_4+a_2a_3 & 0 & a_3 \\ \cdots & \cdots & \ddots & -a_5+a_1a_4 & 0 & -a_3+a_1a_2 & 0 \\ \cdots & \cdots & a_5 & 0 & a_3 & 0 & a_1 \end{bmatrix} \tag{8}$$

in which the (i, j) element is

$$H_{ij} = \begin{cases} \sum_{k=0}^{n-i} (-1)^{k+n-i} a_k a_{2n-i-j-k+1} & (i+j \text{ even}, i \geqslant j), \\ 0 & (i+j \text{ odd}, i > j), \\ H_{ji} & (i < j), \end{cases}$$

where $a_0 = 1$ and $a_r = 0$ for $r > n$. Here H is Hermite's (1856) matrix, and

$$\dot{V} = x^{\mathrm{T}}(HA + A^{\mathrm{T}}H)x = -2(a_1x_n + a_3x_{n-2} + \cdots)^2.$$

Clearly, $\dot{V}$ is negative semidefinite and, provided that $a_j \neq 0$ for some odd j, can be shown not be identically zero on any trajectory other than $x = \mathbf{0}$, from which a necessary and sufficient condition for stability is that H is positive definite.

The principal minors of H starting from the bottom right-hand corner are $\Delta_1, \Delta_1\Delta_2, \Delta_2\Delta_3, \ldots, \Delta_{n-1}\Delta_n$ where the Δ_i are the Hurwitz determinants of the equation (6). For stability, the Δ_i must all be positive and $p(s)$, appearing as $p(\mathrm{D})$ in (6), is then said to be a 'Hurwitz polynomial'. The Hermite matrix was used by B. D. O. Anderson (1972) to prove the Liénard–Chipart stability criterion and by M. Mansour and Anderson (1992) to provide a Lyapunov proof of the celebrated Kharitonov theorem (1978).

4.3 *Construction of Lyapunov functions*

The first applications of Lyapunov's second method used integrals of motion, such as energy and momentum, or quadratic forms and integrals of the nonlinearity as in the Lur'e and Letov technique. In the 1960s, considerable efforts were made to devise systematic methods of construction of Lyapunov functions.

The Lyapunov matrix equation Given nonlinear equations of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{9}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, one can usually extract the linear part in the form of the matrix equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}. \tag{10}$$

If $\boldsymbol{A}$ has all its eigenvalues with negative real parts, then the origin $\boldsymbol{x} = \mathbf{0}$ is an asymptotically stable equilibrium point of (10) and of (9) also. A suitable Lyapunov function for (9) may however yield other useful information such as an approximation to the region of attraction surrounding $\boldsymbol{x} = \mathbf{0}$. Such a Lyapunov function can be a quadratic form $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}$ is found by solving the Lyapunov matrix equation

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{Q}, \tag{11}$$

in which $\boldsymbol{Q}$ is a given arbitrary symmetric positive-definite matrix. This involves solving $\frac{1}{2}n(n+1)$ linear equations for the elements p_{ij} $(j \geqslant 1)$ of the symmetric matrix $\boldsymbol{P}$. (This is always possible, provided that $\lambda_i + \lambda_j \neq 0$ for any i and j, where the λ_i are the eigenvalues of $\boldsymbol{A}$.)

S. Barnett and C. Storey (1967) introduced a new way of solving (11) in the form

$$\boldsymbol{P} = (\boldsymbol{S} - \tfrac{1}{2}\boldsymbol{Q})\boldsymbol{A}^{-1}$$

where $\boldsymbol{S}$ is the unique skew-symmetric matrix satisfying

$$\boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} = \tfrac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{A}). \tag{12}$$

Solving equation (12) involves the solution of $\frac{1}{2}n(n-1)$ linear equations for the elements of $\boldsymbol{S}$: a reduction of n compared with the solution of (11). More importantly, Barnett & Storey (1968) were able to exploit this idea to investigate the sensitivity or robustness of the stability of (10).

An early application of the Lyapunov matrix equation was due to A. G. J. MacFarlane (1963). He constructed performance functionals for the stable linear system (10) in the form

$$\int_0^\infty t^r x^{\mathrm{T}} Q x \,\mathrm{d}t = (-1)^{r+1} x^{\mathrm{T}}(0) P_{r+1} x(0),$$

where $P_{s+1}A + A^{\mathrm{T}}P_{s+1} = P_s$ $(s = 1,\ldots, r)$ and $P_1 A + A^{\mathrm{T}} P_1 = Q$.

A typical application of the Lyapunov matrix equation to examine the effect of bounded disturbances is the paper by P. A. Cook (1980). As an example, he considered the damped pendulum equation

$$\ddot{x} + 2\varepsilon\dot{x} + \sin x = u(t)$$

with $|u(t)| \leqslant k$ for all t. A suitable quadratic Lyapunov function V is found using (11) with $Q = 2\varepsilon I$ and then, from $\dot{V}$, a sufficient condition on k to ensure that $|x| \leqslant \nu$ for all t is

$$k \leqslant \nu\varepsilon\sqrt{(1-\varepsilon^2)} - \tfrac{1}{2}(\nu - \sin\nu).$$

Integration by parts An important development in the construction of Lyapunov functions was introduced by R. W. Brockett (1970). His result may be stated as follows: given the linear constant-coefficient nth-order differential equation

$$p(\mathrm{D})y = 0, \tag{13}$$

where $p(\mathrm{D})$ takes the form of equation (6) given earlier, then a Lyapunov function for (13) is

$$V(x) = \int \{q(\mathrm{D})y\,p(\mathrm{D})y - [r(\mathrm{D})y]^2\}\,\mathrm{d}t,$$

where x is the phase-space vector $[y, \mathrm{D}y, \ldots, \mathrm{D}^{n-1}y]^{\mathrm{T}}$, $q(s)/p(s)$ is a positive-real function, and

$$\mathrm{Ev}[q(s)p(-s)] \equiv r(s)r(-s),$$

with $\mathrm{Ev}(\bullet)$ denoting the even part of the polynomial $(\bullet)$ and $r(s)$ being a Hurwitz polynomial. The importance of this technique is its connection with frequency response, since it is a property of the positive-real function $q(s)/p(s)$ that $\mathrm{Re}[q(\mathrm{j}\omega)/p(\mathrm{j}\omega)] \geqslant 0$ for all real ω.

We note that some special choices of $q(s)$ are possible: if $q(s)$ is chosen as the polynomial $a_1 s^{n-1} + a_3 s^{n-3} + a_5 s^{n-5} + \cdots$ formed from $p(s)$ by deleting alternate terms, then $V = \frac{1}{2}x^{\mathrm{T}}Hx$, using the special Hermite form H given earlier in equation (8). On the other hand, if $q(s)$ is chosen as $(\mathrm{d}/\mathrm{d}s)p(s) \equiv ns^{n-1} + a_1(n-1)s^{n-2} + \cdots + a_1$, then it turns out that $\dot{V} = -x^{\mathrm{T}}Hx$; i.e. H turns up in $\dot{V}$ rather than V (Parks 1969), so that the sufficient conditions obtained from $\dot{V}$ are in fact *necessary* as well.

The variable-gradient method Of more general methods of constructing Lyapunov functions, the so-called variable-gradient method of D. G. Schultz and J. E. Gibson (1962) is prominent. Here the gradient of V, that is the vector

$$g = \mathrm{grad}\, V = \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right]^{\mathrm{T}},$$

is considered as $g = [g_1(x), \ldots, g_n(x)]^{\mathrm{T}}$, with the $g_i(x)$ given as

$$g_i(x) = \sum_{j=1}^{n} a_{ij}(x)x_j,$$

that is, as quasi-linear forms in $x_1, \ldots, x_n$. Two sets of conditions are imposed on g: first, it has to be the gradient of a scalar function (V) so that the $n \times n$ curl matrix of g is zero, i.e.

$$(\mathrm{curl}\, g)_{ij} = \frac{\partial g_j(x)}{\partial x_i} - \frac{\partial g_i(x)}{\partial x_j} = 0 \quad (i, j = 1, \ldots, n),$$

and secondly $\dot{V} = g^{\mathrm{T}}x \leqslant 0$, so that $\dot{V}$ is negative-definite or negative-semidefinite. V may be found by integration along any path from the origin to the general point x (usually the path is chosen, for convenience of the integration, as parallel to each x_i axis in turn).

A variant of this procedure is due to D. R. Ingwerson (1961): here the Jacobian matrix

$$J(x) = \left[\frac{\partial f_i(x)}{\partial x_j}\right]_{i,j=1}^{n}$$

is calculated from (9) to give

$$\ddot{x} = J(x)\dot{x}. \tag{14}$$

The Lyapunov matrix equation $J^{\mathrm{T}}(x)P(x) + P(x)J(x) = -Q$ is solved for $P(x)$, and a new matrix $P_1(x)$ is formed from $P(x)$ by putting all $x_k = 0$ in $(P(x))_{ij}$ except x_i and x_j. The procedure then follows that of the variable-gradient method, taking

$$g_i(x) = \sum_{j=1}^{n} \int_0^{x_j} (P_1(x_i, \xi_j))_{ij}\, \mathrm{d}\xi_j,$$

the curl matrix being identically zero from the construction of g_i.

The Zubov Method In a series of papers culminating in a book, V. I. Zubov (1962) developed a scheme for solving the partial differential equation

$$(\mathrm{grad}\, V)^{\mathrm{T}} f(x) = -\theta(x), \tag{15}$$

where $\theta(x)$ is a positive-definite or positive-semidefinite function. In particular, if $f(x)$ can be developed as a power series in the components x_i of x, then V can also be so developed. In general, there is the danger of a combinatorial explosion in doing this, but in simple examples the method works well and, in some cases, exact domains of attraction can be found.

4.4 *Popov's stability criterion*

Important developments by V. M. Popov (in Romania) and R. E. Kalman (in the USA) led to a connection between frequency response and the existence of Lyapunov

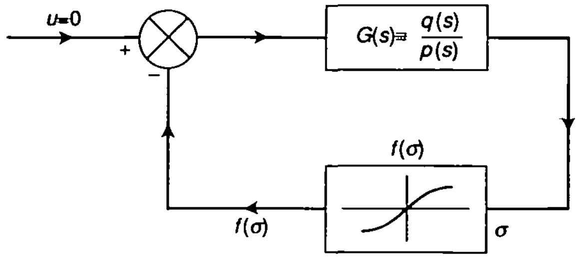

FIG. 8. Feedback loop for Popov's criterion.

functions. A neat form of these results is due to Brockett (1964) for the feedback loop shown in Fig. 8, and may be stated as a theorem:

> The null solution of the system shown in Fig. 8 is asymptotically stable for all admissible $f(\sigma)$ if there is an $\alpha > 0$ such that $(1 + \alpha s)G(s)$ is a positive-real function.

The key property of a positive-real function $Z(s)$ is that Re $Z(j\omega) \geqslant 0$ for all real ω. The theorem leads to the graphical interpretation of Fig. 9, showing the *modified* Nyquist plot of $G(j\omega)$, in which ω Im $G(j\omega)$ is plotted against Re $G(j\omega)$. In Fig. 9, this modified Nyquist plot must lie to the right of a straight line through the origin with slope $1/\alpha$—the 'Popov line'—for asymptotic stability, as stated in the theorem. In fact, this condition ensures the existence of a Lyapunov function of the type 'quadratic form plus integral of the non-linearity', the quadratic form being found by Brockett's 'integration by parts' technique.

If the non-linearity can be confined so that

$$0 < \sigma f(\sigma) < k\sigma^2 \quad (\sigma \neq 0),$$

then a relaxation of the Popov condition may be made by moving the Popov line a distance $1/k$ to the left of the origin of the modified Nyquist diagram as shown in Fig. 10. For example, if $G(s)$ in Fig. 8 is $1/(s^3 + as^2 + bs)$, with $a,b > 0$,

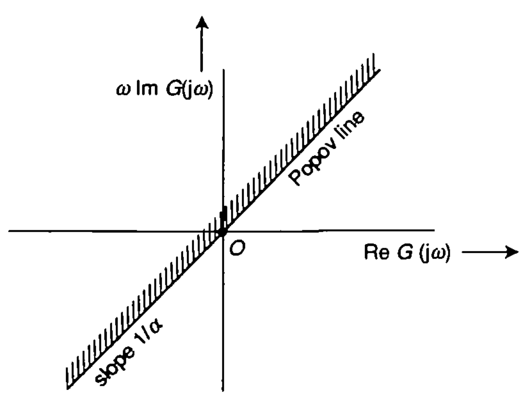

FIG. 9. Popov line.

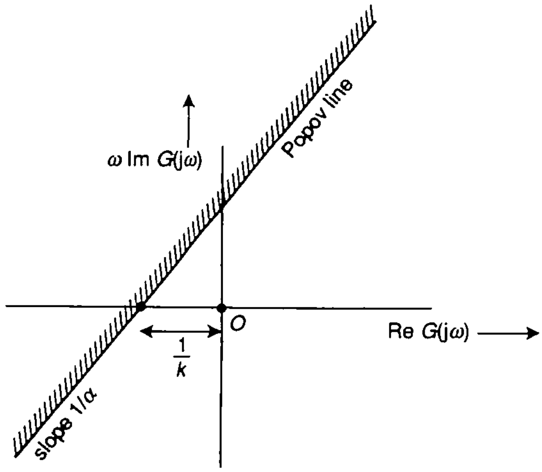

FIG. 10. Displaced Popov line.

then the closed loop is asymptotically stable, provided that $0 < \sigma f(\sigma) < ab\sigma^2$ for $\sigma \neq 0$.

4.5 *The circle criterion*

Another form of time-varying feedback loop is shown in Fig. 11. This is a linear problem since

$$p(\mathrm{D})x + k(t)q(\mathrm{D})x = u, \tag{16}$$

where $p(\mathrm{D})x = e$ and $y = q(\mathrm{D})x$. Brockett's technique may be used here to prove the 'circle criterion':

Let $G(s)$ be a rational function with more poles than zeros, no common factors, and no poles in $\operatorname{Re} s > 0$. If the *conventional* Nyquist plot $G(\mathrm{j}\omega)$ does not encircle or enter the circular disc with diameter $[-1/\beta, -1/\alpha]$, then the null solution

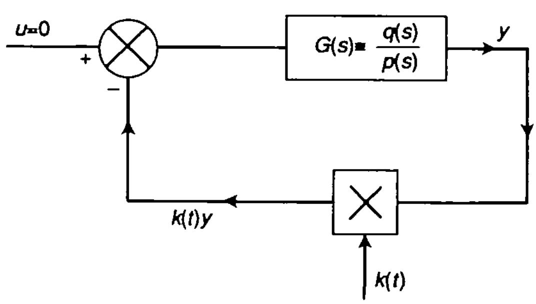

FIG. 11. Feedback loop for the circle criterion.

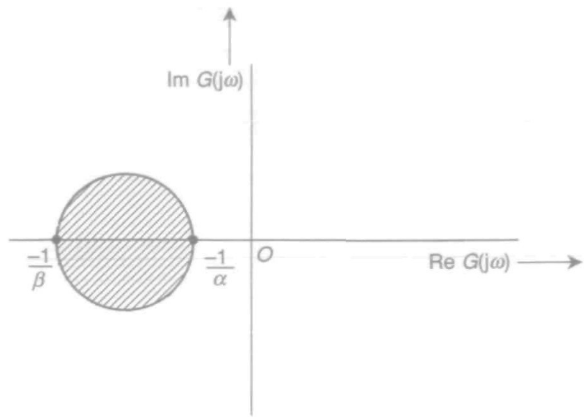

FIG. 12. The circle criterion.

of equation (16) (with $u = 0$) is uniformly asymptotically stable if

$$0 \leqslant \beta + \varepsilon \leqslant f(t) \leqslant \alpha - \varepsilon < \infty$$

for some $\varepsilon > 0$.

This criterion is illustrated in Fig. 12, which shows the location of the circular disc in the complex plane.

4.6 *Sampled-data or discrete-time systems*

Lyapunov functions may be constructed also for discrete-time systems. If the discrete time is taken as $\ldots, t-1, t, t+1, t+2, \ldots$, then the 'derivative' of the Lyapunov function V_t is taken as the difference $V_t - V_{t-1}$. Again quadratic forms may often be used; so, for example, the stable linear discrete-time system

$$\boldsymbol{x}_t = \boldsymbol{A}\boldsymbol{x}_{t-1} \tag{17}$$

has a Lyapunov function $\boldsymbol{x}_t^{\mathsf{T}}\boldsymbol{P}\boldsymbol{x}_t$, where $\boldsymbol{P}$ satisfies the discrete-time Lyapunov matrix equation

$$\boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = -\boldsymbol{Q}, \tag{18}$$

in which $\boldsymbol{Q}$ is positive-definite or positive-semidefinite. By a suitable choice of $\boldsymbol{Q}$, the necessary and sufficient stability condition that $\boldsymbol{P}$ is positive-definite can be identified as the classical Shur–Cohn stability criterion for the eigenvalues of $\boldsymbol{A}$ to lie within the unit circle of the complex plane (Parks 1964a; Jury 1974).

4.7 *Special Lyapunov functions*

Occasionally, special types of Lyapunov function may be employed. A good example is the Lyapunov function

$$V = \sum_{i=1}^{n} |x_i| \tag{19}$$

devised by H. H. Rosenbrock (1963). If this is applied to the linear system

$$\dot{x} = Ax, \tag{20}$$

then it is sufficient to check that the trajectories of (20) are directed inwards at all the vertices of the hypercubes defined by V = constant surrounding $x = 0$. This will be so if

$$a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \leqslant -\varepsilon_j < 0 \quad (j = 1, \ldots, n). \tag{21}$$

An alternative Lyapunov function of a simple type is

$$V = \max_i |x_i|,$$

when the condition corresponding to (21) is

$$a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leqslant -\varepsilon_i < 0 \quad (i = 1, \ldots, n). \tag{22}$$

Inequalities (21) and (22) will be recognized as Gershgorin's (1931) sufficient conditions for the stability of (20).

4.8 *Lyapunov functions for partial differential equations*

Many important physical processes are described by partial differential equations as opposed to ordinary differential equations considered so far. In control-engineering language, these are 'distributed-parameter systems', as opposed to 'lumped-parameter systems'. A. A. Movchan (1959) was among the first to extend Lyapunov's second method to distributed-parameter systems. The precise definitions of stability have to be extended to normed metric spaces in which the Euclidean distance $\sqrt{(x_1^2 + \cdots + x_n^2)}$ used hitherto is replaced by norms such as

$$\left\{\int_0^l \left[z^2 + \left(\frac{\partial z}{\partial t}\right)^2\right] \mathrm{d}x\right\}^{\frac{1}{2}}.$$

A simple example will illustrate these ideas.

Consider the damped vibrating string of Fig. 13 with equation of motion

$$T\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t}, \tag{23}$$

where $y(x, t)$ is the lateral displacement of the string, m its (constant) line density, c (>0) is a viscous damping constant, and T the constant tension in the string; the boundary conditions are $y(0, t) = y(l, t) = 0$ ($t \in \mathbb{R}$). Consider the Lyapunov functional

$$V = \int_{x=0}^{l} \left[\tfrac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 + \tfrac{1}{2}m\left(\frac{\partial y}{\partial t}\right)^2\right] \mathrm{d}x, \tag{24}$$

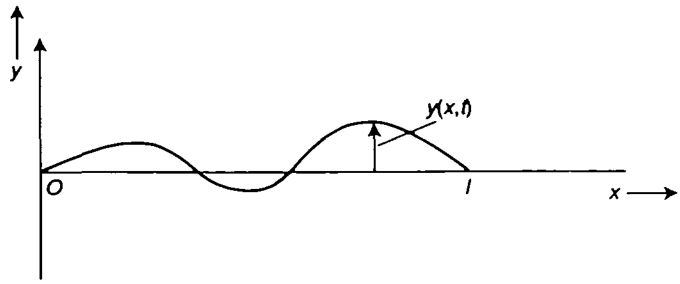

FIG. 13. Vibrating string.

which will be recognized as the total potential and kinetic energy in the string. Then

$$\dot{V} = \int_{x=0}^{l} \left(T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \, \partial x} + m \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \right) \mathrm{d}x. \tag{25}$$

Substituting for $m \, \partial^2 y/\partial t^2$ from the differential equation (23), integrating by parts with respect to x, and using the boundary conditions, $\dot{V}$ becomes

$$\dot{V} = \int_{x=0}^{l} -c\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \leqslant 0. \tag{26}$$

Now, if a metric-space norm ρ is considered, where

$$\rho^2 = \int_0^l \left[y^2 + \left(\frac{\partial y}{\partial t}\right)^2 \right] \mathrm{d}x, \tag{27}$$

it can be shown that $V \to 0$ as $t \to \infty$ and, since V bounds $\lambda \rho^2$ from above, where $\lambda = \min\{\frac{1}{2}\pi^2 T/l^2, \frac{1}{2}m\}$, then $\rho^2 \to 0$ also.

The Lyapunov functional technique may also be employed to determine the stability of distributed-parameter control systems, as controls located at the boundary, or at fixed points in space, or distributed in space will enter into the expression for $\dot{V}$. Conditions on the controls can ensure that $\dot{V} \leqslant 0$. Examples of this technique will be found in Parks & Pritchard (1969).

A useful technique for constructing Lyapunov functionals for distributed-parameter systems will also be found in Parks & Pritchard (1969) with further examples in Parks & Pritchard (1972). If the partial differential equation is written in an operator form $Lu = 0$, then a second operator M is formed by 'differentiating' L partially with respect to the symbol '$\partial/\partial t$'. A Lyapunov functional is found by integration by parts applied to the integral $\int_t \int_x (Lu)(Mu) \, \mathrm{d}x \, \mathrm{d}t$ to obtain two groups of sign-definite terms, viz.

$$\int_x q \, \mathrm{d}x + \int_t \int_x Q \, \mathrm{d}x \, \mathrm{d}t.$$

V is now taken as $V = \int q \, \mathrm{d}x$, and its time derivative is $\dot{V} = -\int Q \, \mathrm{d}x$, because $\int_t \int_x (Lu)(Mu) \, \mathrm{d}x \, \mathrm{d}t = 0$ (since $Lu = 0$).

4.9 *Recent Applications of Lyapunov's second method to control problems*

Here adaptive control, robotics, and neural networks will be particularly emphasized.

4.10 *Adaptive control systems*

The history of adaptive control can be traced back to the 1950s, and an early milestone was the flight adaptive control symposium held at Wright Field, Dayton, Ohio in 1959 (Gregory 1959). An important idea at this symposium was the model-reference adaptive control (MRAC) system of Whitaker (Osburn *et al.* 1961). Here an error signal is formed by subtracting the output of the real plant from the output of an ideal model. This error is then used to adjust gains within the plant so that it follows the model (Fig. 14).

The system is essentially nonlinear. Whitaker devised an algorithm for adaption (the 'M.I.T. rule') based on what today would be called 'sensitivity' arguments—these, however, are heuristic rather than exact, and can lead to instability of the adaptive loop. Parks (1966) was one of the first to recast the design of MRAC systems using Lyapunov functions. This work also introduced a role for positive-real functions in MRAC design. The procedure was improved by R. V. Monopoli (1974) and by Y. D. Landau (1979), who introduced Popov's (1973) hyperstability theory as an alternative to the Lyapunov design procedure.

The Lyapunov design procedure This early work neglected some aspects of the problem such as parameter convergence, and later work has emphasized the concepts of 'persistence' of the input signals required to drive the adaptive loops, and of system robustness. This followed a lively discussion initiated by Rohrs *et al.* (1985). Today the subject has achieved a certain maturity and is the topic of several new books by Goodwin & Sin (1984), Anderson *et al.* (1986), Narendra & Annaswamy (1989), Åström & Wittenmark (1989), and Sastry & Bodson (1989). Lyapunov-function techniques are used extensively, but are combined with various results from functional analysis and the use of 'averaging' techniques for solving time-varying differential equations.

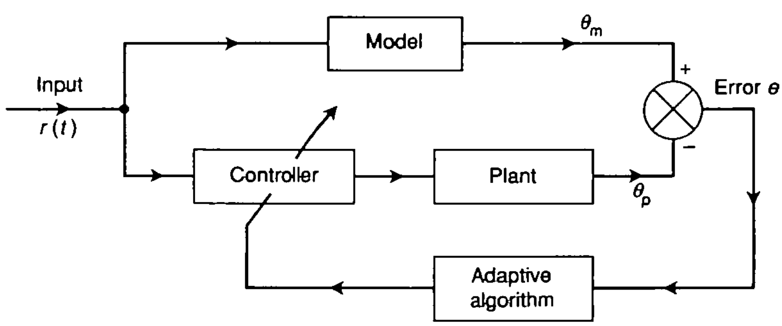

FIG. 14. Model-reference adaptive control.

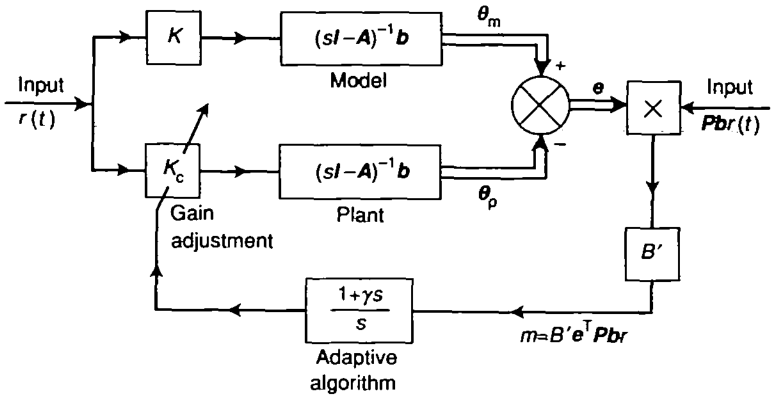

FIG. 15. Lyapunov design of gain adaption loop.

As a simple example of the Lyapunov-function technique, consider the gain adaption loop shown in Fig. 15. The Lyapunov function

$$V = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} + \lambda(x + \gamma m)^2,$$

where $\boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{Q}$, $x = K - K_c$, $m = B'\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Pbr}$, λ and γ are positive constants, and $B' = 1/\lambda$, has

$$\dot{V} = -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{e} - 2\lambda\gamma m^2,$$

using the stable adjustment law for K_c:

$$\dot{K}_c = m + \gamma\dot{m}. \tag{28}$$

If $\gamma = 0$, the original problem considered by Parks (1966) is obtained. Here a matrix differential equation may be written in terms of $\boldsymbol{e} = \boldsymbol{\theta}_m - \boldsymbol{\theta}_p$, that is,

$$\begin{bmatrix} \dot{\boldsymbol{e}} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}r \\ -B'\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}r & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ x \end{bmatrix}. \tag{29}$$

This is a linear equation, but with time-varying coefficients on account of $r(t)$, the system input. In Anderson *et al.* (1986), a more general equation of this type is considered, i.e.

$$\begin{bmatrix} \dot{\boldsymbol{e}} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b}\boldsymbol{\phi}^{\mathrm{T}}(t) \\ -\varepsilon\boldsymbol{\phi}(t)\boldsymbol{c}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{\theta} \end{bmatrix}, \tag{30}$$

or, recognizing $\boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ as a scalar transfer function $H(s)$ and eliminating $\boldsymbol{e}$, the vector differential equation

$$\dot{\boldsymbol{\theta}}(t) = -\varepsilon\boldsymbol{\phi}(t)H(s)\{\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\theta}\}$$

is obtained, where $H(s)\{\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\theta}\}$ denotes that $\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\theta}$, a scalar, is filtered by the transfer function $H(s)$. In this description of the adaptive system, $\boldsymbol{\theta}(t)$ is described as the 'parameter error vector', $\boldsymbol{\phi}(t)$ as a 'regressor vector' generated by the adaptive system and its input, and ε is the 'adaptive gain'.

Now a tentative Lyapunov function for (30) is

$$V = e^{\mathrm{T}} P e + \lambda \theta^{\mathrm{T}} \theta, \tag{31}$$

for which $\dot{V} = -e^{\mathrm{T}} Q e$ if $\lambda = 1/\varepsilon$, while P satisfies the following Lyapunov matrix equation with constraint:

$$PA + A^{\mathrm{T}} P = -Q, \qquad Pb = c. \tag{32}$$

The existence of a solution to the *two* equations (32) is assured by the celebrated 'MKY' or Meyer–Kalman–Yakubovich lemma (Kalman 1963), which requires the transfer function $H(s) \equiv c^{\mathrm{T}}(sI - A)^{-1} b$ to be a positive-real function. In addition, for convergence of the parameter error vector to zero, it is necessary that $\phi(t)$ is 'persistently exciting', or that there is a suitable input continually driving the system and providing signals to operate the adaption mechanism.

4.11 *Robotics*

Typical robotic systems consist of a number of mechanical linkages which are rapidly moving relative to one another, and thus changing their moments and products of inertia as measured about the axes about which controlling torques are applied. Adaptive controls using Lyapunov functions to assure stability seem attractive in this situation. A pioneer in this use of the Lyapunov-function technique has been J. J. E. Slotine (Slotine & Li 1987; Slotine 1988). A simple two-link example of such a robot is shown in Fig. 16, taken from R. Johansson (1990). Typically the equations

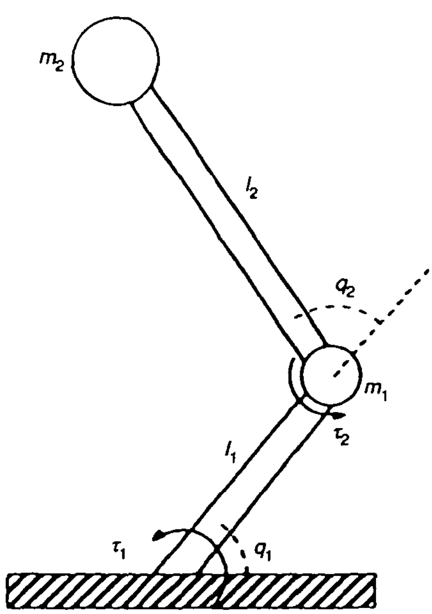

FIG. 16. Two-link manipulator, from Johansson (1990).

of motion are obtained from the classical rigid-body dynamical theory using Lagrange's equation of motion, and may be written

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau. \tag{33}$$

Here q, $\dot{q}$, and $\ddot{q}$ are vectors representing respectively the angular position, velocity, and acceleration of the joints, τ is a vector of torques applied to the joints, $M = M^{\mathrm{T}}$ is a moment-of-inertia matrix, C a matrix representing Coriolis, centripetal, and frictional forces, and $g(q)$ represents a vector of gravitational torques.

In a typical Lyapunov design procedure, such as that of Johansson (1990), the model-reference adaptive control (MRAC) approach is used. A model for q is set up as

$$\ddot{q}_r + K_d\dot{q}_r + K_p q_r = Kr, \tag{34}$$

where K_d, K_p, and K are matrices chosen to give desirable damping, stiffness, and gain characteristics. A difference vector $\tilde{q} = q - q_r$ is then formed, and the design attempts to eliminate the position and velocity errors $\tilde{q}$ and $\dot{\tilde{q}}$. A full error state vector $x(t)$ is constructed consisting of $\tilde{q}$, $\dot{\tilde{q}}$, and $\tilde{\theta}$, the latter being a vector of parameter errors, and a Lyapunov function candidate is the quadratic form in $x(t)$ given by

$$V = x^{\mathrm{T}}P(q)x, \tag{35}$$

for which $\dot{V}$ can be shown to be of the form

$$\dot{V} = -(\dot{\tilde{q}}^{\mathrm{T}}\tilde{q}^{\mathrm{T}})Q(\dot{\tilde{q}}\tilde{q})^{\mathrm{T}} \leqslant 0. \tag{36}$$

Some simulated results for the system of Fig. 16, taken from Johansson (1990) are shown in Fig. 17. A sudden change in the mass m_2 from 1 kg to 10 kg took place at time $t = 1$ s ($m_1 = 1$ kg, $l_1 = l_2 = 1$ m).

A certain maturity is also apparent in this field. Two recent books which both use Lyapunov-function techniques are Craig (1988) and Slotine & Li (1991), but new research continues to appear—see, for example, Qu *et al.* (1992).

4.12 *Artificial neural networks*

Artificial neural networks and their applications are the subject of intensive research activity at the present time, with many new books and journals being published and conference programmes established. Significant applications of neural networks in the control field are also being developed (Miller *et al.* 1990; Warwick *et al.* 1992). Neural networks are essentially nonlinear in their operation, and so it is not surprising that Lyapunov functions have appeared in this literature, in particular to establish stability of equilibrium states of the Hopfield net (Hopfield 1982; Grossberg 1988). This is a fully connected network (see Fig. 18) forming an 'associative memory' which can store patterns and has the ability to recover the exact stored patterns when presented initially with partial or distorted versions of them. Thus the device has a number of stable equilibrium points with surrounding domains of attraction: a distorted pattern presented to the net is an initial point in one of these *domains*; in subsequent operations, the trajectory leads to the domain's equilibrium point: the

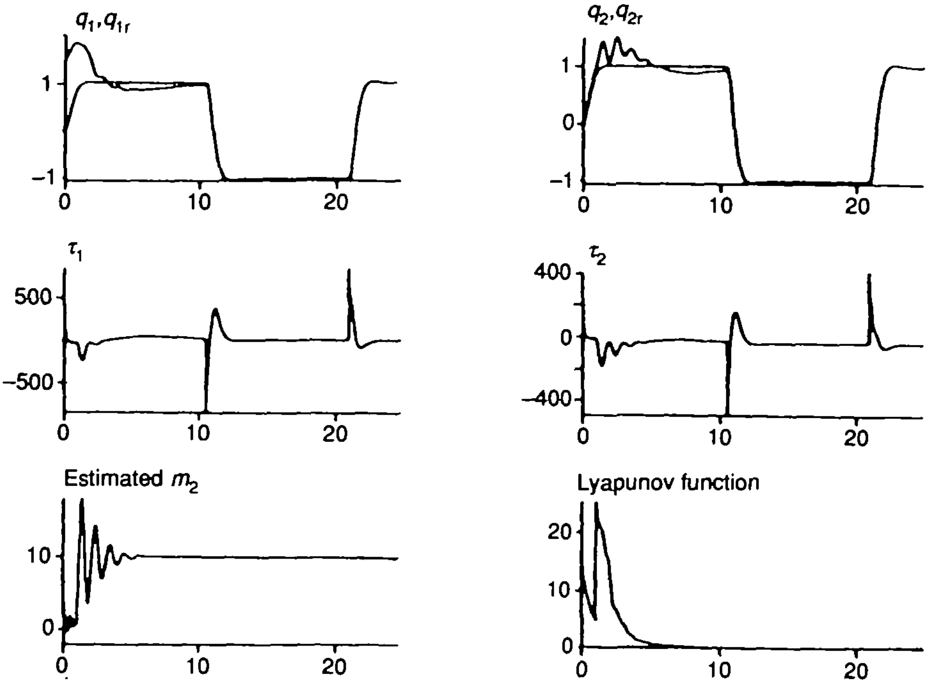

FIG. 17. Simulated results for the robot of Fig. 16.

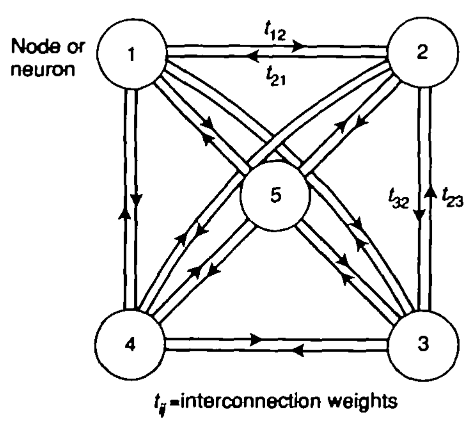

FIG. 18. Hopfield net.

exact stored pattern. The number of patterns that can be stored in this way appears to be given approximately by $0.14N$ where N is the number of neurons in the network.

The input to node or neuron is $\sum_j t_{ij}x_j + I_i$, where t_{ij} are the 'weights' or interactions between nodes, with $t_{ij} = t_{ji}$ and $t_{ii} = 0$; here x_j are the outputs of the nodes and I_i are the external inputs. The output x_i from node i is given as $x_i = \underline{v}_i$ if the input is less than or equal to u_i, and $x_i = \bar{v}_i$ if the input exceeds u_i, where $\bar{v}_i > \underline{v}_i$

and u_i is the 'threshold' of neuron i. A Lyapunov-like function is given by

$$V = -\tfrac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{x} - \sum_i I_i x_i + \sum_i u_i x_i, \tag{37}$$

where $\boldsymbol{T}$ is the matrix with entries t_{ij}. Now the change in V due to a change Δx_i in x_i is

$$\Delta V = -\left(\sum_j t_{ij} x_j + I_i - u_i\right)\Delta x_i; \tag{38}$$

here Δx_i may be 0 or $\pm(\bar{v}_i - \underline{v}_i)$, where the sign is plus if $\sum_j t_{ij}x_j + I_i - u_i > 0$ and minus if $\sum_j t_{ij}x_j + I_i - u_i < 0$. Thus V can only decrease, and stability of the given equilibrium point can be established.

The stability theory of control systems employing artificial neural networks has yet to be developed. K. S. Narendra (Miller *et al.* 1990: p. 126) states that 'our knowledge of the stability of dynamical adaptive systems using artificial neural networks is quite rudimentary at the present time and considerable work remains to be done ... new concepts and methods based on stability theory will have to be explored'. So far, only partial results are available, such as the proof of convergence of the associative memory for the cerebellar model articulation controller (CMAC) of J. S. Albus (1975). This is given by Parks and Militzer (1989), where a rather trivial Lyapunov function $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is employed. V is reduced as (discrete) time proceeds by an orthogonal projection algorithm for adjusting the 'weights' in the memory. The algorithm was originally invented by S. Kaczmarz (1937) for solving sets of linear algebraic equations by iteration, and rediscovered by Albus.

Lyapunov-function arguments appear in a more decisive rôle in a recent paper by R. M. Sanner and J. J. E. Slotine (1991) which uses Gaussian radial basis functions to compensate for plant nonlinearities and to develop a convergence weight-adjustment algorithm.

4.13 *Other new concepts*

A limited number of pages restricts discussion of many other current applications of Lyapunov's stability theory. One may note, however, the concept of 'stability radius' developed in a series of papers on robustness of linear systems by D. Hinrichsen and A. J. Pritchard (1992) involving 'Lyapunov functions of maximum robustness'. The authors remark that they 'believe that this procedure (of going from matrices to polynomials) is both theoretically more productive and numerically more secure than the converse method, which tackles robustness problems for matrices by applying polynomial techniques (e.g. Kharitonov's theorem) to their characteristic polynomials'.

Another current application of Lyapunov stability theory is the proof of stability of so called 'receding-horizon control' of nonlinear systems developed by H. Michalska and D. Q. Mayne (1990, 1991) as the solution of a constrained optimal control problem.

Finally, Lyapunov-like arguments are used by Brockett (1991a)—see also Brockett (1991b)—with the intriguing title 'Dynamical systems that sort lists, diagonalize matrices, and solve linear programming problems'. This concerns equilibrium states of the $n \times n$ matrix differential equation

$$\dot{H} = H^2N - 2HNH + NH^2, \tag{39}$$

where H is a real symmetric and N a real diagonal matrix. One remarkable property of the solution $H(t)$ of (39) is that, if N is a diagonal matrix with differing elements $N_{11}, ..., N_{nn}$, then $H(\infty)$ is a diagonal matrix consisting of the eigenvalues of $H(0)$ *ordered according to the relative magnitudes of* $N_{11}, ..., N_{nn}$.

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