6 Controllability and Observability of Nonlinear Systems

Key points

- Nonlinear observability is intimately tied to the Lie derivative. The Lie derivative is the derivative of a scalar function along a vector field.
- Nonlinear controllability is intimately tied to the Lie bracket. The Lie bracket can be thought of as the derivative of a vector field with respect to another.
- References
  - Slotine and Li, section 6.2 (easiest)
  - Sastry, chapter 11 pages 510-516, section 3.9 and chapter 8
  - Isidori, chapter 1 and appendix A (hard)

Controllability for Nonlinear Systems

<u>The Use of Lie Brackets: Definition</u>

Consider two vector fields f(x) and g(x) in $\Re^n$.

Then the Lie bracket operation generates a new vector field:

$$[f,g] \equiv \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Also, **higher order Lie brackets** can be defined:

$$(ad_f^1, g) \equiv [f, g]$$
$$(ad_f^2, g) \equiv [f, [f, g]]$$
$$\dots$$
$$(ad_f^k, g) \equiv [f, (ad_f^{k-1}, g)]$$

Note: the "ad" is read "adjoint".

Re-writing **controllability conditions for linear systems** using this notation:

$$\dot{x} = Ax + Bu = Ax + B_1 u_1 + B_2 u_2 + \dots + B_m u_m$$
$$f(x) = Ax, \quad g_i(x) = B_i$$

$$\ddot{x} = A\dot{x} = A^2 x + \sum_{i=1}^{m} AB_i u_i = A^2 x - \sum_{i=1}^{m} [f, B_i] u_i$$

How this came about…

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial g}{\partial x} = \frac{\partial B}{\partial x} = 0$$

So for example:

$$[f, B_1] = [Ax, B_1] = \frac{\partial B_1}{\partial x} Ax - \frac{\partial (Ax)}{\partial x} B_1 = -AB_1$$

If we keep going:

$$\dddot{x} = A^3 x + \sum_{i=1}^{m} A^2 B_i u_i = A^3 x + \sum_{i=1}^{m} [ad_f^2, B_i] u_i$$

Notice how this time the minus signs cancel out.

…

$$x^{(n)} = \frac{d^n x}{dt^n} = A^n x + \sum_{i=1}^{m} A^{n-1} B_i u_i = A^n x + (-1)^{n-1} \sum_{i=1}^{m} [ad_f^{n-1}, B_i] u_i$$

Re-writing the controllability condition:

$$C = [B_1, \dots, B_m, [ad_f, B_1], \dots [ad_f, B_m], \dots [ad_f^{n-1}, B_1], \dots [ad_f^{n-1}, B_m]]$$

The condition has not changed – just the notation.

The terms B_1 through B_m correspond to the B term in the original matrix, the terms with ad_f correspond to the AB terms, the terms with ad_f^{n-1} correspond to the $A^{n-1}B$ terms.

Recap – Controllability for Linear Systems

$$\dot{x} = Ax + Bu$$
$$C = \left[B \mid AB \mid \ldots \mid A^{n-1}B\right]$$

Local conditions (linear systems)

$$\dot{x} = Ax + Bu$$

Let u = constant (otherwise no pb, but you get $\dot{u}, \ddot{u}$ etc…)

$$\ddot{x} = A(Ax + Bu) = A^2x + ABu$$
$$\ldots$$
$$x^{(n)} = A^n x + A^{n-1}Bu$$

For linear systems, you get nothing new after the nth derivative because of the Cayley-Hamilton theorem.

Nonlinear Systems

Assume we have an affine system:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

The general case is much more involved and is given in Hermann and Krener.
If we don't have an affine system, we can sometimes ruse:

$$\dot{x} = f(x,u)$$

Let $\tau\dot{u} + u = v$

Select a new state: $z = \begin{bmatrix} x \\ u \end{bmatrix}$ and v is my control $\Rightarrow$ the system is affine in (z,v), and pick τ to be OK.

Theorem

The system defined by:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

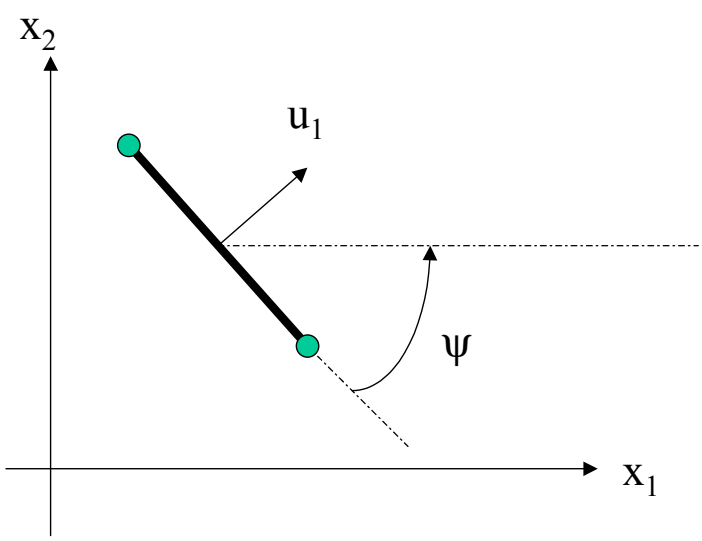
is **locally accessible about x_0** if the accessibility distribution C spans n space, where n is the rank of x and C is defined by:

$$C = \left[g_1, g_2, \ldots, g_m, [g_i, g_j], \ldots, [ad_{g_i}{}^k, g_j], \ldots, [f, g_i], \ldots [ad_f^k, g_i], \ldots\right]$$

The g_i terms are analogous to the B terms, the $[g_i,g_j]$ terms are new from having a nonlinear system, the $[f,g_i]$ terms correspond to the AB terms, etc…

Note: if f(x) = 0 then $\dot{x} = \sum_{i=1}^{m} g_i(x)u_i$ and if in this case C has rank n, then the system is **controllable**.

<u>Example: Kinematics of an Axle</u>



Basically, ψ is the yaw angle of the vehicle, and x_1 and x_2 are the Cartesian locations of the wheels. u_1 is the velocity of the front wheels, in the direction that they are pointing, and u_2 is the steering velocity.

We define our state vector to be:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \psi \end{bmatrix}$$

Our dynamics are:

$$\begin{cases} \dot{x}_1 = \sin\psi u_1 \\ \dot{x}_2 = \cos\psi u_1 \\ \dot{\psi} = u_2 \end{cases}$$

The system is of the form:

$$\dot{x} = f(x) + g_1 u_1 + g_2 u_2$$

$$\text{f(x) = 0,} \quad g_1 = \begin{bmatrix} \sin\psi \\ \cos\psi \\ 0 \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Note:</u>

If I linearize a nonlinear system about x_0 and the linearization is controllable, then the nonlinear system is accessible at x_0 (not true the other way – if the linearization is uncontrollable the nonlinear system may still be locally accessible).

<u>Back to the example:</u>

$$C = [g_1, g_2, [g_1, g_2]]$$

where $[g_1, g_2] = \frac{\partial g_1}{\partial x} g_2 - \frac{\partial g_2}{\partial x} g_1$ and in our case, $\frac{\partial g_2}{\partial x} = 0$

$$[g_1, g_2] = \begin{bmatrix} 0 & 0 & \cos\psi \\ 0 & 0 & -\sin\psi \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix}$$

So $C = \begin{bmatrix} \sin\psi & 0 & \cos\psi \\ \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \end{bmatrix}$

C has rank 3 everywhere, so the system is locally accessible everywhere, and f(x)=0 (free dynamics system) so the system is controllable!

<u>Example 2:</u>

$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases}$$

Note: if I had the linear system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the linear system is controllable.

Back to the example 2:

Is the nonlinear system controllable? Answer is NO, because x_1 can only increase.

But let's show it.

In standard form:

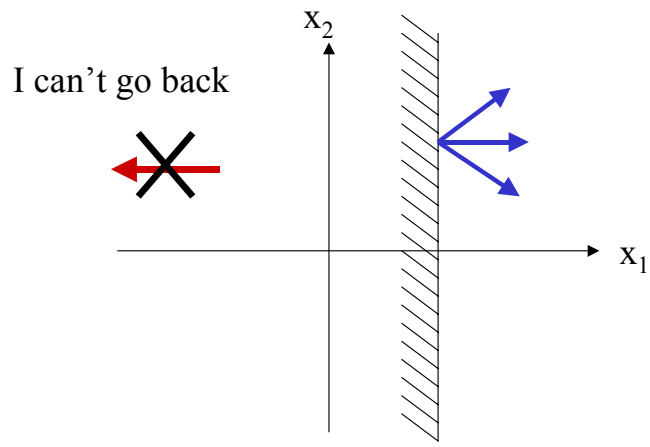
$$f = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [g, [f, g]]$$

$$[f, g] = \begin{bmatrix} -2x_2 \\ 0 \end{bmatrix}$$

So $C = \begin{bmatrix} 0 & -2x_2 \\ 1 & 0 \end{bmatrix}$

Accessible everywhere except where x_2=0



If we tried [f,[f,g]], would we pick up new directions? It turns out they will also be dependent on x_2, and the rank will drop at $x_2 = 0$.

<u>Example 3:</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \cos\theta x_3 + \sin\theta x_4 \\ -\sin\theta x_3 + \cos\theta x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

where $\theta \equiv \theta(\sqrt{x_1^2 + x_2^2})$

The system is of the form:

$$\dot{x} = f(x) + g_1 u_1 + g_2 u_2$$

where

$$f(x) = \begin{bmatrix} \cos\theta x_3 + \sin\theta x_4 \\ -\sin\theta x_3 + \cos\theta x_4 \\ 0 \\ 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We have: $C = [g_1, g_2, [f, g_1], [f, g_2]]$

If C has rank 4, then the system is locally accessible. Have fun…

Observability for Nonlinear Systems

Intuition for observability:
From observing the sensor(s) for a finite period of time, can I find the state at previous times?

Review of Linear Systems

$$\dot{x} = Ax \quad \text{where} \quad x \in \Re^n$$
$$z = Mx \quad \text{where} \quad z \in \Re^p \text{ and } p<n$$

$$x(t) = e^{At} x_0 \text{ (linear system)}$$

$$z(t), \ 0 \le t \le T$$

Can I determine x_0?

$z(t) = Me^{At} x_0$ where M is pxn, e^{AT} is nxn and x_0 is nx1, so z(t) is px1

Using the Cayley-Hamilton theorem:

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$$

Note: the Cayley-Hamilton theorem applies to time-varying matrices as well.

So, I have:

$$z(t) = \left\{ \alpha_0(t)M + \alpha_1(t)MA + \ldots + \alpha_{n-1}(t)MA^{n-1} \right\} x_0$$

So I can solve for x_0 iff the matrix O spans n space, where:

$$O = \begin{bmatrix} M \\ MA \\ \ldots \\ \ldots \\ MA^{n-1} \end{bmatrix}$$

This does not carry over to nonlinear systems, so we take a local approach.

Local Approach to Observability (Linear Systems)

$$z = Mx + v(t)$$

v(t) is the measurement noise, can cause problems.

$$\dot{z} = M\dot{x} = MAx$$
$$\ddot{z} = MA^2 x$$
$$\cdots$$
$$z^{(n-1)} = MA^{n-1}x$$

$$\begin{bmatrix} z \\ \ldots \\ z^{(n-1)} \end{bmatrix} = Ox \quad \Rightarrow \text{ O must have rank n}$$

Lie Derivatives:

Definition

Let f: $\Re^n \rightarrow \Re^n$ be a vector field in $\Re^n$.
Let h: $\Re^n \rightarrow \Re$ be a smooth scalar function.

Then the Lie derivative of h with respect to f is:

$$L_f h = \nabla h . f = \frac{\partial h}{\partial x} . f = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i$$

Dimensions

$$\text{f looks like:} \quad \begin{bmatrix} f_1(x) \\ \ldots \\ \ldots \\ \ldots \\ f_n(x) \end{bmatrix}$$

h looks like: h(x) with $x \in \Re^n \Rightarrow$ associates a scalar to each point in $\Re^n$

The Lie derivative looks like:

$$L_f h = \left[\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ \ldots \\ \ldots \\ \ldots \\ f_n(x) \end{bmatrix} \Rightarrow \text{ } L_f h \text{ is a } \textbf{scalar}.$$

Conventions:

By definition, $L_f^0(h) = h$

We can also define higher-order Lie derivatives:

$$L_f^2(h) = \frac{\partial}{\partial x}[L_f^1(h)].f \text{ etc…}$$

<u>Note:</u> if $\dot{x} = f(x)$, then $\dot{h} = \frac{\partial h}{\partial x}.\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}.f = L_f^1(h)$

And $\ddot{h} = \frac{\partial}{\partial x}[L_f^1(h)].f = L_f^2(h)$

Etc… so $h^{(k)} = L_f^k(h)$

<u>Use of Lie Derivative Notation for Linear Systems</u>

$$\dot{x} = Ax \qquad \text{so f(x)=Ax}$$

$$z_i = M_i x \qquad i \in [1, p], \qquad \text{M}_\text{i} \text{ is 1xn}$$

$$\Rightarrow \; z_i = h_i(x) = L_f^0(h_i)$$

$$\dot{z}_i = L_f^1(h_i)$$
$$\dots$$
$$z_i^{(k)} = L_f^k(h_i)$$

Define $G = \begin{bmatrix} L_f^0(h_1) & \dots & L_f^0(h_p) \\ \dots & \dots & \dots \\ L_f^{n-1}(h_1) & \dots & L_f^{n-1}(h_p) \end{bmatrix} = \begin{bmatrix} M_1 x & \dots & M_p x \\ \dots & \dots & \dots \\ M_1 A^{n-1} x & \dots & M_p A^{n-1} x \end{bmatrix}$

Now, define a gradient operator:

$$dG = \begin{bmatrix} dL_f^0(h_1) & \dots & dL_f^0(h_p) \\ \dots & \dots & \dots \\ dL_f^{n-1}(h_1) & \dots & dL_f^{n-1}(h_p) \end{bmatrix} = O$$

O must have rank n for the system to be observable.

Nonlinear Systems

$$\dot{x} = f(x,u)$$
$$z = h(x) = [h_1(x), h_2(x), ..., h_p(x)]^T$$

Theorem:

Let G denote the set of all finite linear combinations of the Lie derivatives of $h_1,\ldots,h_p$ with respect to f for various values of u = constant. Let dG denote the set of all their gradients. If we can find n linearly independent vectors within dG, then the system is **locally observable**.

The system is **locally observable**, that is **distinguishable at a point x_0** if there exists a neighborhood of x_0 such that in this neighborhood,

$$x_0 \neq x_1 \Rightarrow z(x_0) \neq z(x_1)$$

"if the sensor readings are different, the states are different"

Case of a single measurement:

$$\dot{x} = f(x,u)$$
$$z = h(x)$$

Look at the derivatives of z:

$$z = h = L_f^0(h)$$

$$\dot{z} = \dot{h} = L_f^1(h)$$

$$\ldots$$

$$z^{(n-1)} = L_f^{n-1}(h)$$

Let: $l(x) \equiv \begin{bmatrix} L_f^0(h) \\ ... \\ L_f^{n-1}(h) \end{bmatrix}$

Expand in a first-order series about x_0 for $u = u_0$

$$l(x,u_0) \approx l(x_0,u_0) + \left.\frac{\partial l(x,u_0)}{\partial x}\right|_{x=x_0} (x - x_0) + hot$$

Then $dG = O = \left.\frac{\partial l(x,u_0)}{\partial x}\right|_{x=x_0}$ must have rank n

<u>Example:</u>

$$\begin{cases} \dot{x}_1 = \dfrac{x_1^2}{2} + \exp(x_2) + x_2 \\ \dot{x}_2 = x_1^2 \\ z = x_1 \end{cases}$$

The question we are trying to answer is: "from observation, does z contain enough information on x_1 and x_2?"

Since $x_1 = z$, $\exp(x_2) + x_2 = \dot{z} - \frac{z^2}{2}$ (by substitution in the first line)

If $z = x_2$, then the system would not be distinguishable, since $\dot{z} = x_1^2$

But if you take one more derivative you can get a single valued expression since x_1^3 has only one solution.

$$\dot{z} = x_1^2$$

$$\dot{z} = 2x_1\dot{x}_1 = 2x_1\left(\frac{x_1^2}{2} + \exp(x_2) + x_2\right)$$

Rank test for $z=h=x_1$

$$G = \begin{bmatrix} h \\ L_f(h) \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dfrac{x_1^2}{2} + \exp(x_2) + x_2 \end{bmatrix}$$

$O = dG = \begin{bmatrix} 1 & 0 \\ x_1 & e^{x_2} + 1 \end{bmatrix}$ has rank 2 everywhere

$\Rightarrow$ ***the system is locally observable everywhere***.

SUMMARY: Controllability and Observability for Nonlinear Systems

Controllability

$$\dot{x}=f(x)+\sum_{1}^{m} g_i(x)u_i$$

The system is **locally accessible** about a point $\underline{x}_0$ if and only if

$$C = [\ g_1,\ldots,g_m,\ [g_i,\ g_j],\ldots[ad_{gi}{}^k\ ,g_j],\ldots,\ [f,g_i],\ldots,\ [ad_f{}^k,g_i],\ldots]$$

has rank n where n is the rank of $\underline{x}$. C is the accessibility distribution.

If the system has the form: $\dot{x}=\sum_{1}^{m} g_i(x)u_i$

If f(x) = 0 and C has rank n, then the system is **controllable**.

Observability

$$\dot{x}=f(x,u)$$
$$z=h(x)$$

Two states $\underline{x}_0$ and $\underline{x}_1$ are **distinguishable** if there exists an input function u* such that: $z(\underline{x}_0) \neq z(\underline{x}_1)$

The system is **locally observable** at $\underline{x}_0$ if there exists a neighbourhood of $\underline{x}_0$ such that every $\underline{x}$ in that neighbourhood other than $\underline{x}_0$ is distinguishable from $\underline{x}_0$.

A test for local observability is that:

$$O(x_0,u^*)\equiv\left.\frac{\partial l(x_0,u^*)}{\partial x}\right|_{x=x_0}$$

must have rank n, where n is the rank of $\underline{x}$ and

$$l(x_0,u^*)\equiv\begin{bmatrix} L_f^0(h_1) \\ \cdots \\ \cdots \\ \cdots \\ L_f^{n-1}(h_1) \end{bmatrix}$$

For a p*x*1 vector,
$\underline{z} = [h_1, \ldots, h_p]^T$,

$$O(x_0,u^*) \equiv \begin{bmatrix} dL_f^0(h_1) \\ \cdots \\ dL_f^0(h_p) \\ \cdots \\ dL_f^{n-1}(h_1) \\ \cdots \\ dL_f^{n-1}(h_p) \end{bmatrix}$$

	LINEAR SYSTEMS $\dot{x}=Ax+Bu$	**NONLINEAR SYSTEMS** $\dot{x}=f(x,u)$
CONTROLLABILITY AND ACCESSIBILITY Intuition: the system is controllable ⇒ you can get anywhere you want in a finite amount of time.	**LINEAR TIME INVARIANT SYSTEMS** $\dot{x}=Ax+Bu$ **CONTROLLABILITY** The system is controllable if: $C = [\ B\ \ AB\ \ \ldots\ A^{n-1}B\ ]$ has rank n, where n is the rank of $\underline{x}$.	**AFFINE SYSTEMS** $\dot{x}=f(x)+\sum_{1}^{m} g_i(x)u_i$ **ACCESSIBILITY** The system is locally accessible about a point $\underline{x}_0$ if and only if $C = [\ g_1,\ldots,g_m,\ [g_i, g_j],\ldots\ [ad_{gi}^{\ k}, g_j],\ldots,\ [f,g_i],\ldots,\ [ad_f^k, g_i],\ldots]$ has rank n where n is the rank of $\underline{x}$. C is the accessibility distribution. **CONTROLLABILITY** $\dot{x}=\sum_{1}^{m} g_i(x)u_i$ If f(x) = 0 and C has rank n, then the system is controllable.

OBSERVABILITY AND DISTINGUISHABILITY Intuition: the system is observable ⇒ from observing the sensor measurements for a finite period of time, I can obtain the state at previous times.	$\dot{x}=Ax$ $z=Mx$ $\underline{x}$ has rank n $\underline{z}$ has rank p p<n **OBSERVABILITY** The system is observable if: $O=\begin{bmatrix} M \\ MA \\ \dots \\ \dots \\ M\ddot{A}^{n-1} \end{bmatrix}$ has rank n, where n is the rank of $\underline{x}$.	$\dot{x}=f(x,u)$ $z=h(x)$ **DISTINGUISHABILITY** Two states $\underline{x}_0$ and $\underline{x}_1$ are distinguishable if there exists an input function u* such that: $z(\underline{x}_0) \neq z(\underline{x}_1)$ **LOCAL OBSERVABILITY** The system is locally observable at $\underline{x}_0$ if there exists a neighbourhood of $\underline{x}_0$ such that every $\underline{x}$ in that neighbourhood other than $\underline{x}_0$ is distinguishable from $\underline{x}_0$. A test for local observability is that: $O(x_0,u^*)\equiv\left.\frac{\partial l(x_0,u^*)}{\partial x}\right\|_{x=x_0}$ must have rank n, where n is the rank of $\underline{x}$ and $l(x_0,u^*)\equiv\begin{bmatrix} L_f^0(h_1) \\ \dots \\ \dots \\ \dots \\ L_f^{n-1}(h_1) \end{bmatrix}$ For a p*x*1 vector, $\underline{z} = [h_1, ..., h_p]^T$, $O(x_0,u^*)\equiv\begin{bmatrix} dL_f^0(h_1) \\ \dots \\ dL_f^0(h_p) \\ \dots \\ dL_f^{n-1}(h_1) \\ \dots \\ dL_f^{n-1}(h_p) \end{bmatrix}$

Remarks

In general the conditions for nonlinear systems are weaker than those for linear systems. Properties for nonlinear systems tend to be local.

What to do for nonlinear controllability if the system is not in affine form?

$$\dot{x}=f(x,u)$$

Let $\tau\dot{u}+u=v$

$\Rightarrow$ $\underline{z}=\begin{bmatrix} x \\ u \end{bmatrix}$, v is my control $\Rightarrow$ the system is now affine in (z,v) and pick τ to be OK.

Marius Sophus Lie

Born: 17 Dec 1842 in Nordfjordeide, Norway
Died: 18 Feb 1899 in Kristiania (now Oslo), Norway

The Lie Derivative and Observability

Definition

Let f: $\Re^n \rightarrow \Re^n$ be a vector field in $\Re^n$.
Let h: $\Re^n \rightarrow \Re$ be a smooth scalar function.

Then the Lie derivative of h with respect to f is:

$$L_f h=\nabla h.f=\frac{\partial h}{\partial x}.f$$

Dimensions

f looks like: $\begin{bmatrix} f_1(x) \\ \cdots \\ \cdots \\ \cdots \\ f_n(x) \end{bmatrix}$

h looks like: h(x) with $x \in \Re^n \Rightarrow$ associates a scalar to each point in $\Re^n$

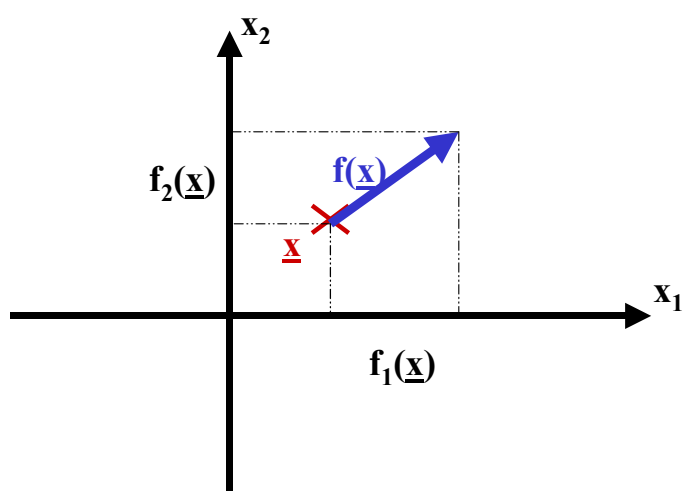
The Lie derivative looks like:

$$L_f h = \left[\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ \cdots \\ \cdots \\ \cdots \\ f_n(x) \end{bmatrix} \Rightarrow L_f h \text{ is a } \textbf{scalar}.$$

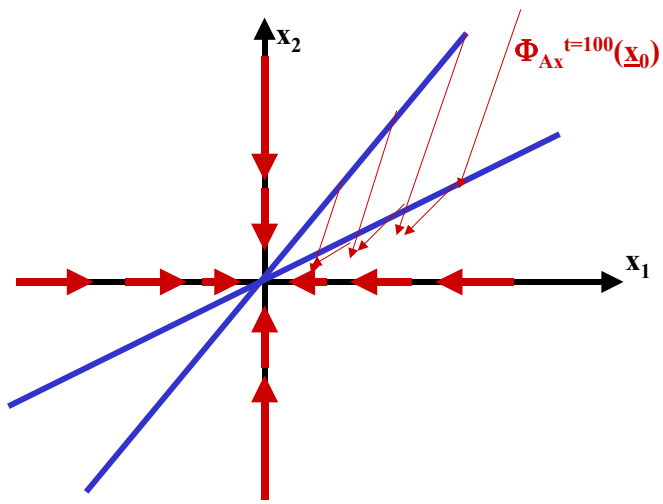
<u>Physically (time for pictures!)</u>

<u>Picture of f</u>
f associates an n-dimensional vector to each point in $\Re^n$

In $\Re^2$:



For example, let $f(\underline{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

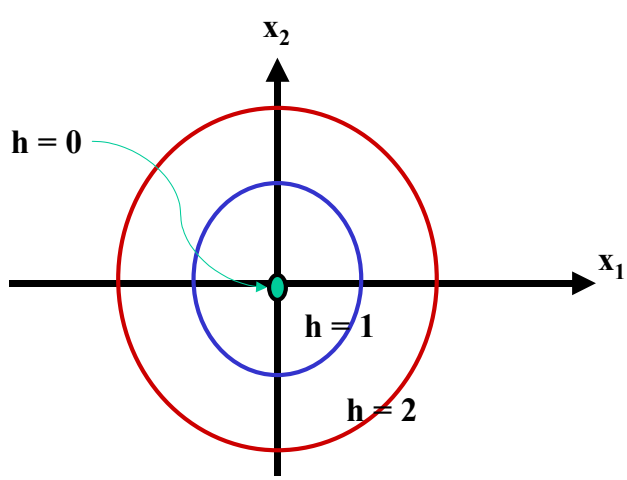


$\Phi_f^t(\underline{x}_0)$ = "flow along the vector field for time t, starting at x_0 "
$\Rightarrow$ "tangent" to the phase plane plot at every single point

Picture of h

For example, in $\Re^2$, pick h to be the distance to the origin:

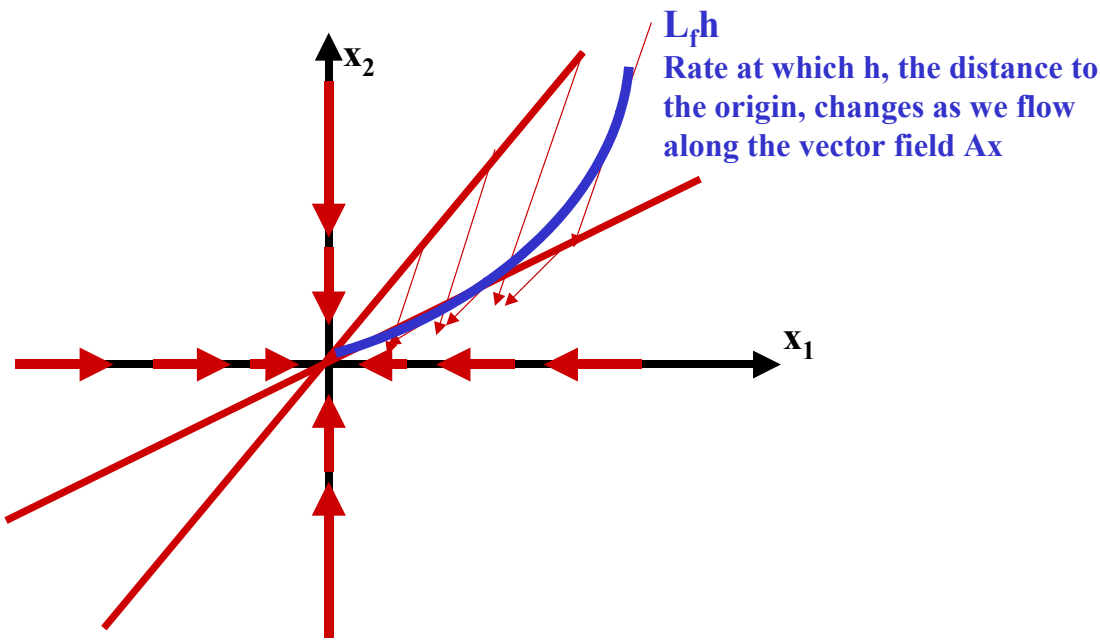
$$h(\underline{x}) = \|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2}$$



Lie derivative picture

Using this example:

$$L_f h = \begin{bmatrix} \frac{\partial \|x\|_2}{\partial x_1} & \frac{\partial \|x\|_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$



So, the Lie derivative gives the rate of change in a scalar function h as one flows along the vector field f.

In a control systems context:

$\dot{x} = f(x)$ $x \in \Re^n$ $\quad$ f: $\Re^n \rightarrow \Re^n$

y=h(x) $\quad$ $y \in \Re$ $\quad$ h: $\Re^n \rightarrow \Re$

$$L_f h = \begin{bmatrix} \frac{\partial h}{\partial x_1} \cdots \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \cdots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \cdots \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \cdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \cdots \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \cdots \\ \frac{\partial x_n}{\partial t} \end{bmatrix} = \frac{\partial h}{\partial t}$$

along the flow of f

How does this tie into observability?

Imagine: $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$ and we can only "see" y, a scalar, and we wish to find $x \in \Re^n$

y = Cx
y' = Cx' = CAx
…
$y^{(n-1)} = CA^{n-1}\,x$

and solve for x (n equations)

$\Rightarrow$ if [C CA ... CA^{n-1}] has rank n, we have n independent equations in n variables $\Rightarrow$ OK

Using the Lie derivative

f(x) = Ax, h(x) = Cx

$$L_f h = \frac{\partial}{\partial x}(Cx)Ax = CAx$$

…

$$L_f^{n-1} h = CA^{n-1}x$$

and by convention, $L_f^0 h = C$

The Lie Bracket and Controllability

Definition

Let f: $\Re^n \rightarrow \Re^n$ be a smooth vector field in $\Re^n$.
Let g: $\Re^n \rightarrow \Re^n$ be a smooth vector field in $\Re^n$.

Then the Lie bracket of f and g is a third-order vector field given by:

$$[f,g] = \frac{\partial g}{\partial x}.f - \frac{\partial f}{\partial x}.g$$

Dimensions

f looks like: $\begin{bmatrix} f_1(x) \\ \dots \\ \dots \\ \dots \\ f_n(x) \end{bmatrix}$, g also looks like: $\begin{bmatrix} g_1(x) \\ \dots \\ \dots \\ \dots \\ g_n(x) \end{bmatrix}$

$$[f,g]=\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \cdots \\ \cdots \\ \cdots \\ f_n(x) \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} g_1(x) \\ \cdots \\ \cdots \\ \cdots \\ g_n(x) \end{bmatrix}$$

So [f,g] is a vector field.

<u>How does this tie into controllability?</u>

Consider: $\dot{x}=g_1(x)u_1+g_2(x)u_2$
u_1, u_2 are scalar inputs

$\underline{x}\in\Re^3$

$$g_1=\begin{bmatrix} g_{11}(x) \\ g_{12}(x) \\ g_{13}(x) \end{bmatrix},\ g_2=\begin{bmatrix} g_{21}(x) \\ g_{22}(x) \\ g_{23}(x) \end{bmatrix}$$

<u>What directions can we steer x if we start at some point $\underline{x}_0$?</u>

Clearly, we can move anywhere in the span of $\{g_1(\underline{x}_0),g_2(\underline{x}_0)\}$.

Let's say that: $g_1(x_0)=\begin{bmatrix} * \\ * \\ 0 \end{bmatrix},\ g_2(x_0)=\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$

<u>Can we move in the x_3 direction?</u>

The directions that we are allowed to move in by infinitesimally small changes are $[g_1,g_2]$.