1 Feedback Linearization

Similarly to the approach taken in sliding mode control, in this section we wish to exploit the form of the system equations to modify the dynamics to something more convenient.

We consider a class of nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned}$$

where $f : D \to \Re^n$ and $G : D \to \Re^{n \times p}$ are defined on a domain $D \subset \Re^n$, which contains the origin, and pose the question whether there exists a state feedback control

$$u = \alpha(x) + \beta(x)v$$

and a change of variables

$$z = T(x)$$

that transforms the nonlinear system into an equivalent linear system.

If the answer to this question is positive, we can induce linear behavior in nonlinear systems and apply the large number of tools and the well established theory of linear control to develop stabilizing controllers.

1.1 Motivation

Example 1.1: To introduce the idea of feedback linearization, let us start with the problem of stabilizing the origin of the pendulum equation.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\left[\sin(x_1 + \delta) - \sin\delta\right] - bx_2 + cu \end{aligned}$$

If we choose the control

$$u = \frac{a}{c}\left[\sin(x_1 + \delta) - \sin\delta\right] + \frac{v}{c}$$

We can cancel the nonlinear term $a\left[\sin(x_1 + \delta) - \sin\delta\right]$. This cancellation results in the linear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_2 + v$$

Thus, the stabilization problem for the nonlinear system has been reduced to a stabilization problem for a controllable linear system. We can proceed to design a stabilizing linear state feedback control

$$v = -k_1 x_1 - k_2 x_2$$

to locate the eigenvalues of the closed loop system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k_1 x_1 - (k_2 + b) x_2$$

in the open left half plane. The overall state feedback control law is given by

$$u = \left(\frac{a}{c}\right)\left[\sin(x_1 + \delta) - \sin\delta\right] - \frac{1}{c}(k_1 x_1 + k_2 x_2)$$

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Clearly, we should not expect to be able to cancel nonlinearities in every nonlinear system. There must be a certain structural property of the system that allows us to perform such cancellation. Therefore, the ability to use feedback to convert a nonlinear state equation into a controllable linear state equation by canceling nonlinearities requires the nonlinear space equation to have the structure

$$\dot{x} = Ax + B\gamma(x)\left[u - \alpha(x)\right]$$

Where A is $n \times n$, B is $n \times p$, the pair (A, B) is controllable, the functions $\alpha : \Re^n \to \Re^p$ and $\gamma : \Re^n \to \Re^{p \times p}$ are defined in the domain $D \subset \Re^n$ that contains the origin, and the matrix $\gamma(x)$ is nonsingular for every $x \in D$. If the state equation takes the form $\dot{x} = Ax + B\gamma(x)\left[u - \alpha(x)\right]$, then we can linearize it via the state feedback

$$u = \alpha(x) + \beta(x) v$$

where $\beta(x) = \gamma^{-1}(x)$, to obtain the linear state equation

$$\dot{x} = Ax + Bv$$

For stabilizing, we design $v = -Kx$ such that $A - BK$ is Hurwitz. The overall nonlinear stabilizing state feedback control is

$$u = \alpha(x) - \beta(x) Kx$$

Suppose the nonlinear state equation does not have the required structure. Does this mean we cannot linearize the system via feedback? The answer is no. Even if the state equation does not have the required structure for one choice of variables, it might do so for another choice.

Example 1.2: Consider, for example the system

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

We cannot simply choose u to cancel the nonlinear term $a \sin x_2$. However, if we first change the variables by the transformation

$$z_1 = x_1$$
$$z_2 = a \sin x_2 = \dot{x}_1$$

then z_1 and z_2 satisfy

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 . \dot{x}_2 = a \cos\left(\sin^{-1} \frac{z_2}{a} \right)\left(u - z_1^2\right)$$

and the nonlinearities can be cancelled by the control

$$u = x_1^2 + \frac{1}{a \cos x_2} v$$

which is well defined for $-\pi/2 < x_2 < \pi/2$. The state equation in the new coordinates can be found by inverting the transformation to express (x_1, x_2) in the term of (z_1, z_2); that is,

$$x_1 = z_1$$
$$x_2 = \sin^{-1}\left(\frac{z_2}{a} \right)$$

which is well defined for $-a < z_2 < a$. The transformed state equation is given by

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)\left(-z_1^{\,2} + u\right)$$

which is in the required form to use state feedback

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Definition 1.1: A nonlinear system

$$\dot{x} = f(x) + G(x)u$$

where $f : D \to \Re^n$ and $G : D \to \Re^{n\times p}$ are sufficiently smooth on a domain $D \subset \Re^n$, is said to be feedback linearizable (or input state linearizable) if there exists a diffeomorphism $T : D \to \Re^n$ such that $D_z = T(D)$ contains the origin and a change of variables $z = T(x)$ transforms the system $\dot{x} = f(x) + G(x)u$ into the form

$$\dot{z} = Az + B\gamma(x)\left[u - \alpha(x)\right]$$

with (A, B) controllable and $\gamma(x)$ nonsingular for all $x \in D$.

Definition 1.2: f is called ***smooth*** if $f \in C^\infty$. That is, f is continuous and all derivatives of all order are continuous.

Definition 1.3: T is a ***diffeomorphism*** if T is smooth, and the inverse exists and is also smooth.

Remark 1.1: When certain output variables are of interest, as in tracking control problems, the state model is described by state and output equations. Linearizing the state equation does not necessarily linearize the output equation.

Example 1.3: Consider the previous system

$$\dot{x}_1 = a\sin x_2$$
$$\dot{x}_2 = -x_1^{\,2} + u$$

If the system has an output $y = x_2$, then the change of variables and state feedback control

$z_1 = x_1$, $z_2 = a\sin x_2$, and $u = x_1^{\,2} + \dfrac{1}{a\cos x_2}v$

yield

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$
$$y = \sin^{-1}\left(\frac{z_2}{a}\right)$$

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While the state equation is linear, solving a tracking control problem for y is still complicated by the nonlinearity of the output equation.

1.1.1 Transformation Matrix $T(x)$

With the help of a Matrix $T(x)$ we want to transform

$$\dot{x} = f(x) + G(x)u$$

into the new system

$$\dot{z} = Az + B\gamma(x)(u - \alpha(x))$$

Given the previous system with a coordinate transformation $z = T(x)$ we derive

$$\begin{aligned}\dot{z} &= \frac{\partial T(x)}{\partial x}\dot{x} = \frac{\partial T(x)}{\partial x}(f(x) + G(x)u) \\ &= Az + B\gamma(x)(u - \alpha(x)) \\ &= AT(x) + B\gamma(x)(u - \alpha(x))\end{aligned}$$

and obtain a system of partial differential equations

$$\begin{cases}\dfrac{\partial T}{\partial x} f(x) = AT(x) - B\gamma(x)\alpha(x) \\ \dfrac{\partial T}{\partial x} G(x) = B\gamma(x)\end{cases}$$

By solving this set of partial differential equations for *T(x)*, the transformation is found.

Remark 1.2: $T(x)$ is not uniquely defined by this system of differential equations as shown when we consider any linear transformation $\tilde{z} = Mz$. Then $\tilde{T}(x) = MT(x)$ will also satisfy this system but with $\tilde{A} = MAM^{-1}$ and $\tilde{B} = MB$.

We use this fact to choose the matrix A and B to be in the canonical controllability form. For simplicity we consider this for a single-input system. In this case, the control canonical form is as follows.

$$A_C = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} \end{bmatrix}, \; B_C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Where the α_i are the coefficients of the characteristic polynomial of A:

$$\det(sI - A) = \sum_{i=0}^{n-1} \alpha_i s^i$$

Note that (A_C, B_C) is feedback equivalent to a chain of integrators. *i.e.* with the mapping $u \mapsto u + \sum_{i=0}^{n-1} \alpha_i s^i$, the system is transformed to a chain of integrators. Thus we consider the following system matrices:

$$A_0 = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then with

$$T(x) = \begin{bmatrix} T_1(x) \\ \cdots \\ T_n(x) \end{bmatrix}$$

we obtain

$$A_0 T(x) - B\gamma(x) = \begin{bmatrix} T_2(x) \\ \vdots \\ T_n(x) \\ -\alpha(x)\gamma(x) \end{bmatrix} = \frac{\partial T}{\partial x} f(x)$$

$$B_0 \gamma(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{bmatrix} = \frac{\partial T}{\partial x} G(x)$$

Solving this system of partial differential equations we can find $T(x)$.

The following theorem, stated without proof, gives necessary and sufficient conditions for feedback linearizability of a single-input affine-in-control nonlinear system.

We need some notations first. Let f, g be smooth vector fields on $\mathbb{R}^n$. The Lie-bracket of f and g is defined as $[f,g](x) = \langle \partial f/\partial x(x), g(x)\rangle - \langle \partial g/\partial x(x), f(x)\rangle$.
We also define
$ad_f g(x) := [f,g](x)$,
$ad_f^k g(x) := [f, ad_f^{k-1} g](x)$.
A distribution D is a mapping that assigns to each point in the state space a subspace of the tangent space at that point. In other words, a distribution is a family of smooth vector fields that span a subspace of $\mathbb{R}^n$ at each point. A distribution is involutive if the Lie bracket of any two vector fields in the distribution lies in the distribution. To wit, if D is spanned by $\{f_1, ..., f_m\}$, then for each i, j, x, we have
$[f_i, f_j](x) = \sum_{k=1}^{m} c_{ij}^k(x) f_k(x)$,
where $\{c_{ij}^k\}_{k=1}^m$ are smooth functions. Note that this is a natural generalization of the concept of linear dependence of vectors. We define the following special distributions:
$D^1(x) := span\{g(x)\}, \;\; D^j(x) := span\{ad_f^{q-1} g(x) : 1 \le q \le j\}$.

Theorem 1.1: A single input system $\dot{x} = f(x) + g(x)u$ at a point x if and only if

- $\{g(x), ..., ad_f^{n-1} g(x)\}$ is a linearly independent family of vectors, and
- D^{n-1} is involutive.

1.2 Input-Output Linearization

Consider the single-input-single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where f, g, and h are sufficiently smooth in a domain $D \subset \Re^n$. The mappings $f : D \to \Re^n$ and $g : D \to \Re^n$ are called vector fields on D.

Derive conditions which allow us to transform the system such that the input output map is linear.

The derivative $\dot{y}$ is given by

$$\dot{y} = \frac{\partial h}{\partial x}\left[f(x) + g(x)u \right] = L_f h(x) + L_g h(x) u$$

where

$$\frac{\partial h}{\partial x} f(x) = L_f h(x)$$

is called the Lie Derivative of h with respect to f.

If $L_g h(x) = 0$, then $\dot{y} = L_f h(x)$, independent of u. If we continue to calculate the second derivative of y, denoted by $y^{(2)}$, we obtain

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x}\left[f(x) + g(x)u \right] = L_f^2 h(x) + L_g L_f h(x) u$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$, independent of u. Repeating this process, we see that if $h(x)$ satisfies

$$L_g L_f^{i-1} h(x) = 0,\ i = 1, 2, ..., \rho - 1;\ L_g L_f^{\rho-1} h(x) \neq 0$$

then u does not appear in the equations of y, $\dot{y}, \ldots, y^{(\rho-1)}$ and appears in the equation of $y^{(p)}$ with a nonzero coefficient:

$$y^{(p)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$

The forgoing equation shows clearly that the system is input-output linearizable, since the state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)}\left[-L_f^{\rho} h(x) + v \right]$$

reduces the input-output map to

$$y^{(\rho)} = v$$

which is a chain of ρ integrators. In this case, the integer ρ is called the ***relative degree*** of the system.

Example 1.4: Consider the controlled van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\left(1 - x_1^2\right) x_2 + u \end{aligned}$$

with output $y = x_1$. Calculating the derivatives of the output, we obtain

$$\dot{y} = \dot{x}_1 = x_2$$
$$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon\left(1 - x_1^{\,2}\right)x_2 + u$$

Hence, the system has relative degree two in $\Re^2$. For the output $y = x_1 + x_2^{\,2}$,

$$\dot{y} = x_2 + 2x_2\left[-x_1 + \varepsilon\left(1 - x_1^{\,2}\right)x_2 + u\right]$$

and the system has relative degree one in $D_0 = \left\{x \in \Re^2 \middle| x_2 \neq 0\right\}$

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Example 1.5: Correspondence to relative degree for Linear Systems

Consider a linear system represented by the transfer function

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

where $m < n$ and $b_m \neq 0$. A state model for the system can be taken as

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & & \cdots & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & 1 & 0 \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_m & \cdots & -a_{n-1} \end{bmatrix}_{n\times n}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

$$C = \begin{bmatrix} b_0 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}_{1\times n}$$

This linear space model is a special case of $\dot{x} = f(x) + g(x)u$, $y = h(x)$, where $f(x) = Ax$, $g = B$, and $h(x) = Cx$. To check the relative degree of the system, we calculate the derivative of the output. The first derivative is

$$\dot{y} = CAx + CBu$$

If $m = n-1$, then $CB = b_{n-1} \neq 0$ and the system has relative degree one. Otherwise, $CB = 0$ and we continue to calculate the second derivative $y^{(2)}$. Noting that CA is a row vector obtained by shifting the elements of C one position to the right, while CA^2 is obtained by shifting the elements of C two positions to the right, and so on, we see that

$$CA^{i-1}B = 0 \text{, for } i = 1, 2, ..., n-m-1 \text{, and } CA^{n-m.1}B = b_m \neq 0$$

Thus, u appears first in the equation of $y^{(n-m)}$, given by

$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu$$

and the relative degree of the system is $n-m$ (the difference between the degrees of the denominator and numerator polynomials of $H(s)$).

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Now let

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \overset{def}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \overset{def}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

where $\phi_1(x)$ to $\phi_{n-\rho}(x)$ are chosen such that

$$\frac{\partial \phi_1}{\partial x} g(x) = 0 \text{, for } 1 \leq i \leq n-\rho$$

This condition ensures that when we calculate

$$\dot{\eta} = \frac{\partial \phi}{\partial x}\left[f(x) + g(x)u \right]$$

the term u cancels out. It is now easy to verify that the change of variables $z = T(x)$ transforms the system into

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta,\xi) \\
\dot{\xi} &= A_c\xi + B_C\gamma(x)\left[u-\alpha(x)\right] \\
y &= C_C\xi
\end{aligned}
$$

where $\xi \in \Re^p$, $\eta \in \Re^{n-\rho}$, (A_C, B_C, C_C) is a canonical form representation of a chain of ρ integrators, where

$$
f_0(\eta,\xi) = \frac{\partial\phi}{\partial x} f(x)\Big|_{x=T^{-1}(z)}
$$

$$
\gamma(x) = L_g L_f^{\rho-1} h(x) \text{ and } \alpha(x) = -\frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)}
$$

We have kept α and γ expressed in the original coordinates. These functions are uniquely determined in terms of f, g, and h. They are independent of the choice of ϕ. They can be expressed in the new coordinates by setting

$$
\alpha_0(\eta,\xi) = \alpha\left(T^{-1}(z)\right) \text{ and } \gamma_0(\eta,\xi) = \gamma\left(T^{-1}(z)\right)
$$

which, of course, will depend on the choice of ϕ. In this case, the equation can be rewritten as

$$
\dot{\xi} = A_c\xi + B_C\gamma_0(\eta,\xi)\left[u-\alpha_0(\eta,\xi)\right]
$$

The three equations of the new system are said to be in the normal form. This form decomposes the system into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$
u = \alpha(x) + \beta(x)v
$$

where $\beta(x) = \gamma^{-1}(x)$, while the internal part is made unobservable by the same control. The internal dynamics are described by $\dot{\eta} = f_0(\eta,\xi)$. Setting $\xi = 0$ in that equation results in

$$
\dot{\eta} = f_0(\eta,0)
$$

which is called the ***zero dynamics*** of the system. If the zero dynamics of the system are (globally) asymptotically stable, the system is called ***minimum phase***.

Theorem 1.2: The origin $z=0$ is an asymptotically stable equilibrium point of the linearized system if $\eta=0$ is an asymptotically stable equilibrium point of

$$\dot{\eta}=f_0(\eta,0)$$

In other words, a minimum phase input-output linearizable system can be stabilized by a feedback law

$$u=\alpha(x)-\beta(x)KT(x)$$

Proof

The idea is to construct a special Lyapunov function. By the converse Lyapunov theorem $\exists V_1(\eta)$ such that

$$\frac{\partial V_1}{\partial \eta} f(\eta,0)<-\alpha_3(\|\eta\|)$$

in some neighborhood of $\eta=0$. Let $P=P^T>0$ be the solution of the Lyapunov equation

$$P(A-BK)+(A-BK)^T P=-I$$

Construct a second Lyapunov function V_2

$$V_2=\xi^T P\xi$$

Then use $V(\eta,\xi)=V_1(\eta)+k\sqrt{\xi^T P\xi}$, $k>0$

We want to check that the derivative of $V(\eta,\xi)$ is negative

$$\begin{aligned}\dot{V}&=\frac{\partial V_1}{\partial \eta} f(\eta,\xi)+\frac{k}{2\sqrt{\xi^T P\xi}}\xi^T\underbrace{\left[P(A-BK)+(A-BK)P\right]}_{-I}\xi\\&=\frac{\partial V_1}{\partial \eta} f_0(\eta,0)+\frac{\partial V_1}{\partial \eta}\left(f_0(\eta,0)-f(\eta,\xi)\right)-\underbrace{\frac{k}{2\sqrt{\xi^T P\xi}}}_{k_1}\xi^T\xi\\&\le-\alpha_3(\|\xi\|)+k\|\xi\|-k_1\|\xi\|\end{aligned}$$

We see that if $k<k_1$, then $\dot{V}<0$.

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Remark 1.1: This is a local result

Remark 1.2: If the origin $\eta = 0$ is globally asymptotically stable for $\dot{\eta} = f_0(\eta, 0)$ one might think that system can be globally stabilized. This is not the case.

Remark 1.3: Global stability can be guaranteed if the system is input to state stable and linearizable.

Remark 1.4: One may think that the system can be globally stabilized, or at least semi-globally stabilized, by designing the linear feedback control $v = -K\xi$ to assign the eigenvalues of $(A - BK)$ far to the left in the complex plane so that the solution of $\dot{\xi} = (A - BK)\xi$ decay to zero arbitrarily fast. Then, the solution of $\dot{\eta} = f_0(\eta, \xi)$ will quickly approach the solution of $\dot{\eta} = f_0(\eta, 0)$, which is well behaved, because its origin is globally asymptotically stable.

Example 1.6: Consider the third-order system

$$\begin{aligned} \dot{\eta} &= -\frac{1}{2}(1 + \xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \end{aligned}$$

The linear feedback control

$$v = -k^2\xi_1 - 2k\xi_2 \overset{def}{=} -K\xi$$

assigns the eigenvalues of

$$A - BK = \begin{bmatrix} 0 & 0 \\ -k^2 & -2k \end{bmatrix}$$

at $-k$ and $-k$. The exponential matrix

$$e^{(A-BK)t} = \begin{bmatrix} (1 + kt)e^{-kt} & te^{-kt} \\ -k^2te^{-kt} & (1 - kt)e^{-kt} \end{bmatrix}$$

shows that as $k \to \infty$, the solution $\xi(t)$ will decay to zero arbitrarily fast. Notice, however, that the coefficient of the $(2,1)$ element of the exponential matrix is a quadratic function of k. It can be shown that the absolute value of this element reaches a maximum

value k/e at $1/k$. While this term can be made to decay to zero arbitrarily fast by choosing k large, its transient behavior exhibits a peak of the order of k. The interaction of peaking with nonlinear growth could destabilize the system. In particular, for the initial states $\eta(0)=\eta_0$, $\xi_1(0)=1$, and $\xi_2(0)=0$, we have $\xi_2(t)=-k^2te^{-kt}$ and

$$\dot{\eta}=-\frac{1}{2}\left(1-k^2te^{-kt}\right)\eta^3$$

During the peaking period, the coefficient of η^3 is positive, causing $|\eta(t)|$ to grow. Eventually, the coefficient of η^3 will become negative, but that might not happen soon enough , since the system might have a finite escape time. Indeed the solution

$$\eta^2(t)=\frac{\eta_0^{\ 2}}{1+\eta_0^{\ 2}\left[t+(1+kt)e^{-kt}-1\right]}$$

shows that if $\eta_0^{\ 2}>1$, the system will have a finite escape time if k is chosen large enough.

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Remark 1.5: It is useful to know that the zero dynamics can be characterized in the original coordinates. Noting that

$$y(t)\equiv 0\Rightarrow \xi(t)\equiv 0\Rightarrow u(t)\equiv \alpha(x(t))$$

we see that if the output is identically zero, the solution of the state equation must be confined to the set

$$Z^*=\left\{x\in D_0 \middle| h(x)=L_fh(x)=\cdots=L_f^{\rho-1}h(x)=0\right\}$$

and the input must be

$$u=u^*(x)\overset{def}{=}\alpha(x)\big|_{x\in Z^*}$$

Example 1.7: The system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2+x_3^{\ 2}}{1+x_3^{\ 2}}u \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 x_3 + u \\
y &= x_2
\end{aligned}
$$

has an open-loop equilibrium point at the origin. The derivatives of the output are

$$
\begin{aligned}
\dot{y} &= \dot{x}_2 = x_3 \\
\ddot{y} &= \dot{x}_3 = x_1 x_3 + u
\end{aligned}
$$

Therefore, the system has relative degree two in $\Re^3$. Using $L_g L_f h(x) = 1$ and $L_f^2 h(x) = x_1 x_3$, we obtain

$\gamma = 1$ and $\alpha(x) = -x_1 x_3$

To characterize the zero dynamics, restrict to

$$
Z^* = \{x \in \Re \,|\, x_2 = x_3 = 0\}
$$

and take $u = u^*(x) = 0$. This process yields

$$
\dot{x}_1 = -x_1
$$

which shows that the system is minimum phase. To transform it into the normal form, we want to choose a function $\phi(x)$ such that

$$
\phi(0) = 0, \ \frac{\partial \phi}{\partial x} g(x) = 0
$$

and

$$
T(x) = \begin{bmatrix} \phi(x) & x_2 & x_3 \end{bmatrix}^T
$$

is a diffeomorphism on some domain containing the origin. The partial differential equation

$$
\frac{\partial \phi}{\partial x_1} \frac{2+x_3^{\ 2}}{1+x_3^{\ 2}} + \frac{\partial \phi}{\partial x_3} = 0
$$

can be solved by separating variables to obtain

$$\phi(x) = -x_1 + x_3 + \tan^{-1} x_3$$

which satisfies the condition $\phi(0) = 0$. The mapping $T(x)$ is a global diffeomorphism, as can be seen by the fact that for any $z \in \Re^3$, the equation $T(x) = z$ has a unique solution. Thus, the normal form

$$\begin{aligned}
\dot{\eta} &= \left(-\eta + \xi_2 + \tan^{-1} x_3\right)\left(1 + \frac{2 + \xi_2^{\,2}}{1 + \xi_2^{\,2}}\xi_2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \left(-\eta + \xi_2 + \tan^{-1} \xi_2\right)\xi_2 + u \\
y &= \xi_1
\end{aligned}$$

is defined globally.

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Remark 1.6: (Robustness) Feedback linearization is based on exact cancellation of nonlinearities, which is, in practice, not often practically possible. Most likely we have only approximations $\hat{\alpha}$, $\hat{\beta}$ and $\hat{T}(x)$ of the true α, β, and $T(x)$. The feedback control law has the form

$$u = \hat{\alpha} - \hat{\beta} K \hat{T}_2$$

and the closed loop system

$$\begin{aligned}
\dot{\eta} &= f(\eta, \xi) \\
\dot{\xi} &= A\xi + B\gamma(x)\left[(\hat{\alpha} - \alpha) - \hat{\beta} K \hat{T}_2\right]
\end{aligned}$$

Adding and subtracting $BK\xi$ to the $\dot{\xi}$ equation we obtain

$$\begin{aligned}
\dot{\eta} &= f(\eta, \xi) \\
\dot{\xi} &= (A - BK)\xi + B\delta(z)
\end{aligned}$$

where

$$\delta(z) = \gamma\left[(\hat{\alpha} - \alpha) - \left(\hat{\beta} - \beta\right) K T_2 - \hat{\beta} K \left(\hat{T}_2 - T_2\right)\right]$$

The local closed loop system differs from the nominal one by an additive perturbation. Thus, it is often assumed that in most cases no serious problems are to be expected.

Remark 1.7: In some cases it might be useful not to cancel all nonlinearities. For instance

$$\dot{x} = ax - bx^3 + u$$

Take $u = -\left(K + a\right)x$, $K > 0$ and we obtain

$$\dot{x} = -Kx - bx^3$$

which is asymptotically stable.