AS-767 – SIGNALS AND SYSTEMS

M1 – INTRODUCTION TO SIGNALS AND SYSTEMS

TOPICS

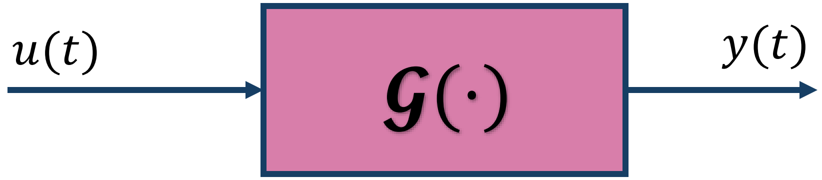
- Introduction to signals theory:
  - classification of continuous- and discrete-time signals;
  - unit impulse;
  - unit step;
  - transformation of the independent variable.
- Introduction to systems theory:
  - continuous- and discrete-time systems;
  - causality;
  - time invariance;
  - linearity;
  - convolution sum and integral;
  - system response to unit impulse and unit step.

NOTATION

- Signals are expressed with lowercase letters. An exception is the family of harmonics: $\phi(t)$. Some examples are: $x(t)$ is a general signal, $\mu(t)$ is the unit step signal, $\delta(t)$ is the unit impulse signal, $u(t)$ is the inpuand t signal, $y(t)$ is the output signal. Constants are also represented by lowercase letters. k and t are usually used to define the independent variables.
- The mathematical operator of the expectation of $(\cdot)$ is denoted by $\mathcal{E}(\cdot)$. Letter $\mathcal{G}(\cdot)$ represents a general system.
- Sets are written using blackboard bold letters. Some examples are $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ that means the sets of natural, integer, real, and complex numbers, respectively.
- Capital letter E means the total energy of a signal or a system, and P the medium potency of a signal or system. T is the period of a signal.

PRELIMINARIES

- Signals and systems in the context of engineering are quite wide.
- In this course, we will analyze both considering continuous- and discrete-time domains.
- Both theory and practical applications are considered.
- **Signals** are functions of independent variables that carry out "**information" or "data**" about some phenomena.
- **Systems** deliver an output signal as a result of a **transformation** applied to input signals.

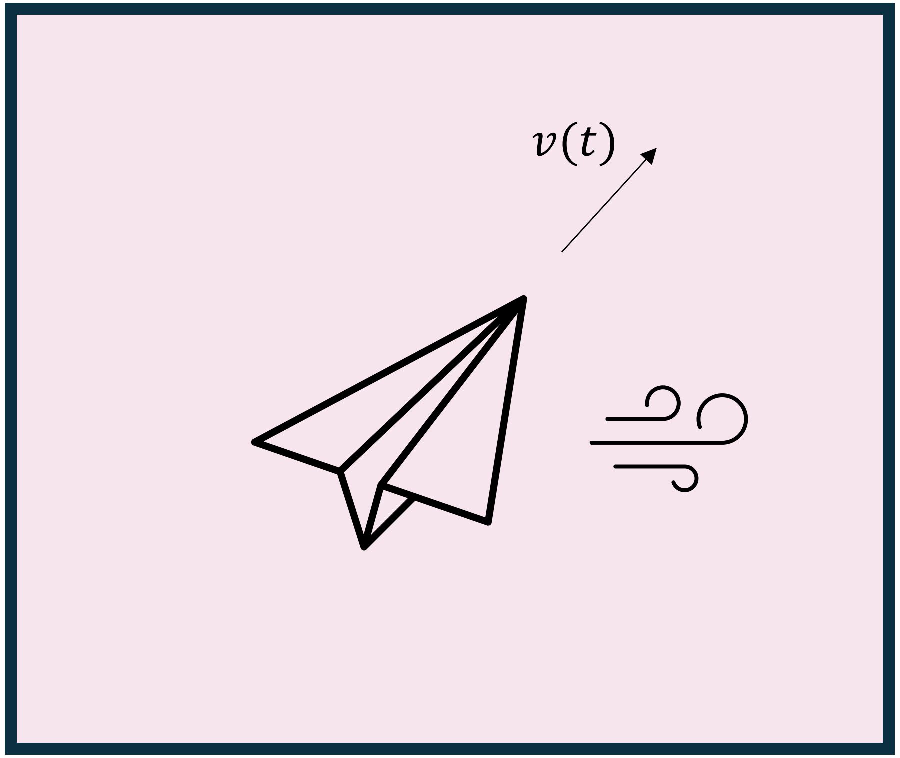


$u(t)$ and $y(t)$ are signals

$\mathcal{G}(\cdot)$ represents a system

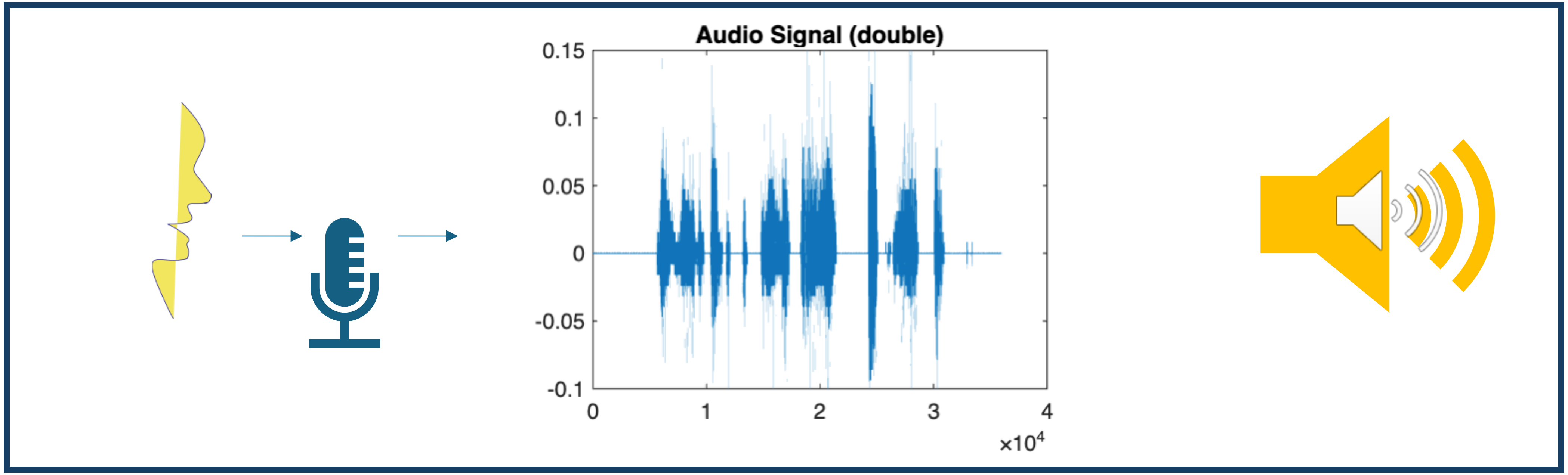
PRELIMINARIES

- Independent of the objective, we are interested in **characterizing systems**.
- It is important to know how systems **respond to different input signals**. An example is the behavior of an aircraft when subject to wind disturbance.
    - Characterization of the disturbance.
    - Understanding the behavior of the aircraft.



INTRODUCTION TO SIGNALS THEORY

- Signals are functions of one or more independent variables. They enclose **information** or **data**.



INTRODUCTION TO SIGNALS THEORY

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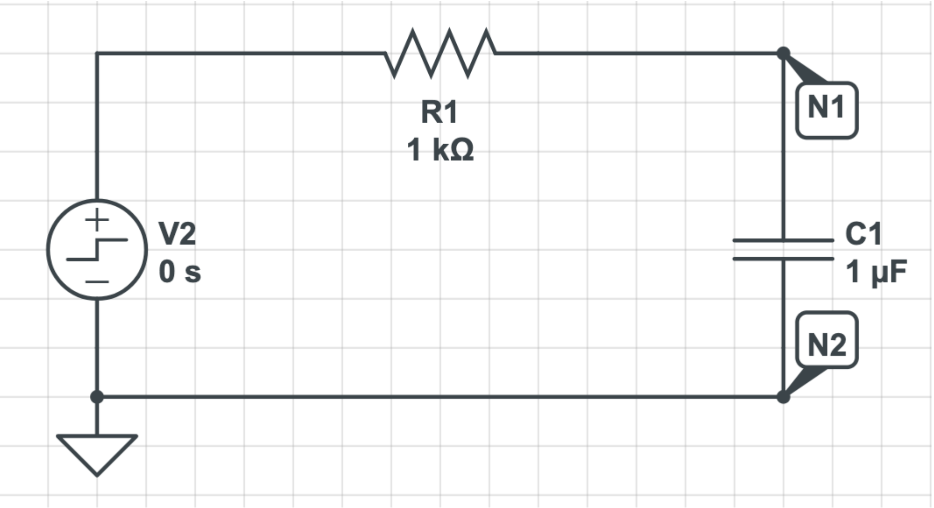
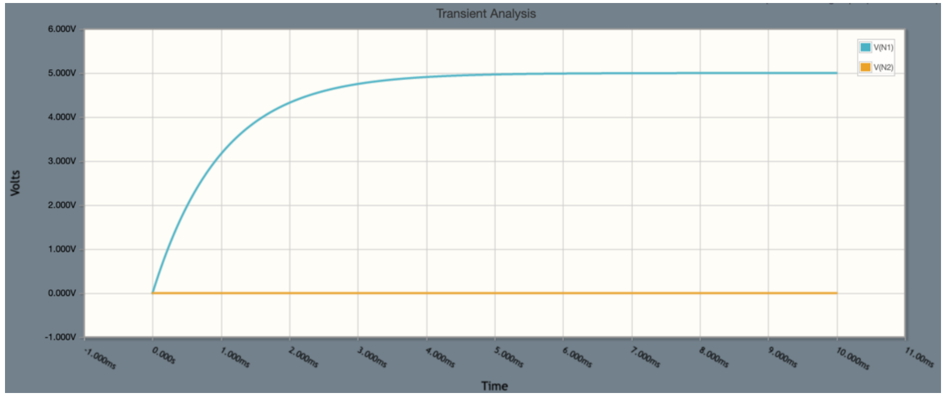




ENERGY AND POWER OF SIGNALS

- Signals refer to different amounts of phenomena. They are associated with power and energy.
- The energy of a signal can be computed as (continuous-time signal)

$$E \triangleq \lim_{t\to\infty} \int_{-t}^{t} |x(\tau)|^2 \, d\tau = \int_{-\infty}^{\infty} |x(\tau)|^2 d\tau$$

or (discrete-time signal)

$$E \triangleq \lim_{n\to\infty} \sum_{i=-n}^{n} |x[i]|^2 = \sum_{i=-\infty}^{\infty} |x[i]|^2$$

ENERGY AND POWER OF A SIGNAL

- Analogously, the power associated with a signal can be computed as (continuous-time signal):

$$P \triangleq \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |x(\tau)|^2 d\tau$$

or (discrete-time signal):

$$P \triangleq \lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} |x[i]|^2$$

ENERGY AND POWER OF A SIGNAL

- In this case, consider the example of a generic signal

$$x(t) = e^{-t}u(t)$$

for it, we have:

$$E = \int_0^{\infty}(e^{-t})^2dt = \frac{1}{2} \quad \text{and} \quad P = \lim_{t\to\infty}\frac{1}{2t}\int_0^{t}(e^{-t})^2dt = 0$$

- Analogously, consider the signal

$$x[n] = 2, \forall n \in \mathbb{Z}$$

hence, $E = \sum_{i=-\infty}^{\infty}(2)^2 = \infty$. However, $P = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n}(2)^2 = 4$

ENERGY AND POWER OF A SIGNAL

- Finally, the signal

$$x(t) = tu(t)$$

has energy and power calculated as

$$E = \int_0^{\infty} (t)^2 dt = \infty \quad \text{and} \quad P = \lim_{t\to\infty} \frac{1}{2t} \int_0^{t} (t)^2 dese = \infty$$

- All these quantities define measures of the intensity of each signal.

ENERGY AND POWER OF A SIGNAL

- These examples shows that signals can be classified as:

  - Finite energy
  - Finite power
  - Infinite energy and power

- The (Root Mean Square – RMS) of $x(t)$ equals the square root of P. This value is also known as the effective value of $x(t)$.

CLASSIFICATION OF SIGNALS

- In addition to classifying signals into finite energy or finite power, signals can be classified into:
  - Continuous-time signal and discrete-time signal (continuity over time)
  - Analogous and digital (continuity of their values)
  - Periodic and aperiodic (periodicity)
  - Energy and Power signals (energy and power)
  - Deterministic and stochastic (stochasticity)
  - Even and Odd (symmetry)

SIGNALS – CONTINUITY OVER TIME

- A signal $x(t)$ is said a **continuous-time signal** if

$$\exists\, x(t)\,, \forall t \in \mathbb{R}$$

- A signal $x[n]$ is said a **discrete-time signal** if

$$\begin{cases} \exists x[n]\,, \forall n \in \mathbb{Z} \\ \nexists x[n]\,, \forall\, n \in \mathbb{R} \backslash \mathbb{Z} \end{cases}$$

- Continuous-time and discrete-time signals can be converted one into another using the **Sampler** and **Holder** elements.

SIGNALS – CONTINUITY OF THEIR VALUES

- **Analogous signals** are those defined in a continuous range of values,

$$x(t) \in \mathbb{C} \; (x[n] \in \mathbb{C})$$

- Instead, **discrete (digital) signals** are defined only in a discrete range of values,

$$x(t) \in \mathbb{K} \subset \mathbb{C} \;\; (x[n] \in \mathbb{K} \subset \mathbb{C})$$

  where $\mathbb{K}$ is a countable set.

- Analogous signals can be converted into digital ones and vice-versa through **A/D** and **D/A converters**, respectively.

SIGNALS – PERIODICITY

- A signal is said to be **periodic** if there is a constant $t_0 \in \mathbb{R}_+^*$ ($k_0 \in \mathbb{Z}_+^*$), such that

$$x(t) = x(t + t_0), \forall t \in \mathbb{R} \text{ (continuous-time signal)} \qquad \text{(Eq. 1)}$$

$$x[n] = x[n + n_0], \forall n \in \mathbb{Z} \text{ (discrete-time signal)} \qquad \text{(Eq. 2)}$$

- The smallest value of $t_0 \in \mathbb{R}$ ($n_0 \in \mathbb{Z}$) that satisfies Eq. 1(2), is called **(fundamental) period** of $x(t)(x[n])$.
- A signal is said to be **aperiodic**, if it is not periodic.

SIGNALS – THEIR ENERGY AND POWER

- A signal is said an **energy signal** if

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \text{ (continuous-time signal)} \qquad \text{(Eq. 3)}$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \text{ (discrete-time signal)} \qquad \text{(Eq. 4)}$$

- And, it is said a **power signal** if

$$0 < P = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |x(\tau)|^2 d\tau < \infty \text{ (continuous-time signal)} \qquad \text{(Eq. 5)}$$

$$0 < P = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} |x[n]|^2 < \infty \text{ (discrete-time signal)} \qquad \text{(Eq. 6)}$$

- If P and E are not finite, the signal is neither an energy nor a power signal.

SIGNALS – STOCHACITY

- A **deterministic signal** is such that signals we can determine at any instant of time.

$$x(t)(x[n]) \text{ is known } \forall\, t \in \mathbb{R}(n \in \mathbb{Z})$$

- A **stochastic signal** is such that we only determine a belief on some characteristic in any instant of time

$$\mathcal{E}\big(x(t)\big) = \bar{x}(t) \text{ (First order moment)}$$

$$\mathcal{E}\left(\big(x(t) - \bar{x}(t)\big)^2\right) = P \text{ (Second order central moment)}$$

$\mathcal{E}(\cdot)$ is a mathematical operator that means the expectation of $(\cdot)$.

SIGNALS - SYMMETRY

- A signal is said to be **even** if

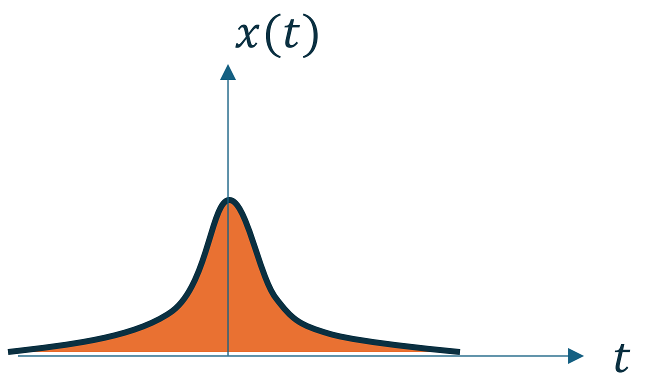
$$x(t) = x(-t), \forall t \in \mathbb{R}$$
$$x[n] = x[-n], \forall n \in \mathbb{Z}$$

- A signal is said to be **odd** if

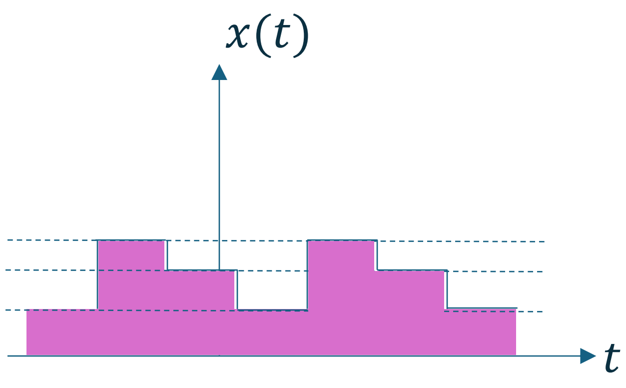
$$x(t) = -x(-t), \forall t \in \mathbb{R}$$
$$x[n] = -x[-n], \forall n \in \mathbb{Z}$$

- Any odd signal is such that $x(0) = 0(x[0] = 0)$.
- A signal can be even, odd, not even nor odd.

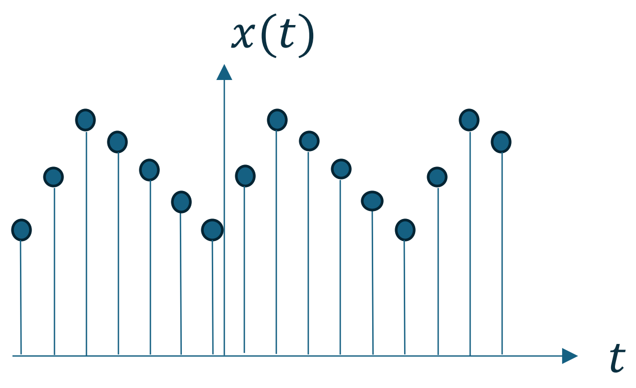
SIGNALS – EXAMPLES



Even
Deterministic
Energy
Continuous-time
Analog
Aperiodic



Not Even Nor Odd
Deterministic
Power
Continuous-time
Digital
Periodic



Not Even Nor Odd
Deterministic
Power
Discrete-time
Analog
Periodic

TRANSFORMATION OF VARIABLES

- For signals, it is important to determine some transformations regarding the independent variable, such as:
  - Translation in time
  - Reflection in time
  - Change of scale
- **Translation** in time is the displacement in time of a value of b, leading to another signal in the form of

$$y(t) = x(t - b)$$

TRANSFORMATION OF VARIABLES

- The **reflection** in time of a signal $x(t)$ is another signal defined as

$$y(t) = x(-t)$$

- A **change of scale** in time is a signal such that

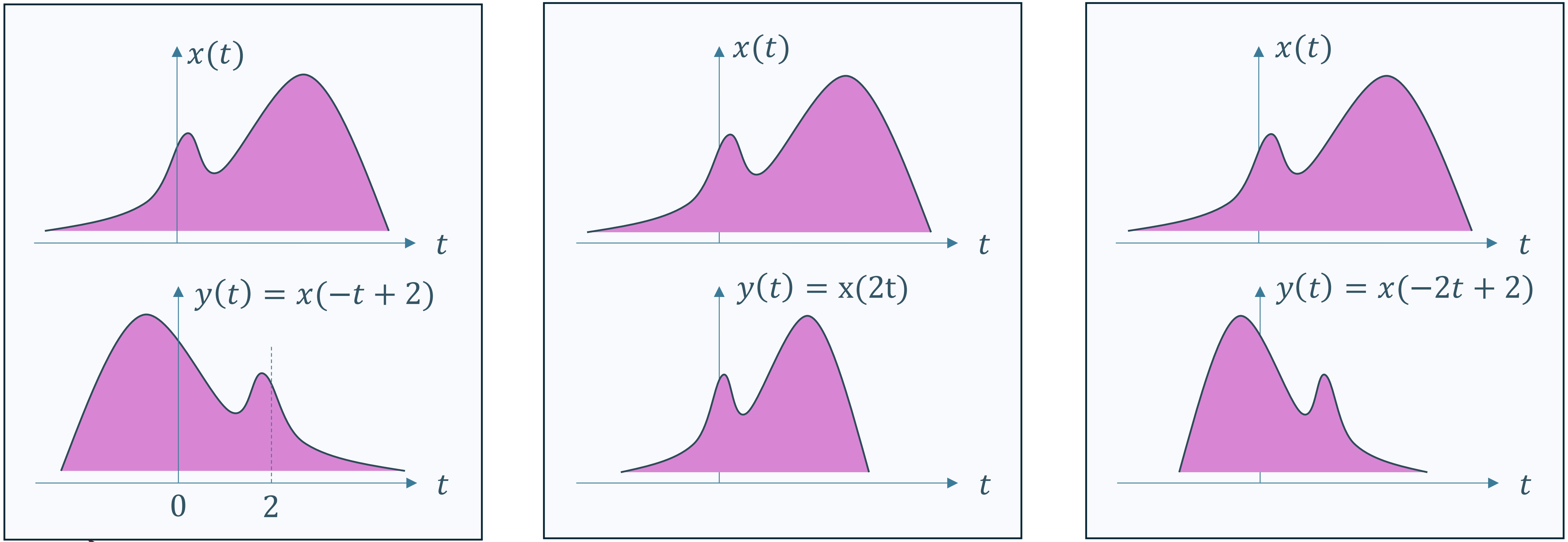
$$y(t) = x(a \cdot t)$$

- In this way, we can define any transformation in the form of

$$y(t) = x(at + b)$$

TRANSFORMATION OF VARIABLES

- Some examples are shown below.



SIGNALS – SYMMETRY

- **Any signal** can be decomposed into the sum of an even signal and an odd signal. Indeed,

$$x(t) = x_e(t) + x_o(t) = \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)]$$

- It is easily seen that

$$x_e(-t) = \frac{1}{2}[x(-t) + x(t)] = \mathrm{x}_e(t) \text{ (EVEN signal)}$$

$$x_o(-t) = \frac{1}{2}[x(-t) - x(t)] = -\mathrm{x}_o(t) \text{ (ODD signal)}$$

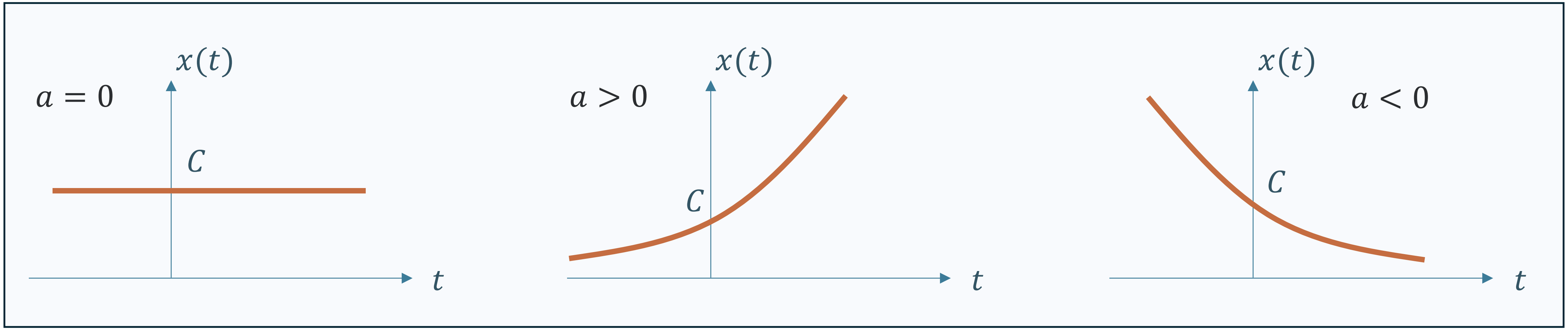
- The same property can be verified for a discrete-time signal.

CONTINUOUS-TIME EXPONENTIAL SIGNAL

- Exponential signals are defined as

$$x(t) = Ce^{at}, C \in \mathbb{C}, a \in \mathbb{C}$$

- Exponential with $\boldsymbol{a = r \in \mathbb{R}}$ **and** $\boldsymbol{C \in \mathbb{R}}$.



CONTINUOUS-TIME EXPONENTIAL SIGNAL

- The exponential with $\boldsymbol{C} = \mathbf{1}$ and $\boldsymbol{a} = \boldsymbol{j\omega_0}, \boldsymbol{\omega_0} \in \mathbb{R}$, is not a real signal but can be used to understand other real signals.
- The signal $x(t) = e^{j\omega_0 t}$ is periodic, and hence

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} \Longrightarrow e^{j\omega_0 T} = 1$$

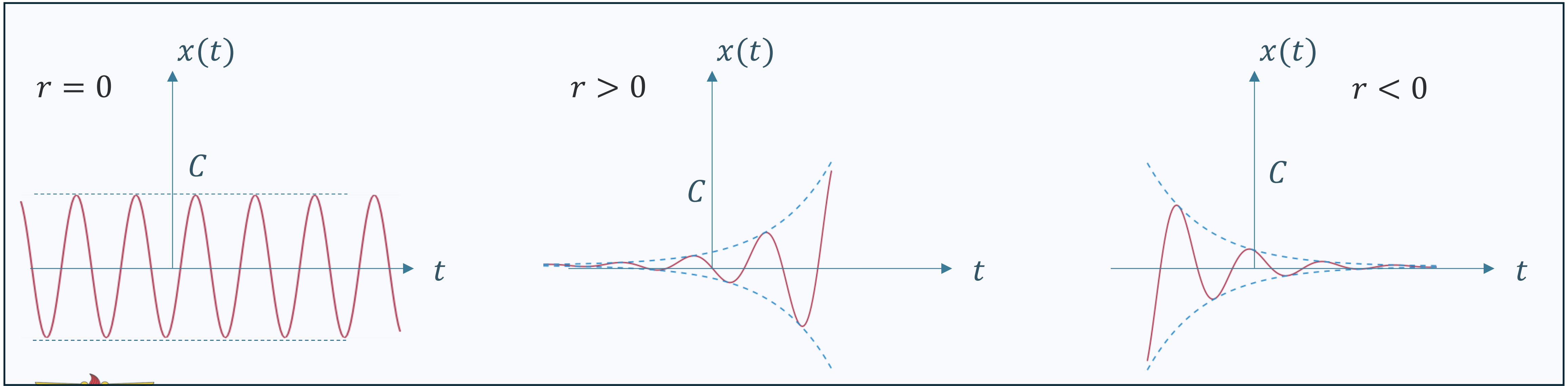
what happens for $\omega_0 T = 2\pi k$, with $k = 0, \pm 1, \pm 2, \cdots$

- The smallest $T = T_0$ that satisfies $\omega_0 T = 2\pi k$ is called **Fundamental Period**, $T_0 = \frac{2\pi}{|\omega_0|}$.

CONTINUOUS-TIME EXPONENTIAL SIGNAL

- Finally, for $a = r + j\omega_0$ and $C = |C|e^{j\theta}$, we have $x(t) = |C|\mathrm{e}^{j\theta}e^{(r+j\omega_0)t}$. That is,

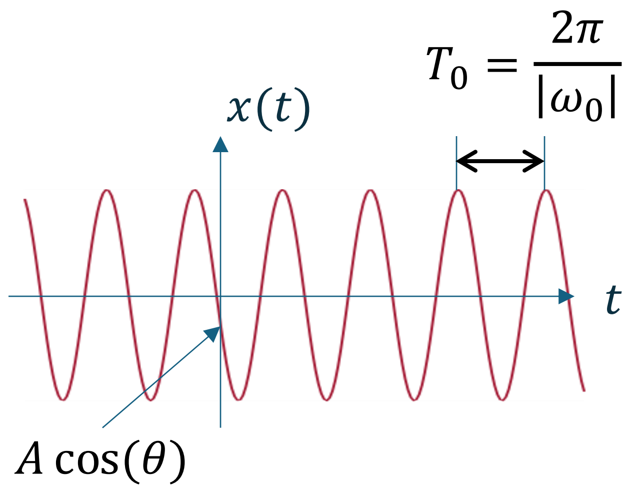
$$x(t) = |C|e^{rt}e^{j(\omega_0 t+\theta)} = |C|e^{rt}(\cos(\omega_0 t + \theta) + j\sin(\omega_0 t + \theta))$$



CONTINUOUS-TIME SINUSOIDAL SIGNAL

- Analogous to the exponential function, the sinusoidal signal is periodic with fundamental frequency $|\omega_0| = 2\pi f_0 = \frac{2\pi}{T_0}$.

$$x(t) = A\cos(\omega_0 t + \theta)$$



- Exponential and Sinusoidal signals are **power signals**

$$\boldsymbol{E = \int_{-\infty}^{\infty} \left|e^{j\omega_0 t}\right|^2 dt = \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)T} \left|e^{j\omega_0 t}\right|^2 dt}$$

$$\boldsymbol{= \sum_{k=-\infty}^{\infty} T_0 = \infty \Rightarrow P = \frac{1}{T_0} \int_{kT}^{(k+1)T} \left|e^{j\omega_0 t}\right|^2 dt = 1}$$

EXPONENTIAL AND SINUSOIDAL CT SIGNALS

- We define a family of harmonic signals such as
$$\phi_k(t) = e^{j|k|\omega_0 t}$$
  In this case, each fundamental period is $\frac{T_0}{|k|} = \frac{2\pi}{|k|\omega_0}$.
- Multiplying the frequency of the signal by an integer value, we compress the signal. The opposite effect is seen when we divide the frequency by an integer value.
- That means that each signal with fundamental period $\frac{T_0}{|k|}$ is also asignal with period T_0.

DISCRETE-TIME EXPONENTIAL (SIGNAL)

- The **discrete-time exponential signal** is defined as

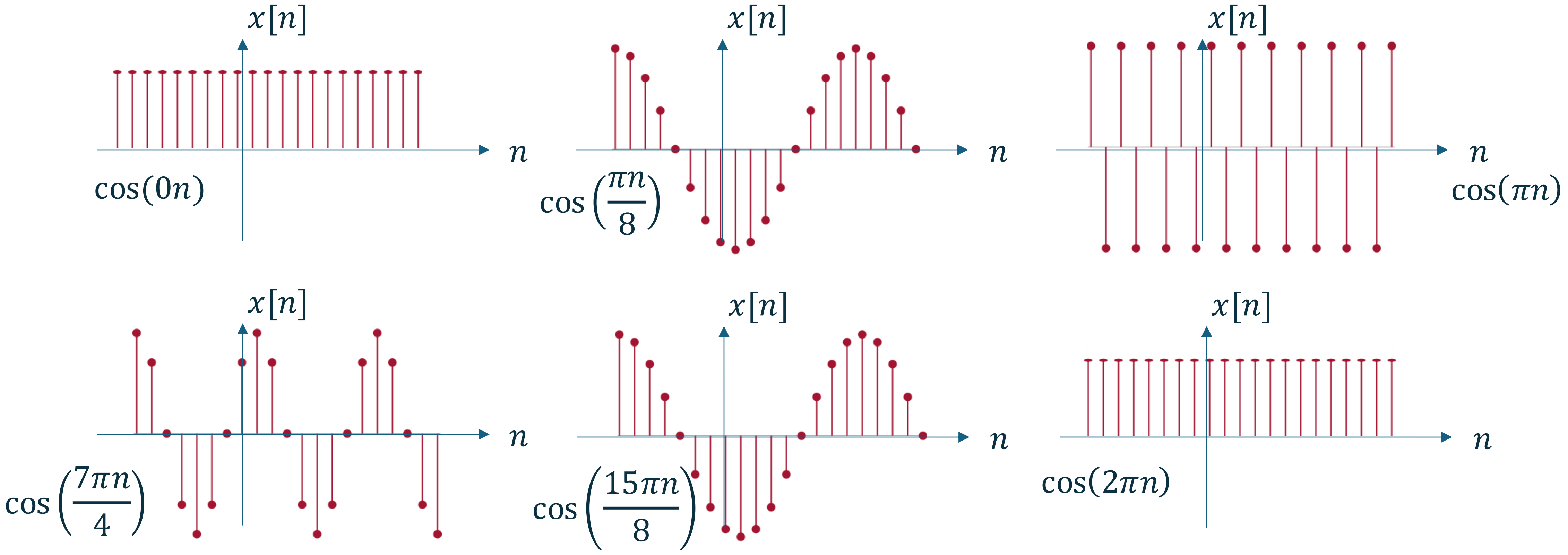
$$x[n] = e^{j\omega_0 n} = e^{j(\omega_0+2\pi)n}, \forall k \in \mathbb{Z}$$

- The oscillation rate does not increase with ω_0. Indeed, it increases with ω_0, for $0 < \omega_0 < \pi$, and decreases with ω_0, for $\pi < \omega_0 < 2\pi$.
- The period of a discrete-time exponential signal is such that

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \implies \omega_0 N = 2\pi m, \text{ for } m \in \mathbb{Z}$$

Hence, **$x[n]$ is periodic if $\frac{\omega_0}{2\pi}$** is rational. The **fundamental period** is calculated as $N = m\left(\frac{2\pi}{\omega_0}\right)$.

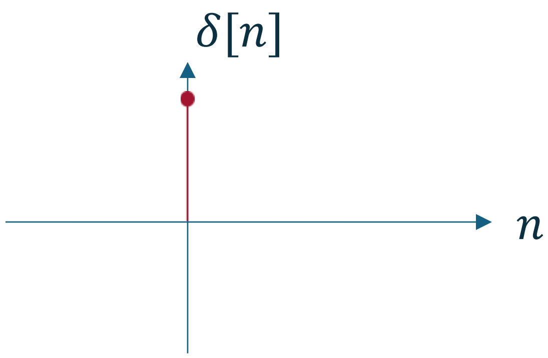
DISCRETE-TIME EXPONENTIAL SIGNAL



DISCRETE-TIME UNIT IMPULSE AND STEP

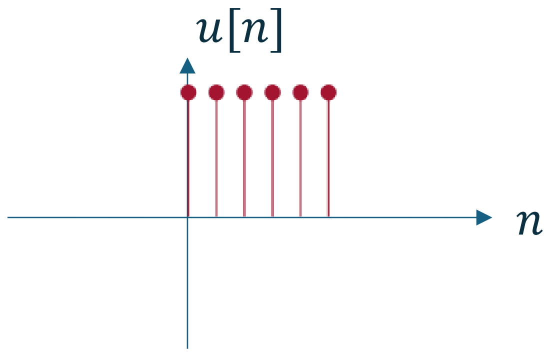
- The **discrete-time unit impulse** is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



- The **discrete-time unit step** is defined as

$$\mu[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



DISCRETE-TIME UNIT IMPULSE AND STEP

- They are related through

$$\delta[n] = \mu[n] - \mu[n-1]$$

$$\mu[n] = \sum_{i=-\infty}^{n} \delta[i] \underset{\boxed{k=n-i}}{=} \sum_{k=0}^{\infty} \delta[n-k]$$

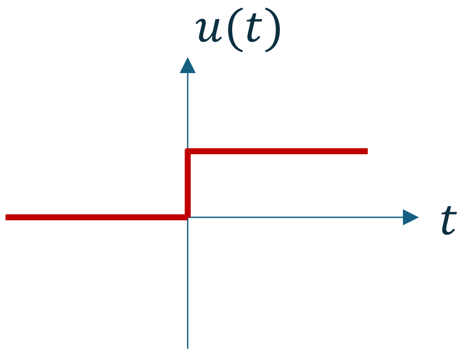
cumulative sum or **sum of impulses**.

- An important property of the unitary impulses is the **selective property**

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

CONTINUOUS-TIME UNIT IMPULSE AND STEP

- The **continuous-time unit step** is defined as

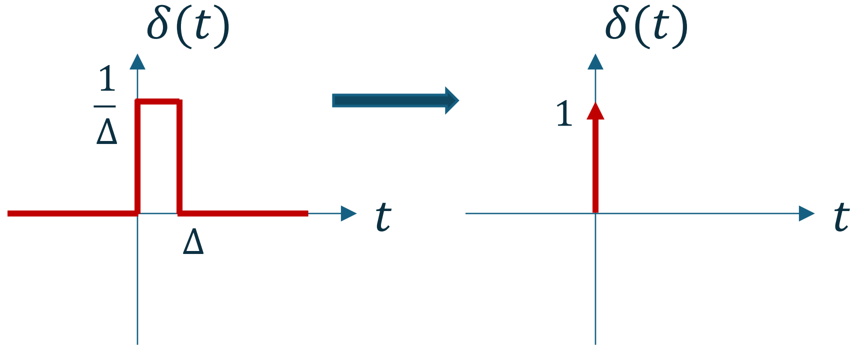


$$\mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

which is discontinuous in $t = 0$.

- The continuous-time unit impulse is defined as a short pulse of infinitesimal duration Δ

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



CONTINUOUS-TIME UNIT IMPULSE AND STEP

- Analogously to the discrete-time case, the CT unit step and CT unit impulse also are related to each other through

$$\mu(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty} \delta(t-\sigma)d\sigma \quad \text{or} \quad \delta(t) = \frac{d\mu(t)}{dt}$$

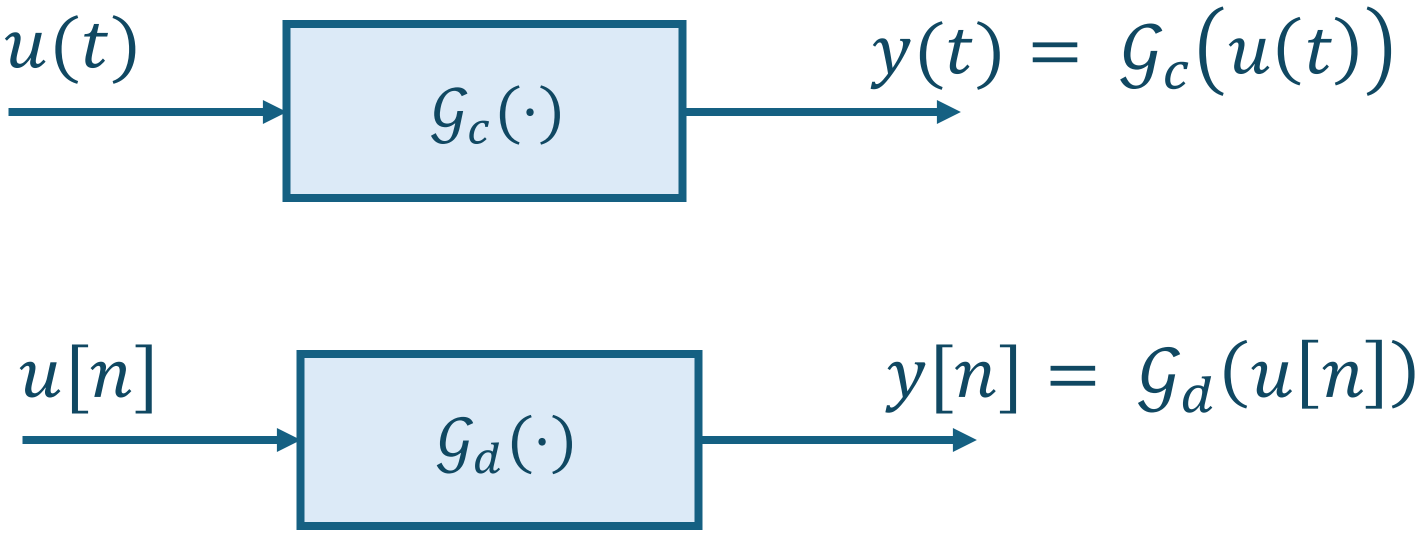
$$\boxed{\sigma = t - \tau}$$

- The **seletive proporty** for continuous-time unit impulse is that

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

INTRODUCTION TO SYSTEMS THEORY

- Systems are processes that modify input signals and respond with some behavior that can be foreseen in their output signals.



INTRODUCTION TO SYSTEMS THEORY

- The symbol $\mathcal{G}(\cdot)$ represents the transformation of the input signal implemented by the process. Generally speaking, it usually is represented by an ordinary differential equation (ODE), for continuous-time systems, or difference equations, for discrete-time systems. This characteristic reflects the **dynamics (memory)** associated with the process of interest. Otherwise, the system is said to be **algebraic (memoryless)**. **(store past information)**

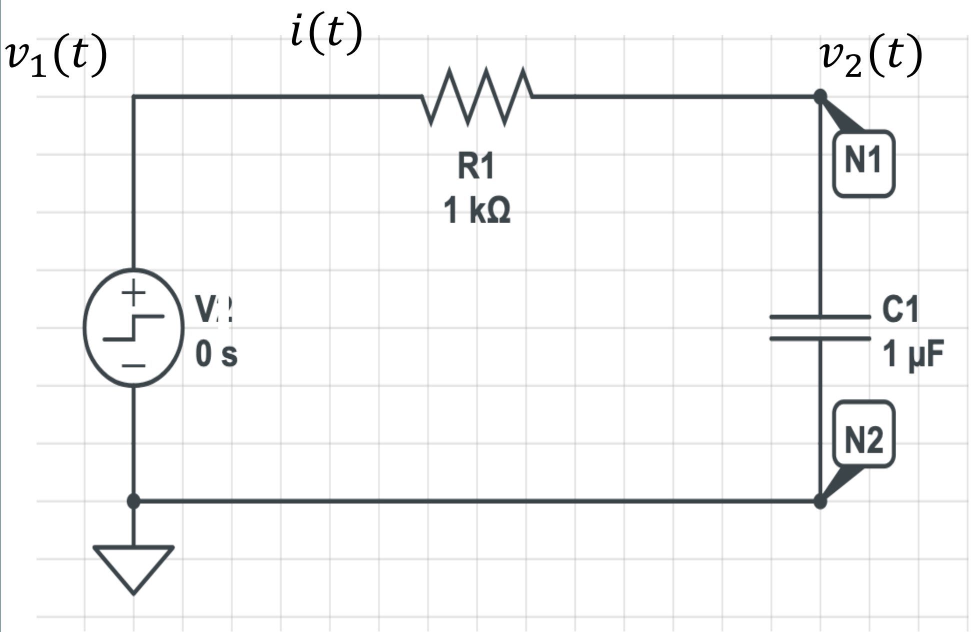
$$
\begin{aligned}
&a_s y[n-s] + a_{s-1} y[n-s+1] + \cdots + a_1 y[n-1] + a_0 y[n] \\
&= b_p u[n-p] + b_{p-1} u[n-p+1] + \cdots + b_1 u[n-1] + b_0 u[n]
\end{aligned}
$$

(Difference Equation)

$$
a_s y^{(s)} + a_{s-1} y^{(s-1)} + \cdots + a_1 y^{(1)} + a_0 y^{(0)} = b_p u^{(p)} + b_{p-1} u^{(p-1)} + \cdots + b_1 u^{(1)} + b_0 u^{(0)}
$$

(ODE)

- Example: RC circuit (continuous-time system)



$$i(t) = \frac{v_1(t) - v_2(t)}{R_1} \text{ and } i(t) = C_1 \frac{dv_2(t)}{dt}$$

$$\frac{di(t)}{dt} + \frac{1}{R_1 C_1} i(t) = \frac{1}{R_1} \frac{dv_1(t)}{dt}$$

Substituting $v_1(t)$ by $u(t)$ and $i(t)$ by $y(t)$ it follows that

$$\dot{y}(t) + \frac{1}{R_1 C_1} y(t) = \frac{1}{R_1} \dot{u}(t)$$

INTRODUCTION TO SYSTEMS THEORY

- Example: Bank saving (discrete-time system)

Month	Interest	Contribution	Value
0	1%	1.000,00	1.000,00
1	1%	0,00	1.010,00
…	1%	…	…

Initial value: $y[0]$

Value in Month 1: $y[1] = 1{,}01y[0] + x[1]$

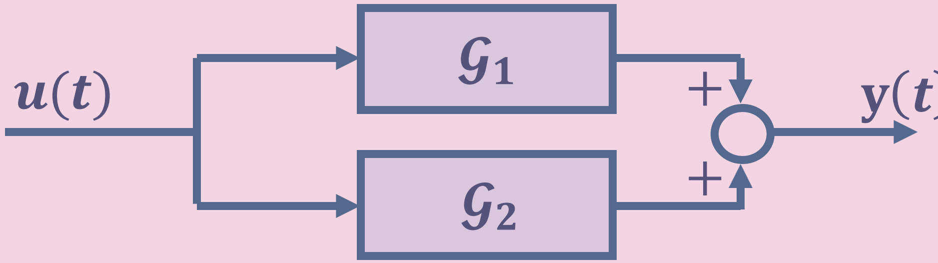
Value in Month 2: $y[2] = 1{,}01y[1] + x[2]$

It is immediate that

$$y[n] - 1{,}01y[n-1] = x[n]$$

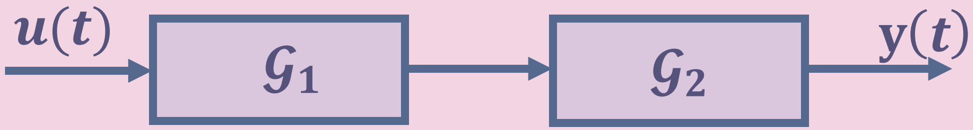
INTRODUCTION TO SYSTEMS THEORY

- Systems can be connected to other systems through **block diagram álgebra** (for continuous- and discrete-time systems):
  - Parallel interconnection



$$y(t) = \mathcal{G}_1(u(t)) + \mathcal{G}_2(u(t))$$

  - Series (cascade) interconnection



$$y(t) = \mathcal{G}_1(u(t)) * \mathcal{G}_2(u(t))$$

INTRODUCTION TO SYSTEMS THEORY

- There is a wide variety of systems, but particularly causal, linear, time-invariant systems play an important class of control systems.
- **Main properties** of systems
  - Causality
  - Time-invariance
  - Linearity
  - Stability

SYSTEM PROPERTY: CAUSALITY

- A system is **causal** if its output $y(t)(y[n])$ depends on only past and present values of the input $u(t)(u[n])$.
  - Examples of causal systems are

$$y[n] + y[n-1] = x[n] + x[n-1]$$

$$y[n] = x[-n]$$

  - Examples of non-causal systems are

$$y[n] = \frac{1}{2M+1} \sum_{i=-M}^{M} x[n-i]$$

SYSTEM PROPERTY: TIME-INVARIANCE

- A system is **time-invariant** if its characteristics and properties do not change with time.

A system $\mathcal{G}_c(\cdot)\big(\mathcal{G}_d(\cdot)\big)$ is time-invariant with output $y(t) = \mathcal{G}_c\big(u(t)\big)$ $\big(y[n] = \mathcal{G}_d(u[n])\big)$ to the input $u(t)(u[n])$, then

$$y(t+t_1) = \mathcal{G}_c\big(u(t+t_1)\big), \forall t, t_1 \in \mathbb{R} \quad \text{(Continuous-time system)}$$

$$y[n+n_1] = \mathcal{G}_c(u[n+n_1]), \forall n, n_1 \in \mathbb{Z} \quad \text{(Discrete-time system)}$$

SYSTEM PROPERTY: LINEARITY

- A system is **linear** if it satisfies the principle of superposition (additivity and homogeneity).

A system $\mathcal{G}_c(\cdot)\big(\mathcal{G}_d(\cdot)\big)$ is linear with outputs $y_1(t) = \mathcal{G}_c\big(u_1(t)\big)\big(y_1[n] = \mathcal{G}_d(u_1[n])\big)$ and $y_2(t) = \mathcal{G}_c\big(u_2(t)\big)\big(y_2[n] = \mathcal{G}_d(u_2[n])\big)$, then

$$y(t) = \mathcal{G}_c\big(au_1(t) + bu_2(t)\big) = ay_1(t) + by_2(t)$$ (Continuous-time system)

$$y[n] = \mathcal{G}_d(au_1[n] + bu_2[n]) = ay_1[n] + by_2[n]$$ (Discrete-time system)

SYSTEM PROPERTY: STABILITY

- A system is **stable** when bounded input produces bounded output. (BIBO stability)
- Examples:
  - $h(t) = e^{u(t)}$ is an stable system
  - $h(t) = \ln\big(u(t)\big)$ is unstable

CONVOLUTION SUM AND INTEGRAL

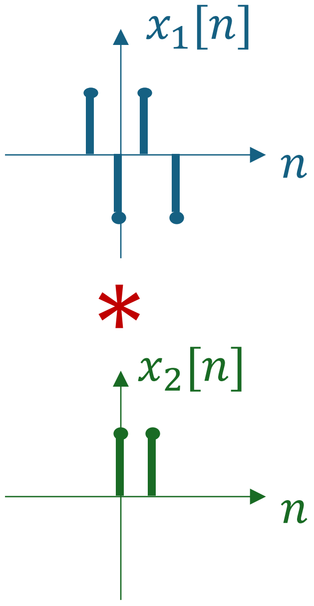
- The convolution sum is defined as

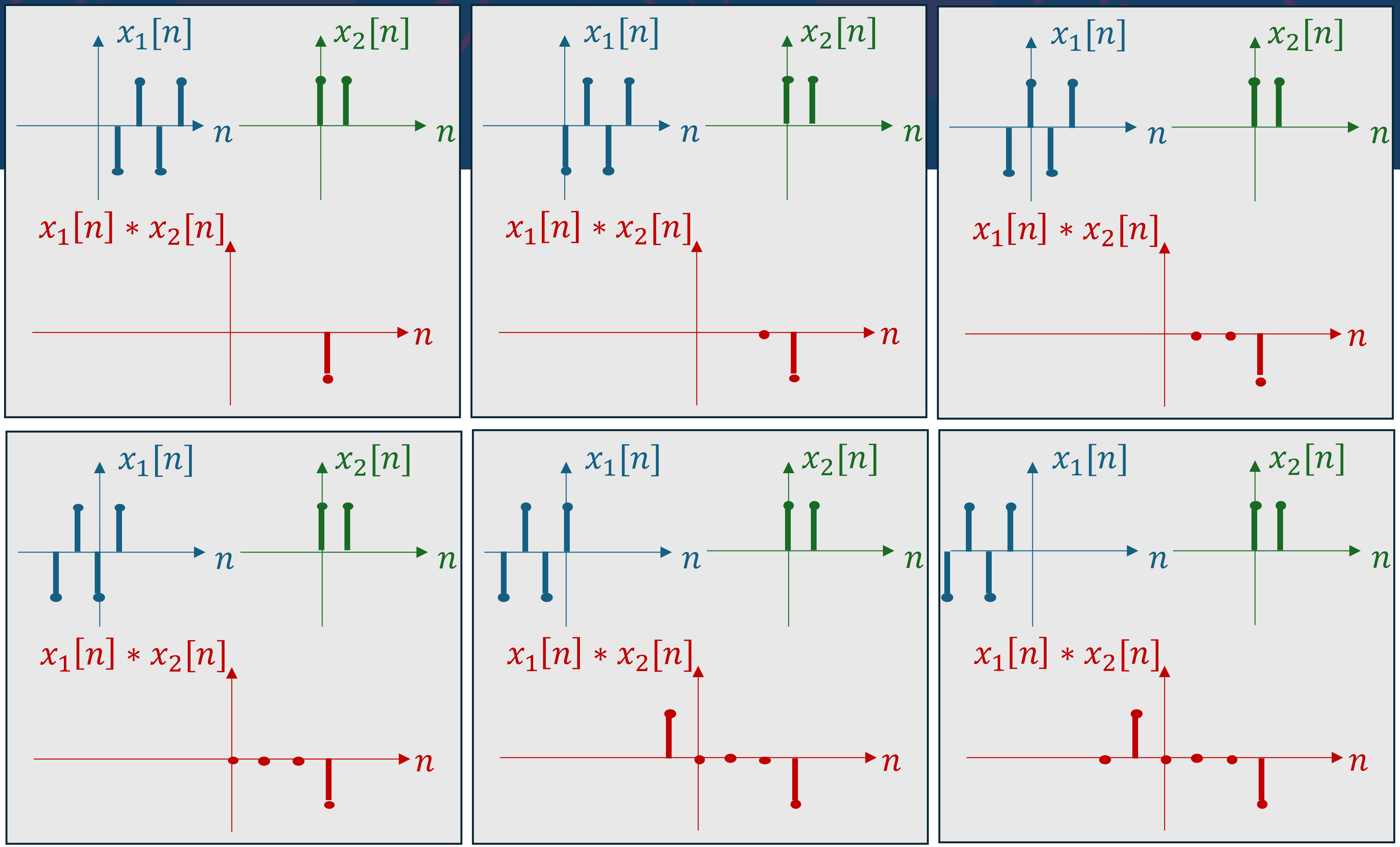
$$x_1[n] * x_2[n] = \sum_{i=-\infty}^{\infty} x_1[n-i]x_2[i]$$

- Analogously, the convolution integral is defined as

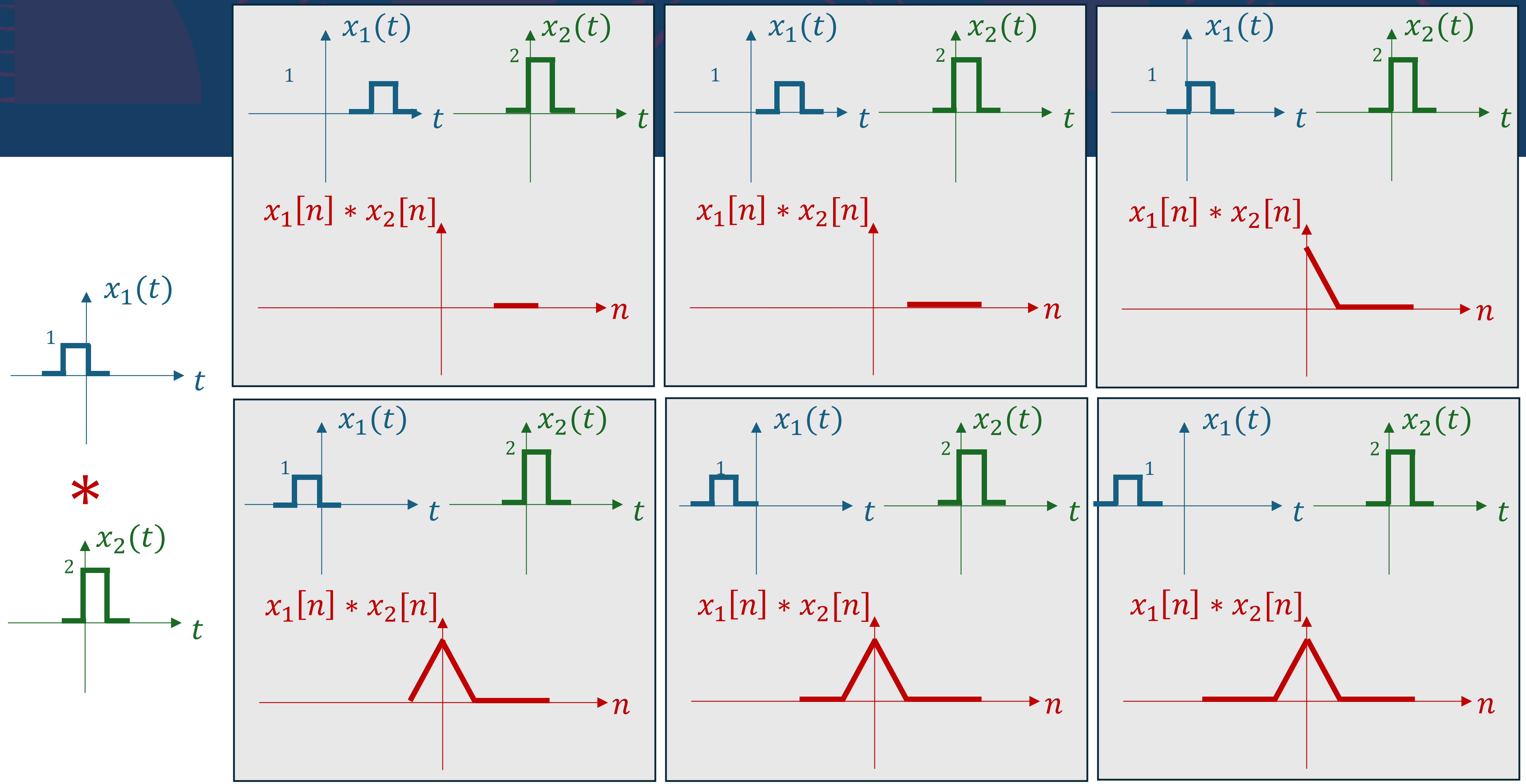
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau$$

CONVOLUTION SUM





CONVOLUTION INTEGRAL



UNIT IMPULSE: SYSTEM RESPONSE

- In this context, any signal can be represented by the **selective property** of the unit impulse, that is,

$$x[n] = x[n] * \delta[n] = \sum_{i=-\infty}^{\infty} x[i]\delta[n-i], \forall n \in \mathbb{Z}$$

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau, \forall t \in \mathbb{R}$$

UNIT IMPULSE: SYSTEM RESPONSE

- Considering previous properties, we can write the response of a discrete-time system to an input $u[n]$ as

$$y[n] = \mathcal{G}_d(u[n]) = \mathcal{G}_d\left(\sum_{i=-\infty}^{\infty} u[i]\delta[n-i]\right)$$

if the system is linear, and hence it is true the superposition principle, then

$$y[n] = \sum_{i=-\infty}^{\infty} \mathcal{G}_d(u[i]\delta[n-i]) = \sum_{i=-\infty}^{\infty} u[i]\mathcal{G}_d(\delta[n-i]) = \sum_{i=-\infty}^{\infty} u[i]h_d[n-i]$$

UNIT IMPULSE: SYSTEM RESPONSE

- Similarly, for the continuous-time system

$$y(t) = \mathcal{G}_c\big(u(t)\big) = \mathcal{G}_c\left(\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau\right)$$

Considering a linear system,

$$y(t) = \int_{-\infty}^{\infty} \mathcal{G}_c\big(u(\tau)\delta(t-\tau)\big)d\tau = \int_{-\infty}^{\infty} u(\tau)\mathcal{G}_c\big(\delta(t-\tau)\big)d\tau = \int_{-\infty}^{\infty} u(\tau)h\_c(t-\tau)d\tau$$

where has been assumed the fact that $u(\tau)$ is the signal $u(t)$ evaluated at $t=\tau$.

UNIT IMPULSE: SYSTEM RESPONSE

- Linear systems are totally characterized by its response to the unit impulse.
- Example: Consider a linear system described by $\ddot{y}(t) + \dot{y}(t) = u(t)$, evolving from $y(0) = 0, \dot{y}(0) = 1$, with $t \in \mathbb{R}_+$. Determine the response of this system to the unit impulse.

UNIT IMPULSE: SYSTEM RESPONSE

- Considering the method of undetermined coefficient:

$$h(t) = y_h(t) + y_p(t) \implies h(t) = 1 - e^{-t}$$

- The **homogeneous solution** is

$$\ddot{y}_h(t) + \dot{y}_h(t) = 0$$

$$y_h(t) = a + be^{-t}$$

- And the **particular solution** is

$$\ddot{y}_h(t) + \dot{y}_h(t) = \delta(t) \implies y_p(t) = 0$$

- Applying the initial conditions follows that

$$y(t) = 1 - e^{-t}$$

UNIT STEP: SYSTEM RESPONSE

- In general, since the impulsive signal is of difficult practical implementation, the response to the unit step is preferred.

- The response of a system to a signal $\mu(t)$ can be written as

$$y(t) = \mu(t) * h(t) \quad (y[n] = \mu[n] * h[n])$$

- Consider a linear system described by $\dot{y}(t) + ay(t) = u(t)$, evolving from $y(0) = 1$, in the time interval $t \in \mathbb{R}_+$. Determine the response of this system to the unit step.

UNIT STEP: SYSTEM RESPONSE

- Two scenarios are analyzed:
  - Response to the unit impulse

$$\dot{h}(t) + ah(t) = \delta(t)$$

and

$$y(t) = \mu(t) * h(t)$$

  - Response to the unit step

$$\dot{y}(t) + ay(t) = \mu(t)$$

(Undetermined Coefficient Method)

$$h(t) = y_h(t) + y_p(t) \implies \boxed{h(t) = e^{-at}}$$

Since,

- The **homogeneous solution** is

$$\dot{y}_h(t) + ay_h(t) = 0$$
$$\frac{d}{dt}\left(y_h(t)e^{at}\right) = 0$$
$$y_h(t) = \mathrm{e}^{-at}y_h(0) \implies y_h(t) = e^{-at}$$

- And the **particular solution** is

$$\dot{y}_p(t) + ay_p(t) = \delta(t) \implies y_p(t) = 0$$

UNIT STEP: SYSTEM RESPONSE

- Two scenarios are analyzed:
  - Response to the unit impulse

$$\dot{h}(t) + ah(t) = \delta(t)$$

and

$$y(t) = \mu(t) * h(t)$$

  - Response to the unit step

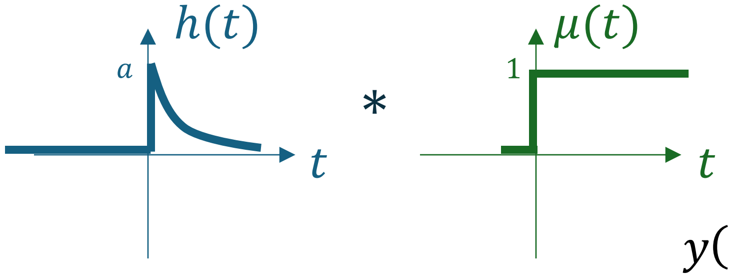
$$\dot{y}(t) + ay(t) = \mu(t)$$

(Undetermined Coefficient Method)

The solution to the unit step is then

$$\boxed{y(t) = \frac{1}{a} - \frac{1}{a}e^{-at}}$$

since, $y(t) = y_\delta(t) * \mu(t)$ is the **accumulation of $y_\delta(t)$ at each instant of time** and



$$y(t) = \int_0^t e^{-a\tau} d\tau$$

UNIT STEP: SYSTEM RESPONSE

- Two scenarios are analyzed:
  - Response to the unit impulse

$$\dot{y}(t) + ay(t) = \delta(t)$$

and

$$y(t) = \mu(t) * h(t)$$

  - Response to the unit step

$$\dot{y}(t) + ay(t) = \mu(t)$$

(Undetermined Coefficient Method)

$$y(t) = y_h(t) + y_p(t) \implies \boxed{y(t) = \frac{1}{a} - \frac{1}{a}e^{-at}}$$

Since,

- The **homogeneous solution** is

$$y_h(t) = e^{-at}$$

- And the **particular solution** is

$$\dot{y}_p(t) + ay_p(t) = \mu(t) \implies y_p(t) = c = \frac{1}{a}$$

EXERCISES

1. Calculate the energy, E, and the power, P, associated with the signals $x(t) = e^{-2t}u(t)$; $x[n] = \left(\frac{1}{2}\right)^n u[n]$; $x[n] = \cos\left(\frac{\pi}{4}n\right)$; and $x(t) = \sum_{k=-\infty}^{\infty} t[u(t+2k+1) - u(t+2k-1)]$. Classify each of them according to energy and power.
2. Consider the discrete-time signal

$$x[n] = 1 - \sum_{i=2}^{\infty} \delta[n-i]$$

Determine the values of a and b, such that $x[n] = u[an - b]$.

EXERCISES

3. Show that: if $x[n]$ is odd, then $\sum_{n=-\infty}^{\infty} x[n] = 0$.
4. Suppose that $x[n]$ is even and $y[n]$ is odd. Then, $x[n]y[n]$ is odd.
5. Suppose $x[n] = x_e[n] + x_o[n]$, with $x_e[n]$ even and $x_o[n]$, odd. Show that

$$\sum_{n=-\infty}^{\infty} (x[n])^2 = \sum_{n=-\infty}^{\infty} (x_e[n])^2 + \sum_{n=-\infty}^{\infty} (x_o[n])^2$$

EXERCISES

6. Suppose $u[n]$ is the input and $y[n]$, the output of a system, such that

$$y[n] = u[n] - (-1)^{n-1}u[n-1]$$

   Verify if this system is linear, time-varying, causal, and stable.

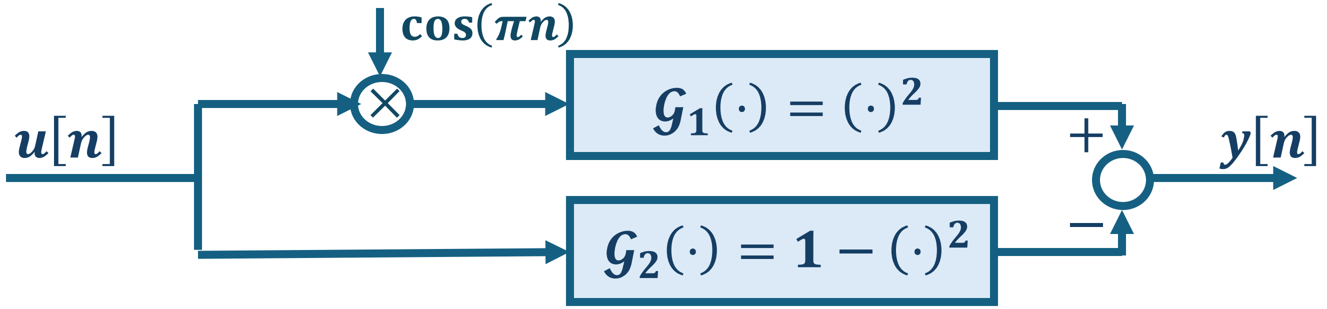
7. Consider a CT time-invariant system with input $u(t)$ and output $y(t)$. Show that, if $u(t)$ is T-periodic, then $y(t)$ is also T-periodic.

8. Repeat the exercise 7, considering DT time-invariant system, with T-periodic input, $u[n]$, and T-periodic output, $y[n]$.

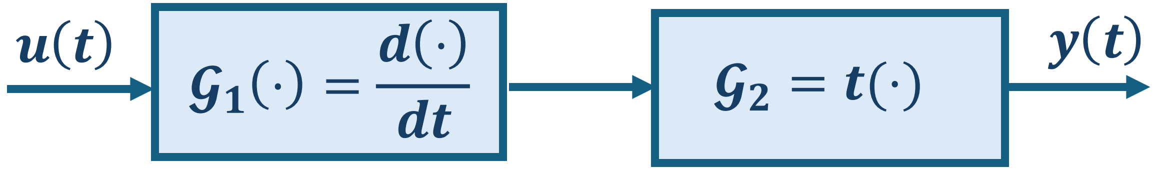
EXERCISES

9. Define the equation that describes the diagrams below. Verify the properties of linearity, time-invariance, causality, and stability for each case.

   a.



   b.



EXERCISES

10. Evaluate the convolution sum or integral, $u[n] * h[n]$ or $u(t) * h(t)$, for each case
    a) $u[n] = 2^n; h[n] = \mu[n] - \mu[n-10]$
    b) $u(t) = e^{-t}; \mathrm{h(t)} = \mu(t)$
11. Determine the solution of the following systems to the unit impulse and to the unit step.
    a) $\dot{y}(t) + y(t) = e^{-t},\ y(0) = 0$
    b) $\dot{y}(t) + y(t) = e^{-t},\ y(0) = 1$
    c) $y[n+1] + y[n] = \mu[n],\ y[0] = 0$
    d) $y[n+1] = \mu[n],\ y[0] = 1$

EXERCISES

12. Determine the output signal of the system

$$\ddot{y}(t) + y(t) = u(t), y(0) = \dot{y}(0) = 0$$

to the input $u(t) = \sin(t)$ in the time interval $t \in [0, +\infty]$.