AS-767 – SIGNALS AND SYSTEMS

M2 – ODE, FDE, FOURIER SERIES, AND TRANSFORM

TOPICS

- Solution to Differential Equations and Finite Difference Equations
- Fourier Series
- Continuous-Time Fourier Transform:
  - Definition
  - Inverse of Fourier Transform
  - Properties
- Discrete-Time Fourier Transform:
  - Definition
  - Inverse of Fourier Transform
  - Properties

NOTATION

- Signals are expressed with lowercase letters. The exception is the family of harmonics: $\phi(t)$. Some examples are: $x(t)$ is a general signal, $\mu(t)$ is the unit step signal, $\delta(t)$ is the unit impulse signal, $u(t)$ is the input signal, and $y(t)$ is the output signal. Constants are also represented by lowercase letters. k and t are usually used to define the independent variables.
- The mathematical operator of the expectation of $(\cdot)$ is denoted by $\mathcal{E}(\cdot)$. Letter $\mathcal{G}(\cdot)$ represents a general system.
- Sets are written using blackboard bold letters. Some examples are $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ which are the sets of natural, integer, real, and complex numbers, respectively.
- Capital letter E means the total energy of a signal or system, and P is the medium potency of a signal or system. T is the period of a signal.

SOLUTION TO DIFFERENTIAL EQUATIONS

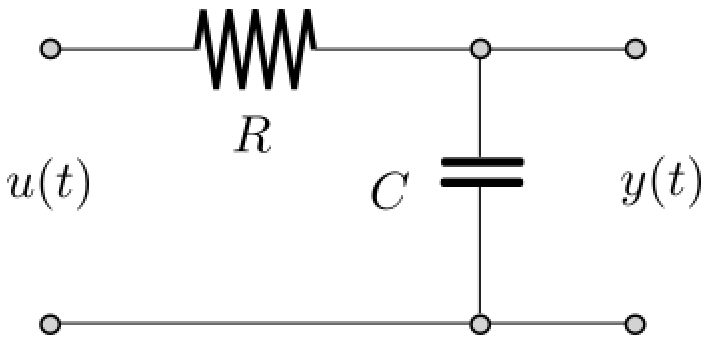
- Let us consider that the model of a dynamical process leads to a differential equation of the form

$$a_n \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$
$$= b_m \frac{d^m}{dt^m} u(t) + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} u(t) + \cdots + b_1 \frac{d}{dt} u(t) + b_0 u(t)$$

which evolves from initial conditions $y(0), \cdots, \frac{d^{n-1}}{dt^{n-1}} y(0)$, $u(0), \cdots, \frac{d^{m-1}}{dt^{m-1}} u(0)$.

FIRST ORDER ODE

❑ Solution to the First Order ODE



RC CIRCUIT

$$u(t) = Ri + y(t); \quad i = C\frac{dy(t)}{dt}$$

$$\dot{y}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}u(t)$$

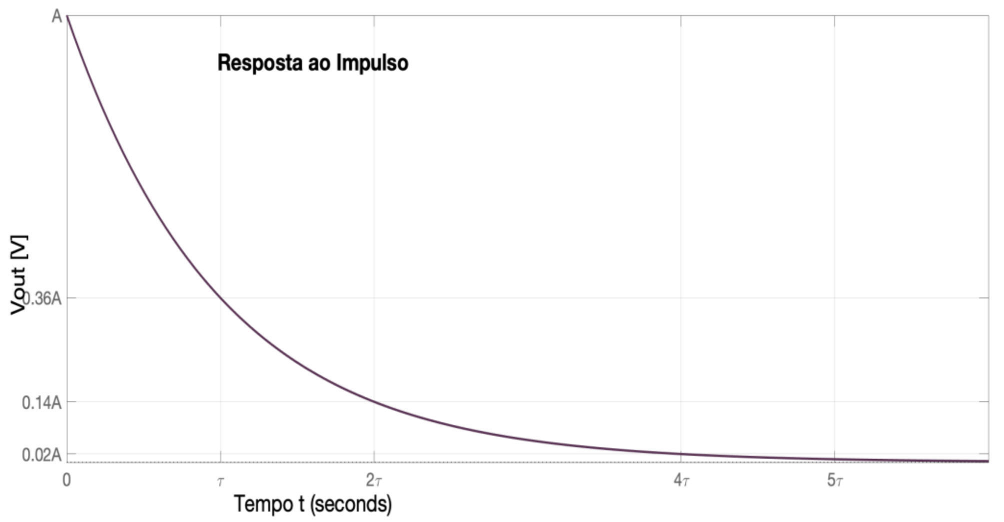
$$y(t) = e^{-\left(\frac{1}{RC}\right)t}y(0) + \int_0^t e^{-\left(\frac{1}{RC}\right)(t-\alpha)}\frac{1}{RC}u(\alpha)d\alpha$$

```
tspan=[0 10];
y0=0; [t,y]=ode45(@ddt,tspan,y0);
plot(t,y)
function dydt = ddt(t,y)
u=1;
```

MATLAB LINK

FIRST ORDER ODE

- The response to the unit impulse is



- $u(t) = A\delta(t)$
- The system evolves from i.c. $y(0) = 0$,

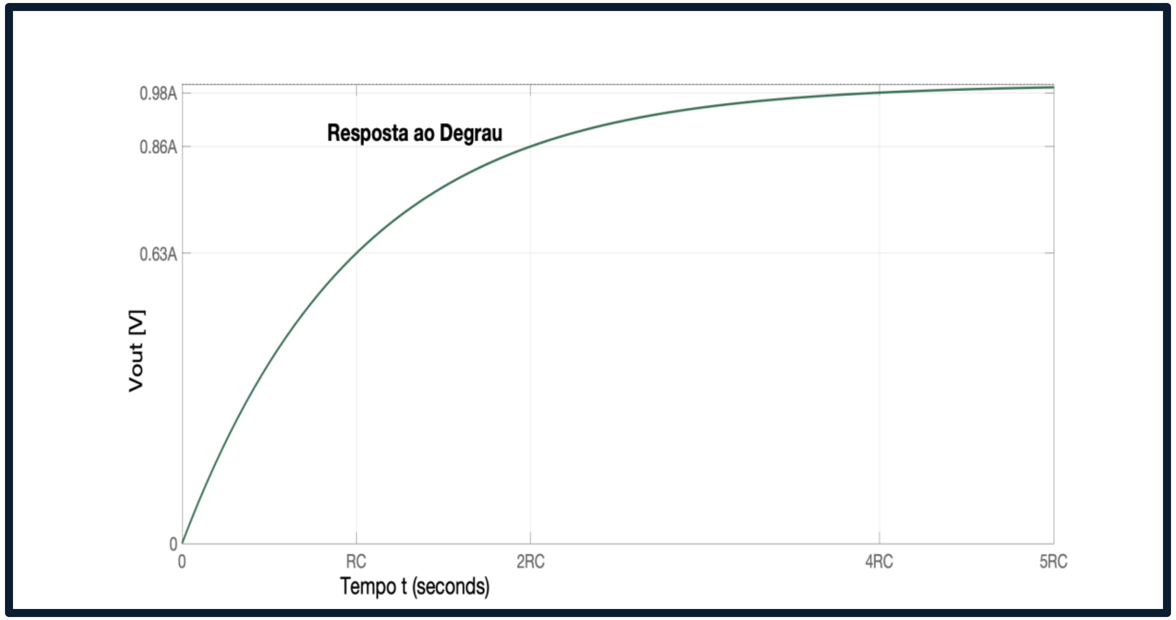
$$y(t) = \int_0^t e^{-(1/RC)(t-\alpha)} \frac{1}{RC} u(\alpha) d\alpha$$

$$y(t) = \frac{A}{RC} e^{-(1/RC)t} = \frac{A}{\tau} e^{-t/\tau}$$

- At $t = \tau$ [s] the signal is about 37% of $y(0)$
- $\tau = RC$ is the **EXPONENTIAL TIME CONSTANT**

FIRST ORDER ODE

- The response to the unit step is



- $u(t) = A\mu(t)$
- The system evolves from i.c. $y(0) = 0$,

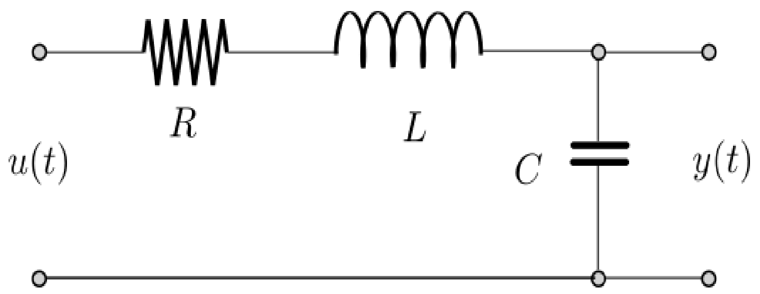
$$y(t) = \int_0^t e^{-\left(\frac{1}{RC}\right)(t-\alpha)} \frac{1}{RC} u(\alpha) d\alpha$$

$$y(t) = A - Ae^{-\left(\frac{1}{RC}\right)t} = A\left[1 - e^{-t/\tau}\right]$$

- At $t = \tau$ [s], the output is about 63% of its steady state value.
- At $t \approx 3\tau$ [s], the output reaches the **region of 5%** of error.
- At $t \approx 4\tau$ [s], the output reaches the **region of 2%** of error.

SECOND ORDER ODE

- Solution to the second order ODE



RLC CIRCUIT

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + y(t); \quad i = C\frac{dy(t)}{dt}$$

$$\ddot{y}(t) + \frac{R}{L}\dot{y}(t) + \frac{1}{LC}y(t) = \frac{1}{LC}u(t)$$

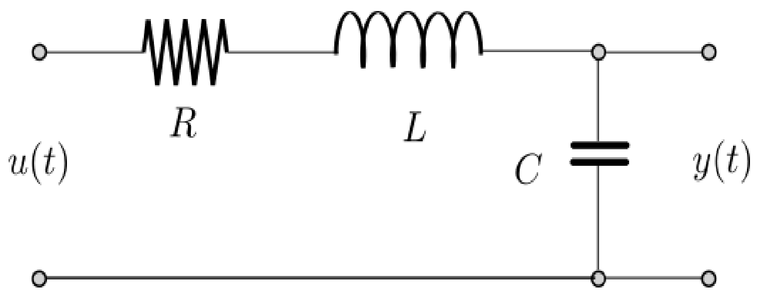
$$y(t) = y_h(t) + y_p(t)$$

$y_p(t)$ **Particular solution**

$y_h(t)$ **Homogeneous solution**

SECOND ORDER ODE

❑ Solution to the second order ODE



RLC CIRCUIT

❑ **Homogeneous Solution**:

❑ Characteristic equation:

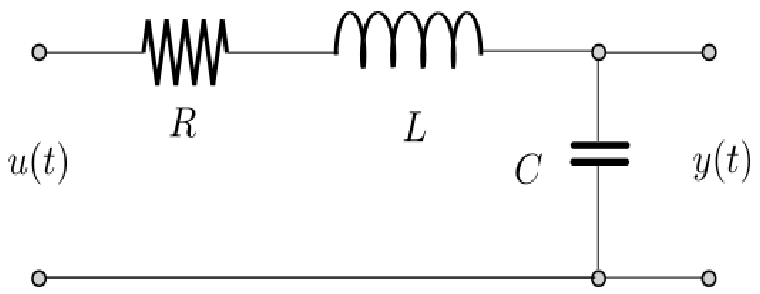
$$L\lambda^2 + R\lambda + \frac{1}{C} = 0$$

$$\lambda = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$y_h(t) = c_1 e^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} + c_2 e^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}$$

SECOND ORDER ODE

❑ Solution to the second order ODE



RLC CIRCUIT

❑ **Particular Solution**:

The solution will depend on the input.

- ❑ The response to the unit impulse is such that

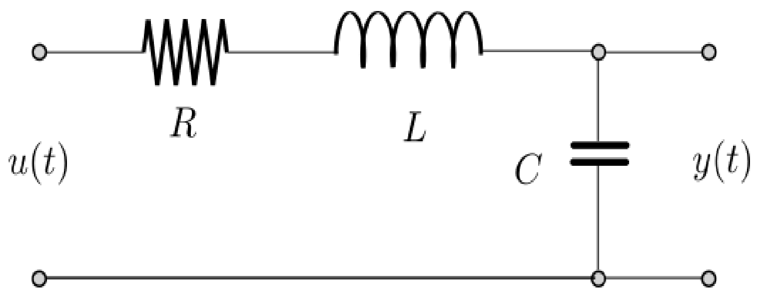
$$u(t) = A\delta(t)$$
$$y_p = 0, \forall t > 0 \; and \; y(0) = 0 \; e \; \dot{y}(0) = A$$

- ❑ Considering $R = 2$ [Ω], $C = 1$ [F] e $L = 1$ [H],

$$y(t) = Ate^{-t}\mu(t)$$

SECOND ORDER ODE

❑ Solution to the second order ODE



RLC CIRCUIT

❑ **Particular Solution**:

The solution will depend on the input.

❑ The response to the unit step is such that

$$u(t) = A\mu(t) \quad \Longrightarrow \quad y_p = c_3\mu(t) = \frac{A}{C}\mu(t)$$

❑ Considering $R = 2$ [Ω], $C = 1$ [F] e $L = 1$ [H],

$$y(t) = -Ae^{-t} - Ate^{-t} + A$$

SOLUTION TO FINITE DIFFERENCE EQUATIONS

- Let us consider that the model of a dynamical process leads to a finite difference equation of the form

$$a_n y[n+s] + \cdots + a_1 y[n+1] + a_0 y[n] = b_p u[n+p] + \cdots + b_1 u[n+1] + b_0 u[n]$$

which evolves from initial conditions $y[0], \cdots, y[s-1]$, $u[0], \cdots, u[p-1]$, where $a_i \in \mathbb{R}$ and $a_n \neq 0$,

$$D(y[n]) = \sum_{i=0}^{s} a_i y[n+i]\,; \quad N(u[n]) = \sum_{i=0}^{p} b_i u[n+i]$$

SOLUTION TO FINITE DIFFERENCE EQUATIONS

- The solution to the difference equation is of the form

$$y[n] = y_h[n] + y_p[n]$$

$y_p[n] = \sum_{i=1}^{m} d_i \, \lambda_i^n$ **is the Particular solution**

$y_h[n] = \sum_{i=1}^{s} \sum_{j=1}^{h_i} c_{ij} \lambda_i^n k^{j-1}$ **is the Homogeneous solution**

with λ_i the solution to $\Delta_D(\lambda) = 0$ and λ_i the solution to $\Delta_N(\lambda) = 0$.

Special attention must be given to repeated roots.

SOLUTION TO THE FDE

- Consider the difference equation

$$y[n+2] - y[n] = 3 \cdot 2^n, \qquad y[0] = 0, \qquad y[1] = 1$$

  - Homogeneous Solution: $\Delta_D(\mu) = \mu^2 - 1 \quad \Rightarrow \quad \mu = \pm 1 \quad \Rightarrow \quad y_h = c_1(1)^n + c_2(-1)^n$

  - Particular Solution: $\Delta_N(\lambda) = \mu - 2 \quad \Rightarrow \quad \mu = 2 \quad \Rightarrow \quad y_p = d(2)^n$

  - Since $y_p = d(2)^n$ hence, $d(2)^{n+2} - d(2)^n = 3 \cdot 2^n \quad \Rightarrow \quad 3d = 3 \quad \Rightarrow \quad d = 1$

  - From Initial Conditions: $\begin{cases} c_1 + c_2 = -1 \\ c_1 - c_2 = -1 \end{cases} \Rightarrow \quad c_1 = -1 \quad e \quad c_2 = 0$

FOURIER SERIES

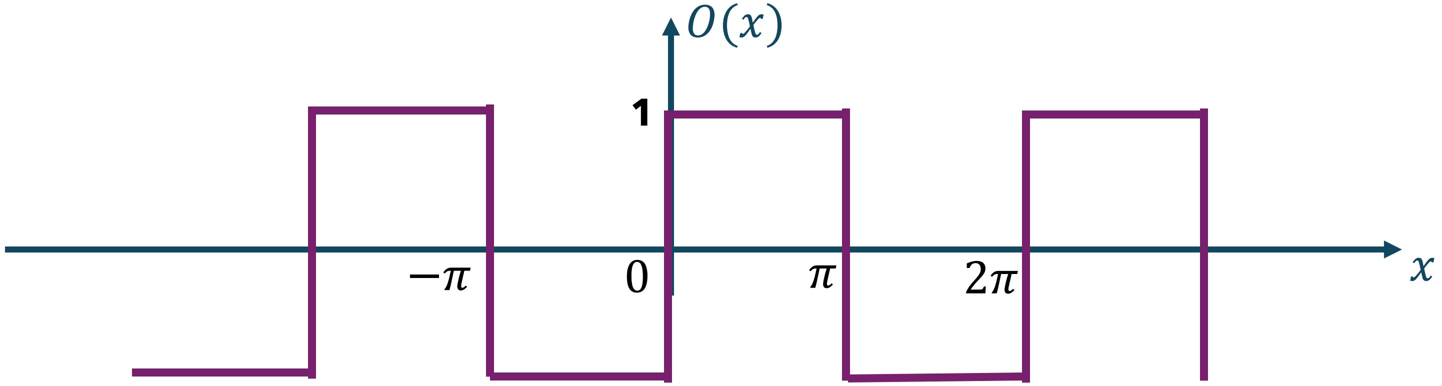
- Fourier, 2 centuries ago, has suggested that **any function can be rewritten as a combination of sines and cosines**.
- Indeed, any **2π-periodic**, $O(x) = O(x + 2\pi)$, **odd function**, $O(-x) = -O(x)$, such that $\boldsymbol{O(0) = O(\pi) = 0}$, can be written in the form

$$O(x) = \sum_{i=1}^{\infty} b_i \sin(ix),$$

$$b_i = \frac{2}{\pi}\int_0^{\pi} O(x)\sin(ix)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} O(x)\sin(ix)\,dx$$

FOURIER SERIES

- Find the Fourier sine coefficients b_i of the square wave



$$b_i = \frac{2}{\pi}\int_0^{\pi} \sin(ix)\, dx = \frac{2}{\pi}\left[-\frac{\cos(ix)}{i}\right]_0^{\pi} = \frac{2}{\pi}\left\{\frac{2}{1}, \frac{0}{2}, \frac{2}{3}, \frac{0}{4}, \frac{2}{5}, \frac{0}{6}, \cdots\right\}$$

$$\boldsymbol{O(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]}$$

FOURIER SERIES

- Analogously, any **2π-periodic**, $E(x) = E(x + 2\pi)$, **even function**, $E(-x) = E(x)$, can be written in the form

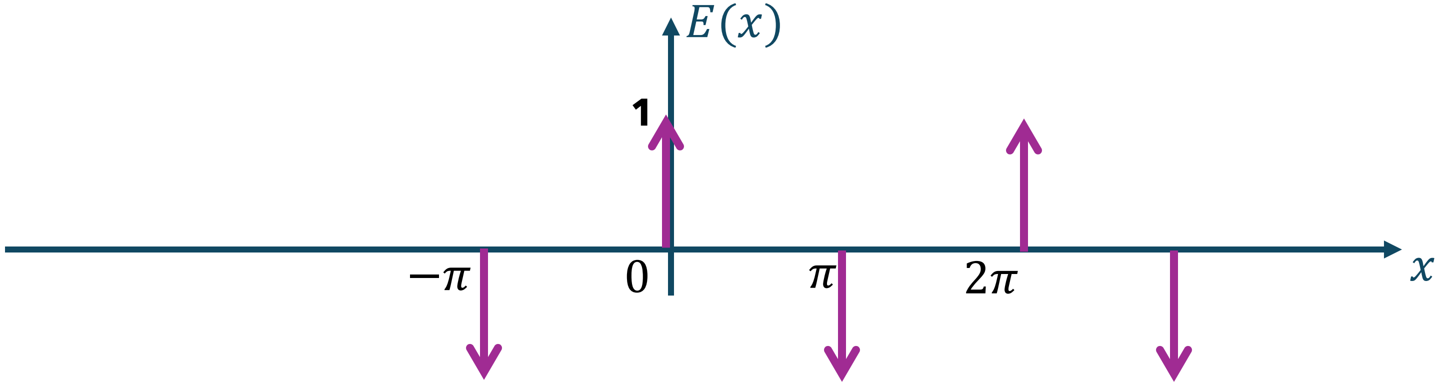
$$\boldsymbol{E(x) = \sum_{i=1}^{\infty} a_i \cos(ix)},$$

$$\boldsymbol{a_i = \frac{2}{\pi}\int_0^{\pi} E(x)\cos(ix)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} E(x)\cos(ix)dx},$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} E(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} E(x)dx$ is the **average value** of $E(x)$.

FOURIER SERIES

- Find the Fourier cosine coefficients a_i of the up-down train



$$a_0 = \frac{1}{\pi}\left(\int_0^{\pi} \delta(x)dx - \int_0^{\pi} \delta(x-\pi)dx\right) = 0; a_i = \frac{2}{\pi}\int_0^{\pi} \left(\delta(x) - \delta(x-\pi)\right)\cos(ix)\,dx = \frac{2}{\pi}\{2,0,2,0,\cdots\}$$

$$\boldsymbol{E(x) = \frac{4}{\pi}[\cos(x) + \cos(3x) + \cos(5x) + \cdots]}$$

CT FOURIER SERIES

- For functions $F(x)$ that are **NOT EVEN or ODD**, but periodic with **period 2π**. (full period $[-\pi, \pi]$ or $[0,2\pi]$).

$$F(x) = a_0 + \sum_{i=1}^{\infty} a_i \cos(ix) + \sum_{i=1}^{\infty} b_i \sin(ix)$$

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos(ix) dx \,, a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) dx \,, b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin(ix)\, dx$$

- Considering that $F(x) = O(x) + E(x)$, with $O(x) = \big(F(x) + F(-x)\big)/2$ and $E(x) = \big(F(x) - F(-x)\big)/2$, **$O(x)$ gives b_i and $E(x)$, a_0 and a_i**.

CT FOURIER SERIES

- Hence a large class of periodic functions can be written as a Fourier series, such that

$$F(x) = a_0 + \sum_{i=1}^{\infty} a_i \cos(ix) + \sum_{i=1}^{\infty} b_i \sin(ix)$$

$$F(x) = a_0 + \sum_{i=1}^{\infty} \frac{a_i\left(e^{jix} + e^{-jix}\right)}{2} + \sum_{i=1}^{\infty} \frac{b_i\left(e^{jix} - e^{-jix}\right)}{2j} = \sum_{i=-\infty}^{\infty} A_i e^{jix}$$

CT FOURIER SERIES

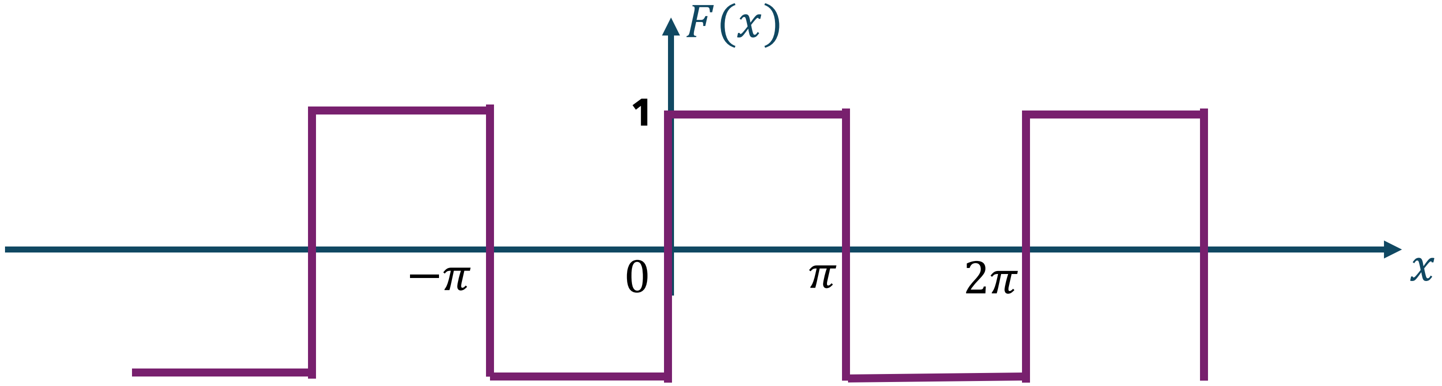
- A Fourier series can represent any signal that satisfies the **three Dirichlet conditions.**
  - **Condition 1** – Function $x(t)$ is **absolutely integrable** inside any interval of one period, that is,

$$\int_T |x(t)| dt < \infty$$

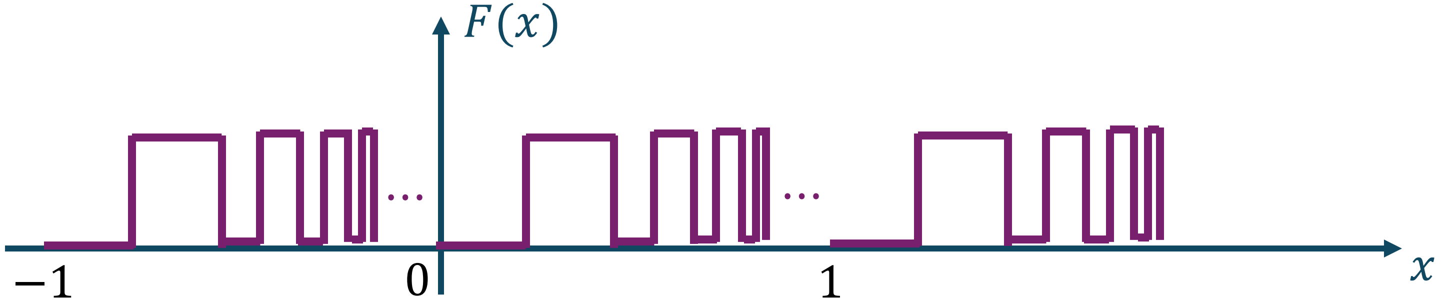
  - **Condition 2** – Inside any finite interval, the function $x(t)$ has a **finite quantity of highs and lows.**
  - **Condition 3** – Inside any finite interval of time the function $x(t)$ has a **finite qunatity of discontinuities**.

CT FOURIER SERIES

- Can be represented by a CT Fourier series



- Can not be represented by a Fourier Series



MAIN CT FOURIER SERIES PROPERTIES

PROPERTY	SIGNAL (PERIOD T)	COEFFICIENTS
Linearity	$Ax(t) + By(t)$	$Aa_i + Bb_i$
Time Displacement	$x(t - t_0)$	$a_i e^{-ji(2\pi/T)t_0}$
Time Reflection	$x(-t)$	a_{-i}
Time Change of Scale	$x(at), a > 0, T_a = T/a$	a_i
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{i-l}$
Frequency Displacement	$e^{jM(2\pi/T)t}$	a_{i-M}
Conjugation	$x^*(t)$	a_{-i}^*
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	Ta_ib_i
Differentiation	$dx(t)/dt$	$ji(2\pi/T)a_i$
Integration	$\int_{-\infty}^{t} x(t)dt\,, a_0 = 0$	$\left(1/\left(ji(2\pi/T)\right)\right) a_i$

DT FOURIER SERIES

- The representation of a DT signal in a Fourier series has **finite quantity of terms**, hence there is no concern about convergence of the series.
- A DT signal in Fourier series is expressed as

$$x[n] = \sum_{i=\langle N \rangle} a_i \phi_i[n] = \sum_{i=\langle N \rangle} a_i e^{ji(2\pi/N)n}$$

- The quantity of terms is finite since the DT signal repeats at each period N.
- The coefficients a_i are determined by

$$a_i = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-ji(2\pi/N)n}$$

MAIN DT FOURIER SERIES PROPERTIES

PROPERTY	SIGNAL (PERIOD N)	COEFFICIENTS
Linearity	$Ax[n] + By[n]$	$Aa_i + Bb_i$
Time Displacement	$x[n - n_0]$	$a_i e^{-ji(2\pi/N)n_0}$
Time Reflection	$x[-n]$	a_{-i}
Time Change of Scale	$\begin{cases} x[n/m] & n \text{ é múltiplo de } m \\ 0 & n \text{ não é múltiplo de } m \end{cases}$ (period mN)	$\frac{1}{m} a_i$ (period mN)
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle}^{\infty} a_l b_{i-l}$
Frequency Displacement	$e^{jM(2\pi/N)n}$	a_{i-M}

MAIN DT FOURIER SERIES PROPERTIES

PROPERTY	SIGNAL (PERIOD N)	COEFFICIENTS
Conjugation	$x^*[n]$	a^*_{-i}
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_ib_i
First Difference	$x[n]-x[n-1]$	$\left(1-e^{ji\left(\frac{2\pi}{N}\right)}\right)a_i$
Accumulated Sum	$\sum_{i=-\infty}^{n} x[i]\,, a_0=0$	$\left(1/\left(1-e^{ji(2\pi/N)}\right)\right)a_i$

FOURIER SERIES

- Parseval for CT Fourier series

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{i=-\infty}^{\infty} |a_i|^2$$

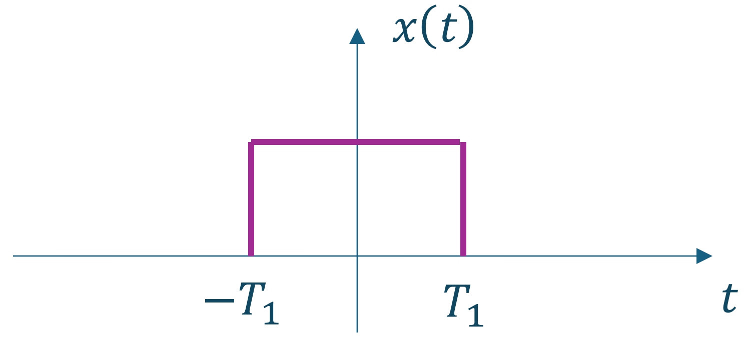
- Parseval for DT Fourier Series

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{i=\langle N\rangle} |a_i|^2$$

CT FOURIER TRANSFORM

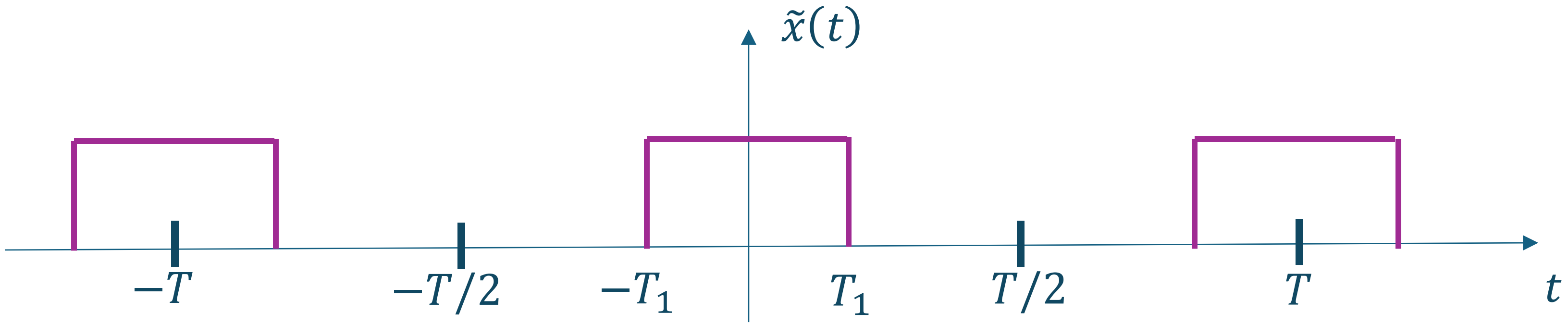
- Many **aperiodic signals** can be represented through an equivalent Fourier Transform. Following Fourier's thoughts, aperiodic signals can be interpreted as **periodic signals with infinite periods**.
- Consider

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$



CT FOURIER TRANSFORM

- An approximate periodic signal is



- This signal can be represented by a Fourier series in the form of

$$\tilde{x}(t) = \sum_{i=-\infty}^{\infty} a_i e^{ji\omega_0 t}; \qquad a_i = \frac{1}{T}\int_T \tilde{x}(t) e^{-ji\omega_0 t} dt$$

CT FOURIER TRANSFORM

- Increasing the period T, the signal $\tilde{x}(t)$ becomes an aperiodic signal.

- In this case, notice that inside the interval $\left[-\frac{\mathrm{T}}{2},\frac{T}{2}\right]$,

$$a_i = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{\boldsymbol{x}}(\boldsymbol{t}) e^{-ji\omega_0 t} dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \boldsymbol{x}(\boldsymbol{t}) e^{-ji\omega_0 t} dt = \frac{1}{\mathrm{T}}\int_{-\infty}^{\infty} x(t) e^{-ji\omega_0 t} dt$$

since $x(t) = 0$ for all $|t| > T$.

CT FOURIER TRANSFORM

- Hence, for an aperiodic signal,

$$a_i = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-ji\omega_0 t}dt = \frac{1}{T}X(ji\omega_0);\;\; X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

On the other hand, for $T \to \infty$ or $\omega_0 \to 0$,

$$\tilde{x}(t) = \sum_{i=-\infty}^{\infty} \frac{1}{\boldsymbol{T}} X(ji\omega_0)e^{ji\omega_0 t} = \frac{1}{\boldsymbol{2\pi}} \sum_{i=-\infty}^{\infty} X(ji\omega_0)e^{ji\omega_0 t}\boldsymbol{\omega_0} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \to x(t)$$

CT FOURIER TRANSFORM

- In this case, for $T \to \infty$ the result leads to the Fourier Transform.

$$\boldsymbol{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega}$$

INVERSE FOURIER TRANSFORM

$$\boldsymbol{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

FOURIER TRANSFORM (FOURIER INTEGRAL)

CT FOURIER TRANSFORM

- As for the Fourier Series, the Fourier Transform converges if $x(t)$ satisfies the **Dirichlet conditions**:

  - **Condition 1** – Function $x(t)$ is **absolutely integrable**, that is,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

  - **Condition 2** – Inside any finite interval, the function $x(t)$ has a **finite quantity of highs and lows.**
  - **Condition 3** – Inside any finite interval of time the function $x(t)$ has a **finite quantity of discontinuities**.

- Alternatively, $x(t)$ has **finite energy**,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

CT FOURIER TRANSFORM

- Verify the the Fourier Transform for the functions:

1. $x(t) = e^{-at}\mu(t)$

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}, a > 0$$

2. $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right) = \frac{2}{\omega}\sin(\omega T_1) = \frac{2T_1\sin(\omega T_1)}{\omega T_1}$$

$$= 2T_1 sinc\,(\omega T_1)$$

CT FOURIER TRANSFORM

3. $X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{1}{2\pi}\frac{1}{jt}\left(e^{jWt} - e^{-jWt}\right) = \frac{1}{\pi t}\sin(Wt) = \frac{2W}{2\pi}\frac{\sin(Wt)}{Wt}$$

$$= \frac{W}{\pi} sinc\,(Wt)$$

4. $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t} d\omega = \frac{1}{2\pi}2\pi e^{j\omega_0 t} = e^{j\omega t}$$

PROPERTIES OF CT FOURIER TRANSFORM

PROPERTY	SIGNAL	FOURIER TRANSFORM
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Time Displacement	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Conjugation	$x^*(t)$ $x(t) \in \mathbb{R}$	$X^*(-j\omega)$ $X^*(j\omega) = X(-j\omega)$
Change of Scale	$x(At)$ $x(-t)$	$\frac{1}{\lvert A\rvert}X\left(\frac{j\omega}{A}\right)$ $X(-j\omega)$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

CT FOURIER TRANSFORM

- **Periodic signals** can be represented by Fourier series, hence from the previous example

$$x(t) = \sum_{i=-\infty}^{\infty} c_i e^{ji\omega_0 t} \Leftrightarrow X(j\omega) = \sum_{i=-\infty}^{\infty} c_i 2\pi\delta(\omega - \omega_0)$$

- As a consequence, the **Fourier Transform of the impulse train**, $x(t) = \sum_{i=-\infty}^{\infty} \delta(t - iT)$, can be determined as

$$X(j\omega) = \frac{2\pi}{T} \sum_{i=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi i}{T}\right), \text{ since } c_i = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-ji\omega_0 t} dt = \frac{1}{T}$$

PROPERTIES OF CT FOURIER TRANSFORM

PROPERTY	SIGNAL	FOURIER TRANSFORM
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Time Displacement	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Conjugation	$x^*(t)$ $x(t) \in \mathbb{R}$	$X^*(-j\omega)$ $X^*(j\omega) = X(-j\omega)$
Change of Scale	$x(At)$ $x(-t)$	$\frac{1}{\|A\|}X\left(\frac{j\omega}{A}\right)$ $X(-j\omega)$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

PROPERTIES OF CT FOURIER TRANSFORM

- Parseval for aperiodic continuous-time signals

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

- Convolution property:

$$y(t) = x_1(t) * x_2(t) \quad \leftrightarrow \quad Y(j\omega) = X_1(j\omega)X_2(j\omega)$$

$$y(t) = x_1(t)x_2(t) \quad \leftrightarrow \quad Y(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

CT FOURIER TRANSFORM FOR LTI SYSTEMS

- Consider an LTI System described by

$$\sum_{i=0}^{N} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i u(t)}{dt^i}$$

by applying the Fourier transform

$$\sum_{i=0}^{N} a_i (j\omega)^i Y(j\omega) = \sum_{i=0}^{M} b_i \, (j\omega)^i X(j\omega)$$

CT FOURIER TRANSFORM FOR LTI SYSTEMS

- For LTI systems, we have

$$y(t) = h(t) * u(t) \quad \Longrightarrow \quad Y(j\omega) = H(j\omega)U(j\omega)$$

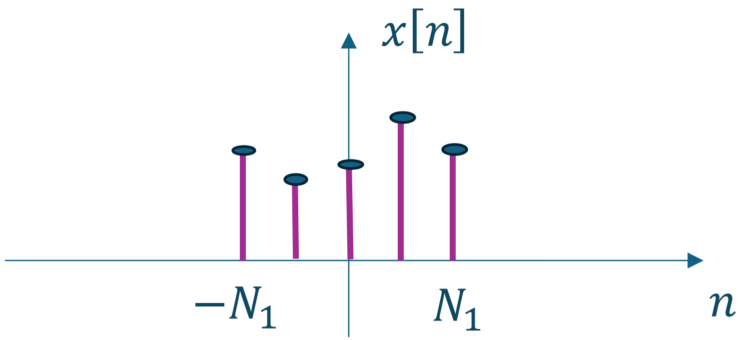
Hence,

$$H(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{\sum_{i=0}^{M} b_i (j\omega)^i}{\sum_{i=0}^{N} a_i (j\omega)^i}$$

- The output is related to the input through a rational function of $(j\omega)$. The denominator is related to the modes of the system, which are the roots of the homogeneous equation.

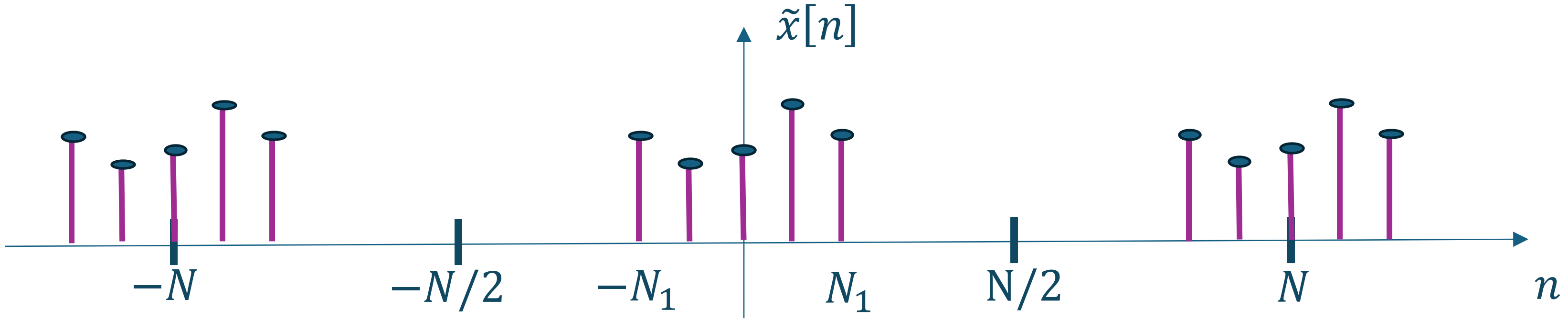
DT FOURIER TRANSFORM

- Many DT **aperiodic signals** can be represented through an equivalent Fourier Transform.
- Consider again an aperiodic signal



DT FOURIER TRANSFORM

- An approximate periodic signal, with $\frac{N}{2} > \max(N_1, N_2)$, is



- This signal can be represented by a Fourier series in the form of

$$\tilde{x}[n] = \sum_{i \in \langle N \rangle} a_i e^{ji\left(\frac{2\pi}{N}\right)n}; \qquad a_i = \frac{1}{N} \sum_{i \in \langle N \rangle} \tilde{x}[n] e^{-ji\left(\frac{2\pi}{N}\right)n}$$

DT FOURIER TRANSFORM

- Increasing the period N, the signal $\tilde{x}[n]$ becomes an aperiodic signal.

- In this case, notice that inside the interval $\left[-\frac{N}{2}, \frac{N}{2}\right]$,

$$a_i = \frac{1}{N} \sum_{i \in \langle N \rangle} \boldsymbol{x[n]} e^{-ji\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-ji\left(\frac{2\pi}{N}\right)n}$$

since $x[n] = 0$ for all $|n| > N$.

DT FOURIER TRANSFORM

- Hence, for an aperiodic signal with $\omega_0 = 2\pi/N$,

$$a_i = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-ji\omega_0 n} = \frac{1}{N}X\left(e^{ji\omega_0}\right); \;\; X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

On the other hand, for $N \to \infty$ or $\omega_0 \to 0$,

$$\tilde{x}[n] = \sum_{i\in\langle N\rangle} \frac{1}{\boldsymbol{N}} X\left(e^{ji\omega_0}\right)e^{ji\omega_0 n} = \frac{1}{\boldsymbol{2\pi}}\sum_{i\in\langle N\rangle} X\left(e^{ji\omega_0}\right)e^{ji\omega_0 n}\,\boldsymbol{\omega_0} \to \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega \to x[n]$$

DT FOURIER TRANSFORM

- In this case, for $T \to \infty$ the result leads to the Fourier Transform.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ **FOURIER TRANSFORM**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ **INVERSE FOURIER TRANSFORM**

- The DT Fourier Transform is 2π periodic due to the exponential $e^{j\omega} = e^{j(\omega+2\pi)}$.

DT FOURIER TRANSFORM

- As before, the existence of the Fourier transform is ensured for **absolutely summable** signals

$$\sum_{n=-\infty}^{\infty} |x(t)| < \infty$$

- Alternatively, $x[n]$ has **finite energy**,

$$\sum_{n=-\infty}^{\infty} |x(t)|^2 < \infty$$

- $X(e^{j\omega})$ for the DT Fourier Transform and $X(j\omega)$ for the CT Fourier Transform are called **SPECTRUM** of $x(t)$ (or $x[n]$).

DT FOURIER TRANSFORM

- Verify the the Fourier Transform for the functions:

1. $x[n] = a^n \mu[n], |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

2. $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

DT FOURIER TRANSFORM

3. $X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{1}{2\pi}\frac{1}{jn}\left(e^{jWn} - e^{-jWn}\right) = \frac{1}{\pi n}\sin(Wn) = \frac{2W}{2\pi}\frac{\sin(Wn)}{Wn}$$

$$= \frac{W}{\pi} sinc\,(Wn)$$

For $W \rightarrow \pi$, $\hat{x}[n] \rightarrow \delta[n]$

4. $X\left(e^{j\omega}\right) = \sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell)$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell)\, e^{j\omega t} d\omega = \frac{1}{2\pi} 2\pi e^{j\omega_0 n} = e^{j\omega_0 n}$$

DT FOURIER TRANSFORM

- **Periodic signals** can be represented by Fourier series, hence from the previous example

$$x[n] = \sum_{i \in \langle N \rangle} c_i e^{ji\left(\frac{2\pi}{N}\right)n} \Leftrightarrow X\left(e^{j\omega}\right) = \sum_{i=-\infty}^{\infty} c_i 2\pi\delta\left(\omega - \frac{2\pi i}{N}\right)$$

PROPERTIES OF DT FOURIER TRANSFORM

PROPERTY	SIGNAL	FOURIER TRANSFORM
Periodicity	–	$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$
Linearity	$Ax_1[n] + Bx_2[n]$	$AX_1\left(e^{j\omega}\right) + BX_2\left(e^{j\omega}\right)$
Displacement	$x[n-n_0]$ $e^{j\omega_0 n}x[n]$	$e^{-j\omega t_0}X\left(e^{j\omega}\right)$ $X\left(e^{j(\omega-\omega_0)}\right)$
Conjugation	$x^*[n]$ $x[n] \in \mathbb{R}$	$X^*\left(e^{-j\omega}\right)$ $X^*\left(e^{j\omega}\right) = X\left(e^{-j\omega}\right)$
Change of Scale	$x_k[n] = \begin{cases} x[n/k], & n/k \in \mathbb{Z} \\ 0, & otherwise \end{cases}$ $x[-n]$	$X\left(e^{jk\omega}\right)$ $X\left(e^{-j\omega}\right)$
Differentiation	$x[n] - x[n-1]$	$\left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$
Accumulation	$\sum_{m=-\infty}^{n} x[m]$	$\frac{1}{1-e^{-j\omega}}X\left(e^{j\omega}\right) + \pi X\left(e^{j0}\right)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$

PROPERTIES OF DT FOURIER TRANSFORM

- Parseval for aperiodic continuous-time signals

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

- Convolution property:

$$y[n] = x_1[n] * x_2[\mathrm{n}] \quad \leftrightarrow \quad Y(j\omega) = X_1\left(e^{j\omega}\right)X_2\left(e^{j\omega}\right)$$

$$y[n] = x_1[n]x_2[n] \quad \leftrightarrow \quad Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right) * X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

DT FOURIER TRANSFORM FOR LTI SYSTEMS

- Consider an LTI System described by

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i\, u[n-i]$$

by applying the Fourier transform

$$\sum_{i=0}^{N} a_i e^{-ji\omega} Y(e^{j\omega}) = \sum_{i=0}^{M} b_i\, e^{-ji\omega} X(e^{j\omega})$$

DT FOURIER TRANSFORM FOR LTI SYSTEMS

- For LTI systems, we have

$$y(t) = h(t) * u(t) \quad \Longrightarrow \quad Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)U\left(e^{j\omega}\right)$$

Hence,

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{U\left(e^{j\omega}\right)} = \frac{\sum_{i=0}^{M} b_i e^{-ji\omega}}{\sum_{i=0}^{N} a_i e^{-ji\omega}}$$

- The output is related to the input through a rational function of $\left(e^{j\omega}\right)$. The denominator is related to the modes of the system, which are the homogeneous solution of the system.

EXERCISES

1. Determine the solution to the ODE
   a. $\ddot{y} + 2\dot{y} + y = \delta t + 2u(t), y(0) = 0, \dot{y}(0) = 0$
   b. $\ddot{y} + 8\dot{y} + 15y - 5 = 2e^{-3t}, y(0) = 0, \dot{y}(0) = 1$
   c. $\ddot{y} + 3\dot{y} + 2y = 8e^{-t}\sin(t), y(0) = 1, \dot{y}(0) = 1$
2. Determine the solution to the FDE
   a. $y[k+2] - 7y[k+1] + 10y[k] = 0, y[0] = 1, y[1] = 0$
   b. $y[k+2] - 6y[k+1] + 5y[k] = 3^k, y[0] = 0, y[1] = 0$
   c. $y[k+2] - 6y[k+1] + 5y[k] = k^2, y[0] = 0, y[1] = 0$
   d. $y[k+2] - 6y[k+1] + 5y[k] = 3^k + 5^k, y[0] = 1, y[1] = 0$
   e. $y[k+2] - 2y[k+1] + y[k] = \delta[k], y[0] = 0, y[1] = 0$

EXERCISES

3. Define the coefficients of the Fourier series to the functions
   a. $F(x) = \begin{cases} 1, & 0 < x < h \\ 0, & h < x < 2\pi \end{cases}$
   b. $F(x) = |x|, -\pi < x < \pi$
4. Consider a discrete-time LTI system whose impulsive response is $h[n] = (1/2)^{|n|}$. Determine the Fourier series that represents the response $y[n]$ to each of the inputs:
   a. $u[n] = \sum_{i=-\infty}^{\infty} \delta[n - 4k]$
   b. $u[n]$ is periodic with period 6 and $x[n] = \begin{cases} 1, & n = 0, \pm 1 \\ 0, & n = \pm 2, \pm 3 \end{cases}$

EXERCISES

5. Consider a causal CT LTI system with input $x(t)$ and output $y(t)$, such that

$$\frac{d}{dt}y(t) + 4y(t) = x(t)$$

Determine the Fourier series that describes the output $y(t)$ to each input:

a. $u(t) = \cos 2\pi t$

b. $u(t) = \sin 4\pi t + \cos\left(6\pi t + \frac{\pi}{4}\right)$

6. Determine the Fourier Transform for the following signals and sketch the magnitude for each of them.

a. $x(t) = \delta(t+1) + \delta(t-1)$

b. $x(t) = \frac{d}{dt}\left(\mu(-2-t) + \mu(t-2)\right)$

EXERCISES

7. Determine the Fourier transform for the signal

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

From this result and the Parseval identity compute

$$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

8. The output of a causal stable LTI system is determined from its input through

$$\frac{d^2y(t)}{dt} + 6\frac{dy(t)}{dt} + 8y(t) = 2u(t)$$

a. Determine its response to the unit impulse.

b. Determine its response to the input $u(t) = te^{(-2t)}\mu(t)$.

EXERCISES

9. Determine the Fourier transform for the signal

   a. $x[n] = \left(\frac{1}{2}\right)^{-n} \mu[-n-1]$

   b. $x[n] = \begin{cases} n, & -3 \leq n \leq 3 \\ 0, & otherwise \end{cases}$

   c. $x[n] = \sin\left(\frac{\pi}{2} n\right) + \cos(n)$

10. Consider a stable causal LTI system described by $y[n] - \frac{1}{6} y[n-1] - \frac{1}{6} y[n-2] = u[n]$. Determine the frequency response $H(e^{j\omega})$ and the response to the unit impulse $h[n]$ for this system.