AS-767 – SIGNALS AND SYSTEMS

M3 – SAMPLING, FILTERING

TOPICS

- ❑ Sampling of Continuous-Time Systems
- ❑ Filtering

SAMPLING OF CT SIGNALS

- Given the context of continuous- and discrete-time systems, sometimes it is necessary to use elements operating in the continuous-time domain and others in the discrete-time domain. For that systems we use sampling.

- Continuous-time systems can be perfectly represented by their sampling in an adequate constant sampling time.

SAMPLING OF CT SIGNALS

- Consider a continuous-time signal $x(t)$, its samplings are represented by

$$x_p(t) = x(t)p(t)$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- T is the **sampling period**, $p(t)$ is the **sampling function**. $\omega_s = 2\pi/T$ is the **sampling frequency**.

SAMPLING OF CT SIGNALS

- From the property $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$, it follows

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

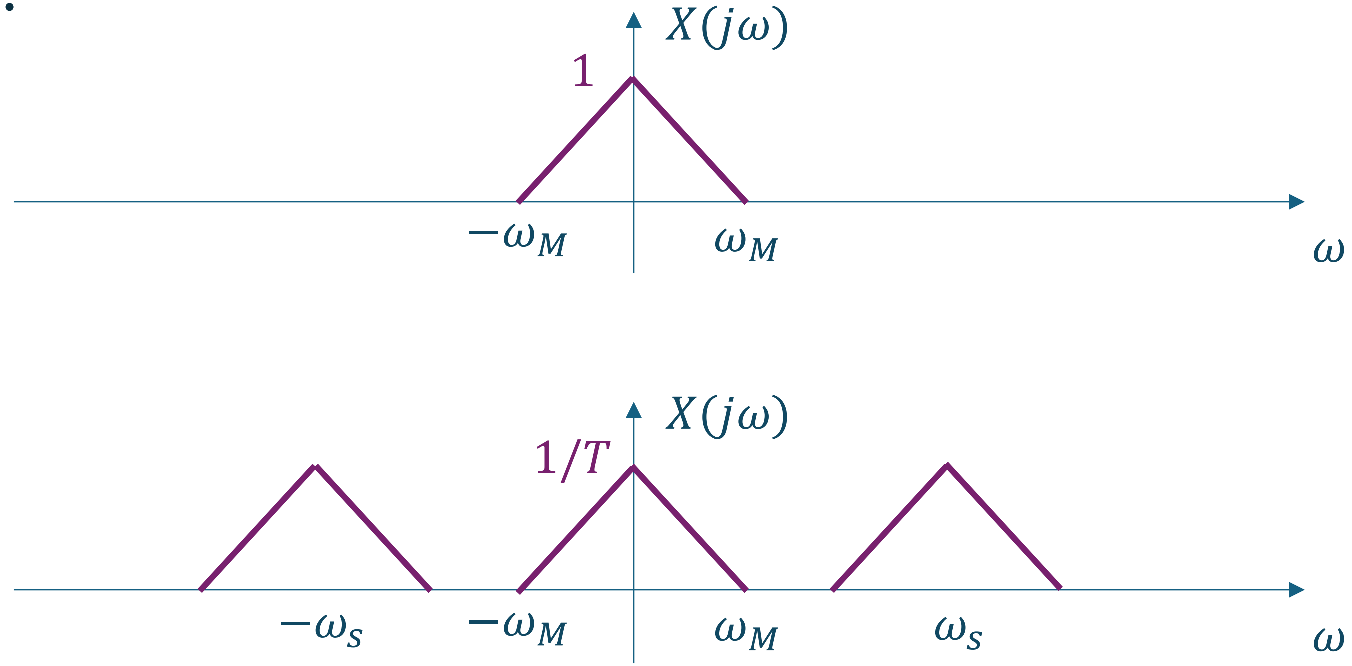
Moreover,

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P\big(j(\omega-\theta)\big)d\theta\,; \quad P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s-\theta)\right]X(j\theta)d\theta = \frac{1}{T}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s)$$

SAMPLING OF CT SIGNALS

- Sampling a signal in the time domain leads to shifted replicas of its spectrum.

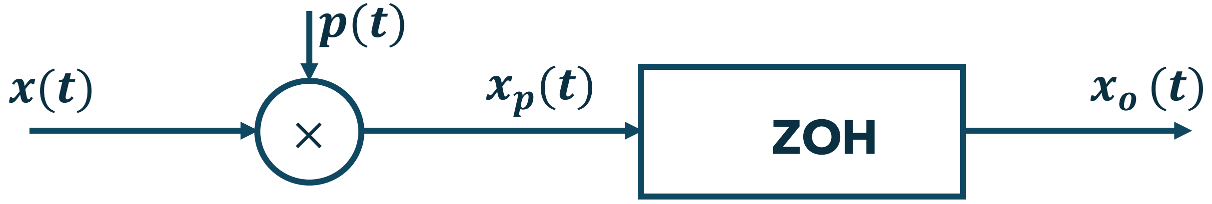


SAMPLING OF CT SIGNALS

- **SAMPLING THEOREM:** Consider the signal $x(t)$ such that $X(j\omega) = 0, \forall|\omega| > \omega_M$. Hence, $x(t)$ is uniquely determined by its samples $x[nT], \mathrm{n} \in \mathbb{Z}$, if $\omega_s > 2\omega_M, \omega_s = 2\pi/T$.
- This signal can be recovered by the adoption of a filter with amplitude T and cutoff frequency $\omega_M < \omega_c < \omega_s - \omega_M$.
- ω_M is the **Nyquist frequency**.
- In practice, **ideal filters** are difficult to implement. Hence, approximations are well accepted to recover the continuous-time signal.

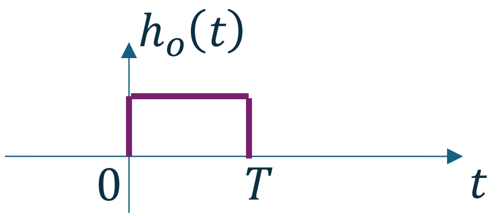
SAMPLING OF CT SIGNALS

- Ideal samplers are also difficult to implement in practice.
  Zero Order Holder is a well-accepted device for this purpose.
- In this case



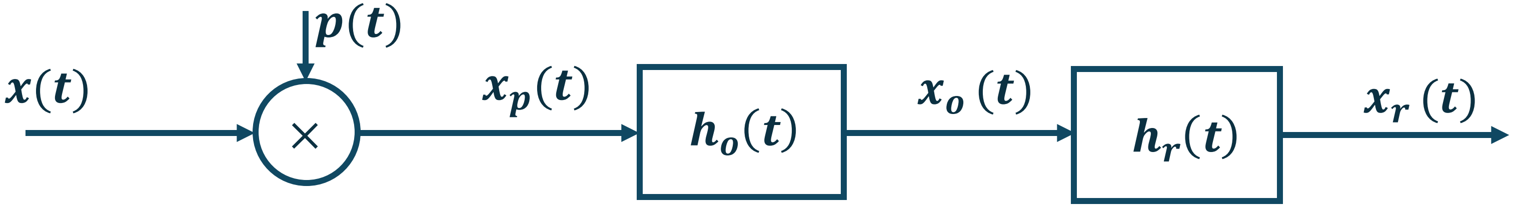
$$x_o(t) = h_o(t) * x_p(t)$$

$$H_o(j\omega) = \int_0^T e^{-j\omega t} dt = e^{-\frac{j\omega T}{2}} \left[\frac{2 \sin\left(\frac{\omega T}{2}\right)}{\omega} \right]$$



SAMPLING OF CT SIGNALS

- To recover the exact signal it becomes necessary to add another cascade device.

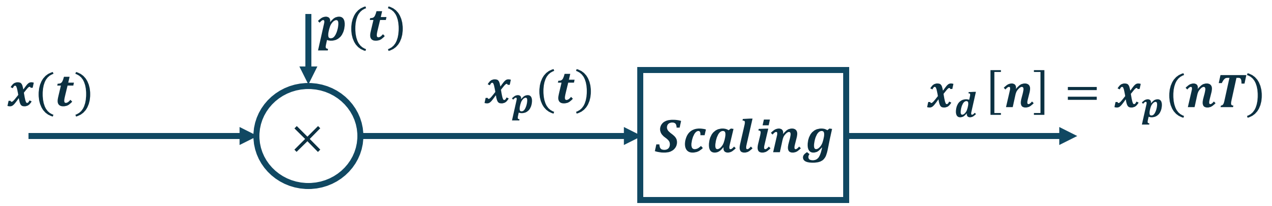


$$H_r(j\omega) = \frac{e^{\frac{j\omega T}{2}} H(j\omega)}{\left[\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}\right]}$$

- Again, due to practical purposes, in general, only the block $h_o(t)$ is required.

DT PROCESSING OF CT SIGNALS

- Sometimes DT processing of CT signals is required. For that purpose, consider the diagram



$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}\,; \quad X_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega n}$$

$$X_d\left(e^{j\Omega}\right) = X_p\left(\frac{j\Omega}{T}\right)$$

DT PROCESSING OF CT SIGNALS

- Since the exponential is related to the impulse,

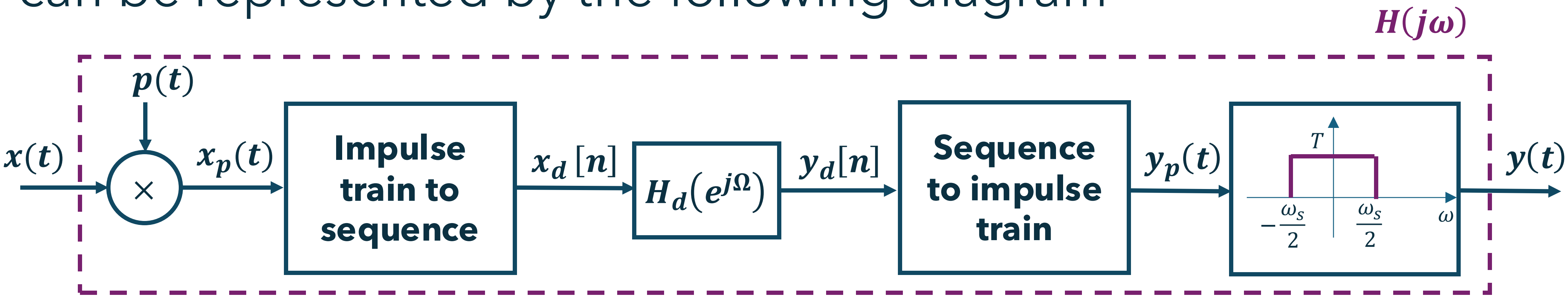
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

$$X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - 2\pi k)/T)$$

which implies a change of scale of the Fourier transform.

DT PROCESSING OF CT SIGNALS

- Finally, discrete-time processing of continuous-time signals can be represented by the following diagram



$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$H(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

FILTERING

- Filtering is the process that changes the amplitude of a signal for a range of frequencies.

- All LTI systems are **frequency-shaping filters**.

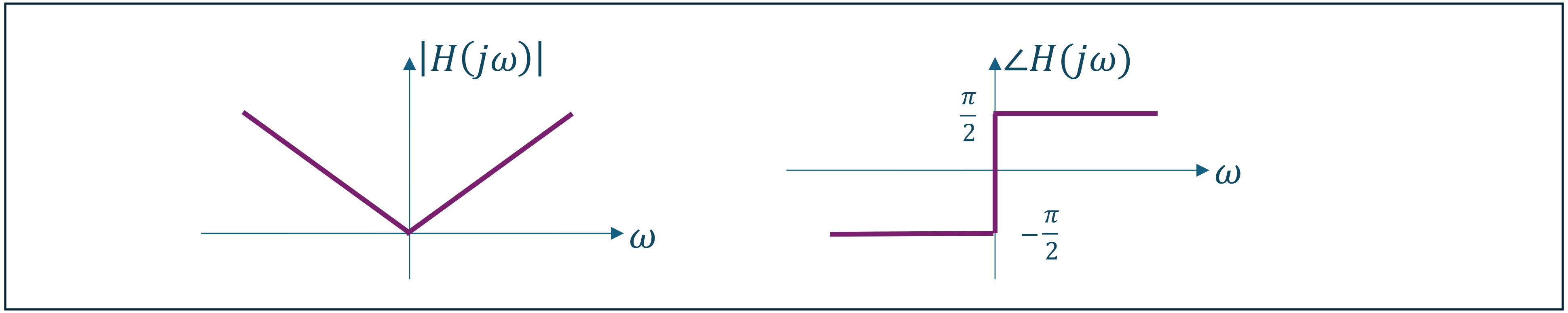
- **Frequency-selective filters** eliminate some range of frequency allowing other frequencies to pass without distortion.

FILTERING

- Consider, for example, a differentiator

$$y(t) = \frac{dx(t)}{dt}$$

$$x(t) = e^{j\omega t} \quad \Longrightarrow \quad y(t) = j\omega e^{j\omega} \quad \Longrightarrow \quad H(j\omega) = j\omega$$



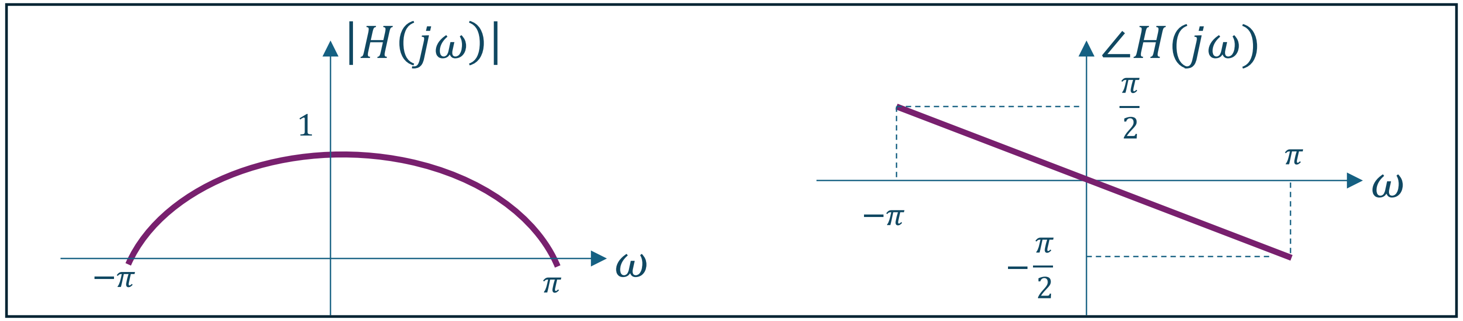
- This filter amplifies high frequencies and attenuate low frequencies.

FILTERING

- Another example is a filter that implements the media of the previous two inputs

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1]) \quad \Longrightarrow \quad H(e^{j\omega}) = \frac{1}{2}\left[1 + e^{-j\omega}\right] = e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$



$$x[n] = k = ke^{j0}$$

$$y[n] = H(e^{j0})ke^{j\omega 0n} = k = x[n]$$

$$x[n] = ke^{j\pi n} = k(-1)^n$$

$$y[n] = H(e^{j\pi})ke^{j\pi n} = 0$$

- This filter allow low frequencies to pass and attenuate high frequencies since the frequency response of discrete-time signals are 2π-periodic.

SELECTIVE-FREQUENCY FILTERS

- ❑ **Low-pass Filters** - allow low frequencies to pass and attenuate high frequencies;
- ❑ **High-pass Filters** - allow high frequencies to pass and attenuate low frequencies;
- ❑ **Band-pass Filters** - allow a range of frequencies to pass and attenuate higher and lower frequencies.

- ❑ The **cut-off frequency** is the frequency at which the behavior changes from the **passband** to the **rejection band** or vice-versa.

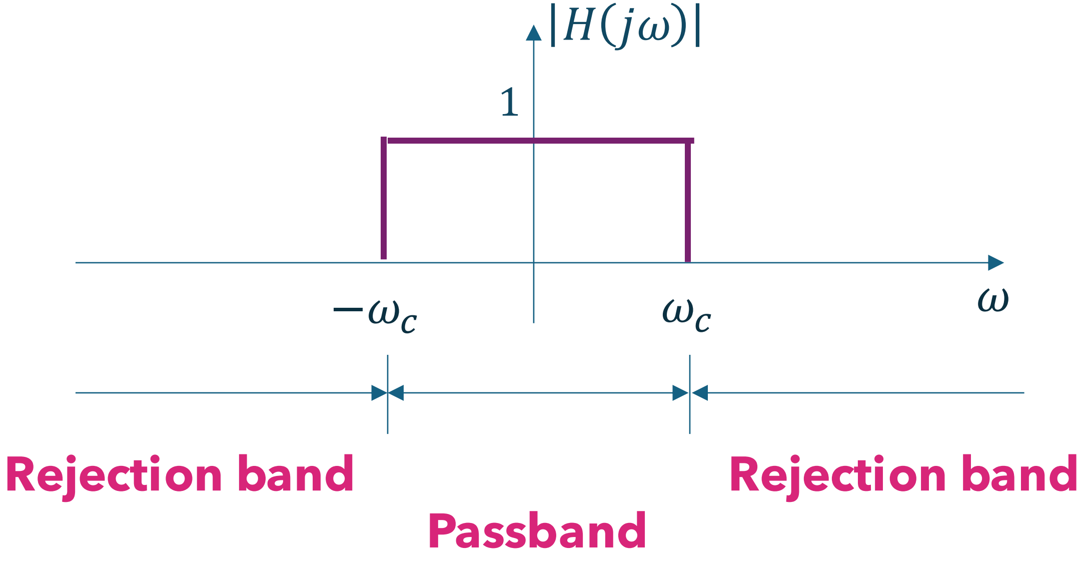
SELECTIVE-FREQUENCY FILTERS

- Ideal filters – do not distort the signal in the passband and have an abrupt change at the cut-off frequency.

- Ideal **low-pass filter**

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

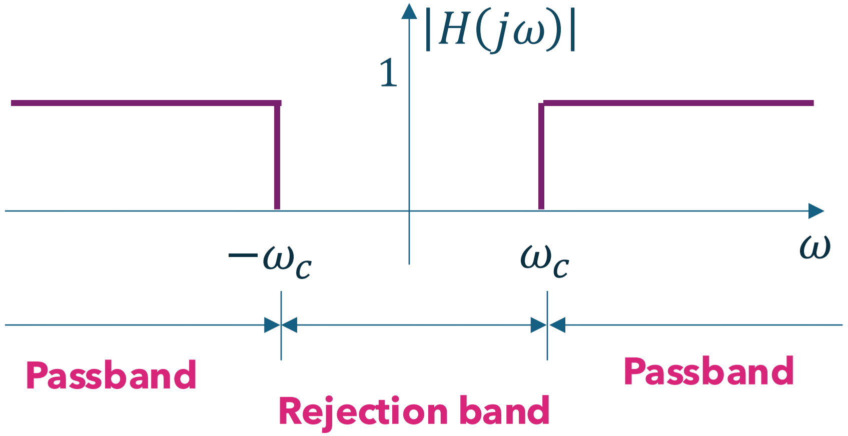
Cut-off frequency: ω_c



SELECTIVE-FREQUENCY FILTERS

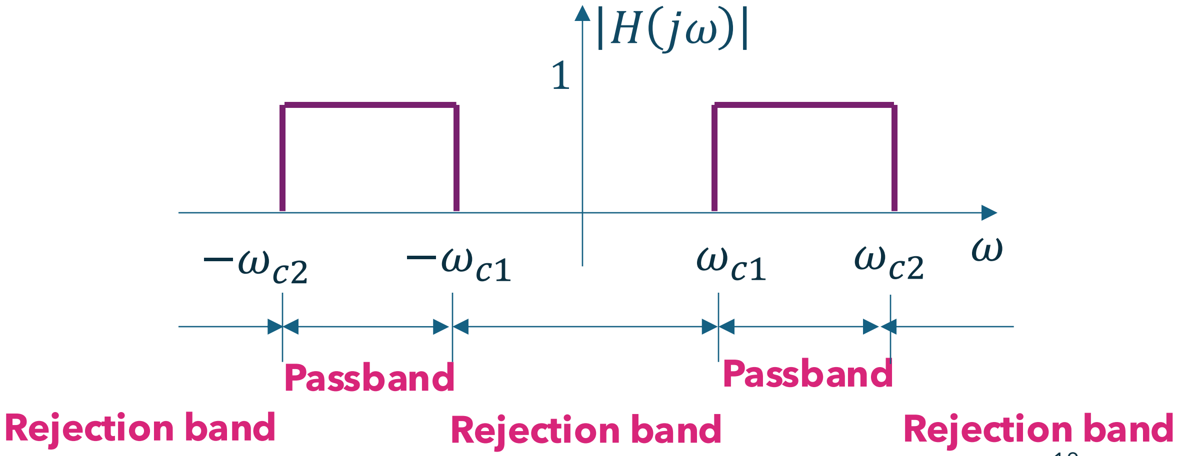
- Ideal **high-pass filter**

  Cut-off frequency: ω_c



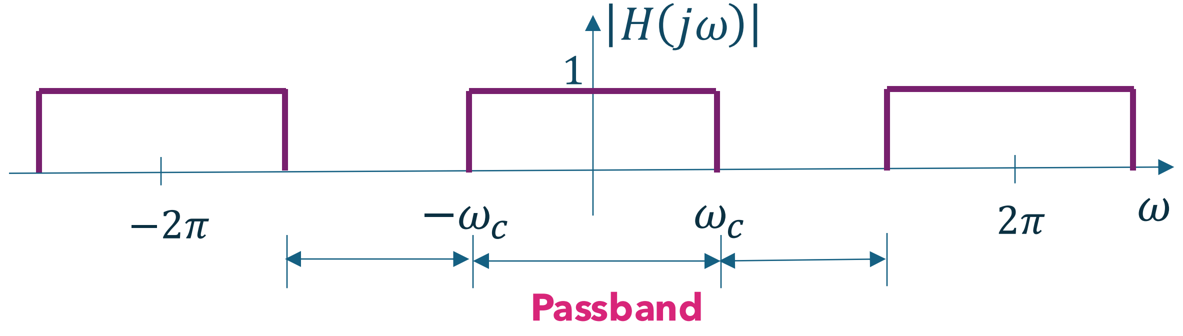
- Ideal **low-pass filter**

  Cut-off frequencies: ω_{c1}, ω_{c2}

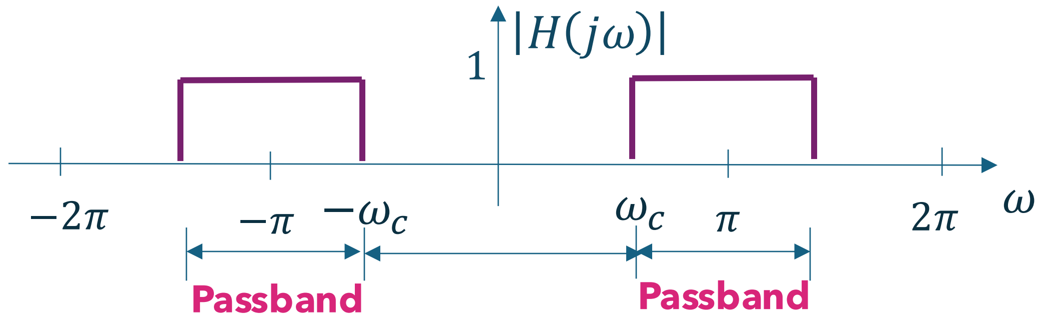


DT SELECTIVE-FREQUENCY FILTERS

- Ideal **low-pass filter**



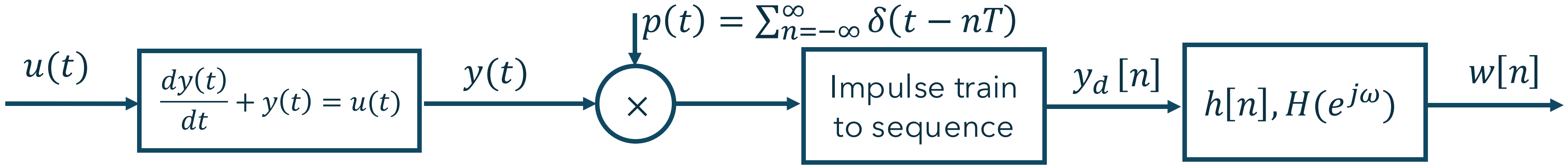
- Ideal **high-pass filter**



- Ideal **band-pass filter**

EXERCISES

1. For the system below and considering $u(t) = \delta(t)$,



Determine the signal $y(t)$, the frequency response $H(e^{j\omega})$ and the response to the unit impulse such that the output $w[n] = \delta[n]$.

2. Consider a signal $x[n]$ with Fourier Transform $X(e^{j\omega})$, such that $X(e^{j\omega}) = 0, for\ \frac{\pi}{4} \leq |\omega| \leq \pi$. Determine the frequency response $H(e^{j\omega})$ of a low-pass filter such that