EE-268

Sampled-Data Systems Through DLMI

Blocks 1 and 2 · Fundamentals and indirect design

Instituto Tecnológico de Aeronáutica ▫ August, 2026 ▫ Prof. Gabriela Werner Gabriel

This content is based on…

GEROMEL (2023) — Differential Linear Matrix Inequalities.

CHEN & FRANCIS (1995) — Optimal Sampled-Data Control Systems.

DUAN & YU (2013) – LMIs in Control Systems: analysis, design and applications.

CHEN, C.-T. (1999) — Linear System Theory and Design.

ROAD MAP

What we are going to study...

1

Fundamentals

Lyapunov stability, LMI, DLMI, and the differential Riccati equation, LMI solvers and mathematical tools.

Geromel, Chs. 1–2; Duan & Yu, Chs. 2–4

2

Discretization and Indirect Design

From the continuous plant to a discrete model, and controller design in discrete time.

Chen & Francis, Chs. 1–4

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BLOCK 1

Fundamentals

Lyapunov stability, LMI, DLMI, and the differential Riccati equation, LMI solvers and mathematical tools.

Geromel, Chs. 1–2; Duan & Yu, Chs. 2–4

Dynamical systems to be studied…

- **Discrete-time Dynamical System:**

$$\mathcal{D} := \begin{cases} \boldsymbol{x[k+1] = Ax[k] + Bu[k] + Ew[k]} & \boldsymbol{, x[0] = x_0} \\ \boldsymbol{z[k] = C_z x[k] + D_z u[k]} & \\ \boldsymbol{y[k] = C_y x[k]} & \end{cases}$$

- **Continuous-time Dynamical System:**

$$\mathcal{S} := \begin{cases} \boldsymbol{\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)} & \boldsymbol{, x(0) = x_0} \\ \boldsymbol{z(t) = C_z x(t) + D_z u(t)} & \\ \boldsymbol{y(t) = C_y x(t)} & \end{cases}$$

These systems evolve from initial condition x_0.

$x(\cdot)/x[\cdot]$: $\mathcal{T} \rightarrow \mathbb{R}^{n_x}$ is the state;
$u(\cdot)/u[\cdot]$: $\mathcal{T} \rightarrow \mathbb{R}^{n_u}$ is the control input;
$w(\cdot)/w[\cdot]$: $\mathcal{T} \rightarrow \mathbb{R}^{n_w}$ is the exogenous input;
$z(\cdot)/z[\cdot]$: $\mathcal{T} \rightarrow \mathbb{R}^{n_z}$ is the controlled/filtered output;
$y(\cdot)/y[\cdot]$: $\mathcal{T} \rightarrow \mathbb{R}^{n_y}$ is the measured output;

where $\mathcal{T} = \{\mathbb{R}_+, \mathbb{N}\}$.

Stability of dynamical systems is related to the eigenvalues of the system's dynamics (state transition) matrix — which, in turn, encompasses the system's modes of operation: the poles of the transfer function between each output and each input of the minimal realization of a system, determined by $(sI - A)^{-1}$ / $(zI - A)^{-1}$.

Lyapunov stability applied to discrete-time systems

- **Quadratic stability.** In the context of dynamical systems, Lyapunov's direct method plays an important role in stability verification and control or filtering design.

Definition 1.1. x_e is an equilibrium point for $\mathcal{D}$ if $x[k] = x_e$, $\forall\, k \geq k_e \geq 0$, whenever $x[k_e] = x_e$. If x_e is unique, then x_e is the global equilibrium point.

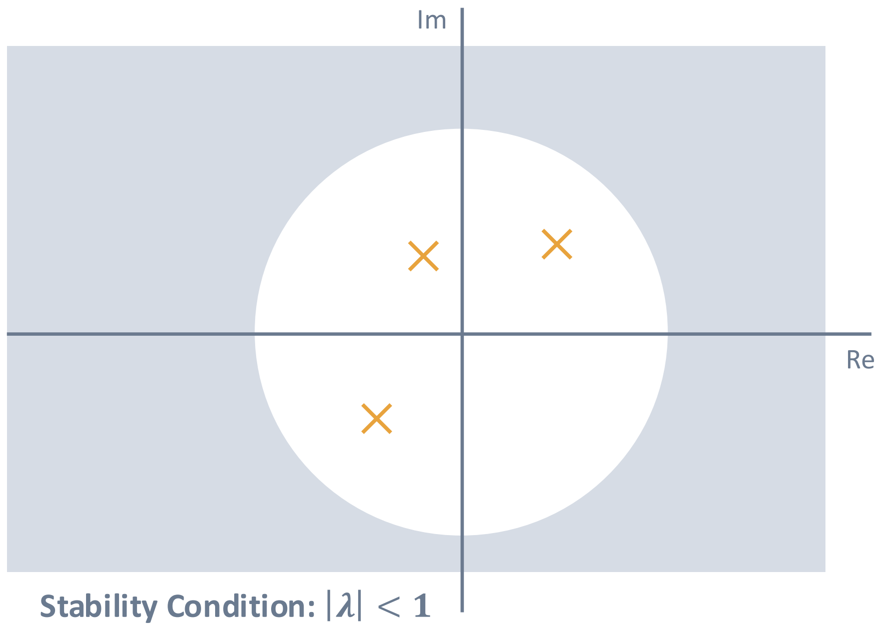
Definition 1.2. Let $V(x[k], k)$ be a real function of $x[k]$. $V(x[k], k)$ is a Lyapunov function if it is such that

$$V(x_e, k) = 0 \text{ and } V(x[k], k) > 0, \text{for } x[k] \neq x_e \qquad (1.1)$$

Moreover, for any $k \in \mathbb{N}$, $V(x[k], k)$ is such that

$$V(x[k+1], k+1) - V(x[k], k) \leq 0. \qquad (1.2)$$

Stability Region (Complex Plane)



Stability Condition: $|\lambda| < 1$

λ(A) — eigenvalues of A

Stability of discrete-time systems via LMIs

- **Lyapunov stability [Gabriel, 2016]**
  - If there exists $V(x[k], k)$, in the sense of Definition 1.2, then x_e is **STABLE**.
  - If there exists $V(x[k], k)$ that strictly satisfies Definition 1.2, then x_e is **ASYMPTOTICALLY STABLE**.
  - If x_e is asymptotically stable and, $\forall k \in \mathbb{N}$, $V(x[k], k)$ is such that $\lim_{|x|\to\infty} V(x, k) \to \infty$, then the equilibrium point is **GLOBALLY ASYMPTOTICALLY STABLE**.

- **LMI Condition:**

> **Theorem 1.3 [Geromel].** Let matrices $B \equiv 0$ and $E \equiv 0$ and $A \in \mathbb{R}^{n\times n}$ in $\mathcal{D}$ be given, along with a symmetric matrix variable $P = P' \in \mathbb{R}^{n\times n}$, such that
> $$A'PA - P \prec 0, \qquad \mathrm{P} \succ 0 \tag{1.3}$$
> is feasible if and only if A is Schur stable.

- **Schur Stability**: Let matrices $B \equiv 0$ and $E \equiv 0$ be given, in the state transition equation of $\mathcal{D}$. If the eigenvalues of $A \in \mathbb{R}^{n\times n}$ are within the unit circle of the complex plane, then A is Schur stable.

- **LMI (Linear Matrix Inequality):** LMI constraints define a convex region of solutions for optimization problems – This allows stability and performance to be addressed via convex optimization.

Lyapunov stability applied to continuous-time systems

- **Now,** consider the continuous-time system $\mathcal{S}$

Definition 1.4. x_e is an equilibrium point for $\mathcal{S}$ if $x(t) = x_e$, $\forall\, t \geq t_e \geq 0$, whenever $x(t_e) = x_e$. If x_e is unique, then x_e is the global equilibrium point.

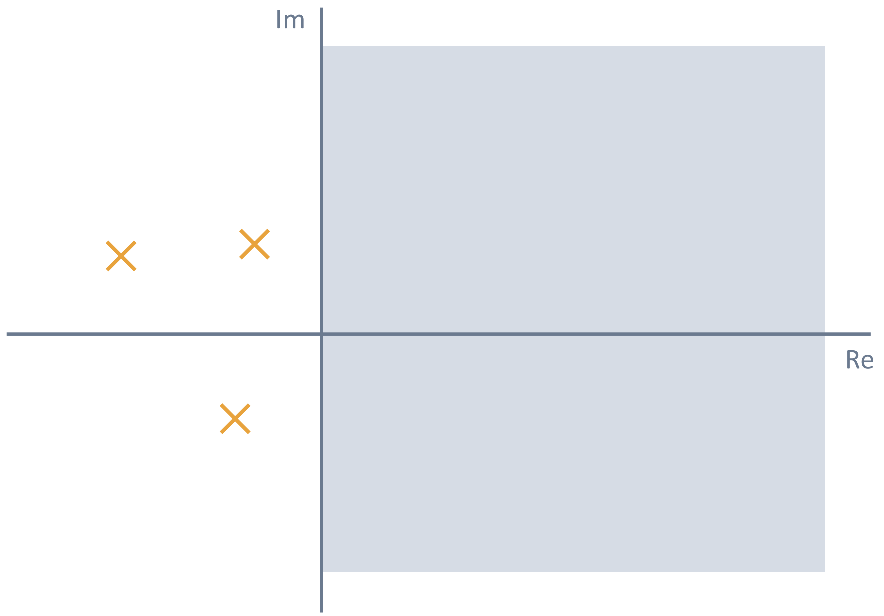
Definition 1.5. Let $V(x(t), t)$ be a real function of $x(t)$. $V(x(t), t)$ is a Lyapunov function if it is such that

$$V(x_e, t) = 0 \text{ and } V(x(t), t) > 0, \text{for } x(t) \neq x_e \qquad (1.4)$$

Moreover, for any t $\in \mathbb{R}_+$, $V(x(t), t)$ is such that

$$\dot{V}(x(t), t) \leq 0. \qquad (1.5)$$

Stability Region (Complex Plane)



Stability Condition: $Re(\lambda) < 0$

λ(A) — eigenvalues of A

Stability of continuous-time systems via LMIs

- **Lyapunov stability [Gabriel, 2016]**
  - If there exists $V(x(t), t)$, in the sense of Definition 1.5, then x_e is **STABLE**.
  - If there exists $V(x(t), t)$ that strictly satisfies Definition 1.5, then x_e is **ASYMPTOTICALLY STABLE**.
  - If x_e is asymptotically stable and, $\forall t \in \mathbb{R}_+$, $V(x(t), t)$ is such that $\lim_{|x| \to \infty} V(x, t) \to \infty$, then the equilibrium point is **GLOBALLY ASYMPTOTICALLY STABLE**.

- **LMI Condition:**

> **Theorem 1.6 [Geromel].** Let matrices $B \equiv 0$ and $E \equiv 0$ and $A \in \mathbb{R}^{n \times n}$ in $\mathcal{S}$ be given, along with a symmetric matrix variable $P = P' \in \mathbb{R}^{n \times n}$, such that
> $$A'P + PA \prec 0, \qquad P \succ 0 \qquad (1.6)$$
> is feasible if and only if A is Hurwitz stable.

- **Hurwitz Stability**: Let matrices $B \equiv 0$ and $E \equiv 0$ be given, in the state transition equation of $\mathcal{S}$. If the eigenvalues of $A \in \mathbb{R}^{n \times n}$ have negative real parts, then A is Hurwitz stable.

Matrix definiteness

- **What does** $M \succ 0$ **actually mean?**

Definition 1.7. A symmetric matrix $M = M^T \in \mathbb{R}^{n \times n}$ is positive definite **($M \succ 0$)** if and only if $x'Mx > 0$ for every $x \neq 0$. Similarly, it is positive semidefinite **($M \succeq 0$)** if and only if $x'Mx \geq 0$ for every $x \in \mathbb{R}^n$.

Equivalent characterization. $\boldsymbol{M \succ 0}$ if and only if every eigenvalue of $M \in \mathbb{R}^{n \times n}$ is strictly positive: $\lambda_i(M) > 0$ for all $i = 1,2,\ldots,n$.
Likewise, $\boldsymbol{M \succcurlyeq 0}$ if and only if $\lambda_i(M) \geq 0$ for all $i = 1,2,\ldots,n$.

EXAMPLE — Consider $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Its eigenvalues are 2 and 3, both positive, so $M \succ 0$ (positive definite).

• • • •

EXERCISE – Find a positive definite matrix $P = P' \in \mathbb{R}^{2\times 2}$ such that

$$A'P + PA + I = 0; A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

LMI definition

- **LMI.** Matrix inequalities which are linear in the matrix variables. Its general form is

$$\mathcal{L}(X) = \sum_{i=1}^{\ell} (E_i{}'XF_i + F_i{}'X'E_i) + Q \qquad (1.7)$$

$\mathcal{L}(\cdot)$ is a linear operator in the variable $X \in \mathbb{R}^{m\times n}$, with given $Q = Q' \in \mathbb{R}^{n\times n}$, $E_i \in \mathbb{R}^{m\times n}$, and $F_i \in \mathbb{R}^{n\times n}$, for $i \in \{1,2,\dots,\ell\}$. The constraint "$\mathcal{L}(X) \succ 0$" requires every eigenvalue of the symmetric matrix $\mathcal{L}(X)$ to be strictly positive.

- **Alternatively**, the matrix inequality is expressed as

$$A(x) = A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n \succ 0 \qquad (1.8)$$

where $A_i = A_i{}' \in \mathbb{R}^{n\times n}, i = 0,1,2\dots,n$. x_i are the decision variables.
Notice all decision variables can be stacked into a vector.

- All sets are convex:

$$\{x : A(x) \succ 0\}$$
$$\{x : A(x) \succcurlyeq 0\}$$
$$\{x : A(x) \prec 0\}$$
$$\{x : A(x) \preccurlyeq 0\}$$

Optimization problem based on LMI

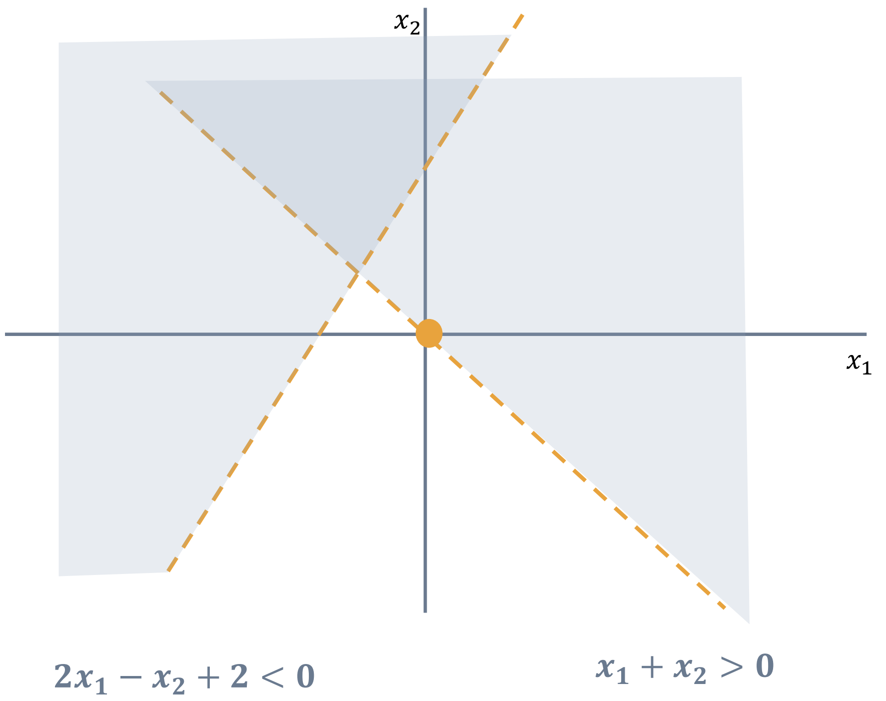
- In general, the main issue regarding convex problems based on LMIs is the feasibility of proposed constraints, which can be expressed through an optimization problem.

- **A linear optimization problem** is expressed by

$$\begin{cases} \min/\inf & J(X) \\ s.t. & \mathcal{L}(X,Q) \succ 0 \end{cases} \tag{1.9}$$

> ***EXAMPLE —*** *Consider the problem of minimizing x_2, such that $x_1 + x_2 > 0$ and $2x_1 - x_2 + 2 < 0$.*
>
> *Should the solution be different if we minimize x_1 instead?*

Feasible Region (x1, x2 Plane)



Programming LMI problems

Interpreter

A modeling layer that lets you describe an optimization problem in natural, algebra-like syntax — variables, constraints, objective — and automatically translates it into the standard form a solver expects.

e.g., YALMIP, CVX

Solver

The numerical engine that actually solves the resulting convex optimization problem and returns a certificate of feasibility or an optimal solution.

e.g., Mosek, SeDuMi, SDPT3, LMILab

Some LMI solvers and interpreter...

1

Using MATLAB

1. Verify if Robust Control Toolbox is available. It contains the LMILab. (https://matlab.mathworks.com/)

Free, open-source interpreter

2

Install YALMIP

1. Download YALMIP from yalmip.github.io.
2. Add the extracted folder and subfolders to your MATLAB path.
3. Run yalmiptest in MATLAB to verify.

Free, open-source interpreter

3

Install Mosek

1. Get a free academic license at mosek.com.
2. Install Mosek and add its toolbox to the MATLAB path.
3. Run mosekopt in MATLAB to verify.

Free academic license required

Programming with LMI toolboxes

Program 1 — from foundation to code

PROG. 1

Introduction to LMI solvers and interpreters

Install YALMIP + LMILab/Mosek. Consider the optimization problem $\begin{cases} \min & x_2 \\ s.t.. & x_1 + x_2 > 0 \\ & 2x_1 - x_2 + 2 < 0 \end{cases}$

1) Obtain one feasible solution for (x_1, x_2).

2) Obtain the minimum value of x_2 that solves the problem above.

3) Investigate the solution of the problem $\begin{cases} \min & x_1 \\ s.t.. & x_1 + x_2 > 0 \\ & 2x_1 - x_2 + 2 < 0 \end{cases}$

Schur complement

Definition 1.8. Let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be a partitioned matrix,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad (1.10)$$

1. $A_{22} - A_{21}A_{11}^{-1}A_{12}$, with A_{11} nonsingular, is the **Schur complement** of A with respect to block A_{11}.
2. $A_{11} - A_{12}A_{22}^{-1}A_{21}$, with A_{22} nonsingular, is the **Schur complement** of A with respect to block A_{22}

- To verify Lemma 1.9 consider $\tilde{A} = T_1 A T_2$, where

$$T_1 = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}, \qquad T_2 = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

Lemma 1.9. Let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10). The following are equivalent.

1. For A_{11} nonsingular,

$$A \mapsto \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \qquad (1.11)$$

2. For A_{22} nonsingular,

$$A \mapsto \begin{bmatrix} A_{22} & 0 \\ 0 & A_{11} - A_{12}A_{22}^{-1}A_{21} \end{bmatrix} \qquad (1.12)$$

Lemma 1.10. Let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10) with $A_{12} = A_{21}'$. The following hold.

1. $A \succ 0 \Leftrightarrow A_{11} \succ 0, A_{22} - A_{12}'A_{11}^{-1}A_{12} \succ 0$
2. $A \succ 0 \Leftrightarrow A_{22} \succ 0, A_{11} - A_{12}A_{22}^{-1}A_{12}' \succ 0$
3. $A \prec 0 \Leftrightarrow A_{11} \prec 0, A_{22} - A_{12}'A_{11}^{-1}A_{12} \prec 0$
4. $A \prec 0 \Leftrightarrow A_{22} \prec 0, A_{11} - A_{12}A_{22}^{-1}A_{12}' \prec 0$

Inversion Lemma

Woodbury matrix identity. Let $\mathcal{A} \in \mathbb{R}^{n\times n}, \mathcal{B} \in \mathbb{R}^{m\times m}, \mathcal{U} \in \mathbb{R}^{n\times m}, \mathcal{V} \in \mathbb{R}^{m\times n}$, then, for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} + \mathcal{U}\mathcal{B}^{-1}\mathcal{V}$ nonsingular matrices,

$$(\mathcal{A} + \mathcal{U}\mathcal{B}^{-1}\mathcal{V})^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{U}(\mathcal{B} + \mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1} \tag{1.13}$$

Lemma 1.11. Now, let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10), then

$$A^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \tag{1.14}$$

By invoking the Woodbury identity other formulas can be derived to handle the cases $where$ A_{11} or A_{22} is not invertible.

Projection Lemma

Definition 1.12. Let $A \in \mathbb{R}^{m\times n}$,

1. $\boldsymbol{A}_{\vdash}$ is the left orthogonal complement if
$$\boldsymbol{A}_{\vdash}\boldsymbol{A} = \mathbf{0}, \qquad \boldsymbol{rank}(\boldsymbol{A}_{\vdash}) = \boldsymbol{m} - \boldsymbol{rank}(\boldsymbol{A})$$

2. $\boldsymbol{A}_{\dashv}$ is the right orthogonal complement if
$$\boldsymbol{A}\boldsymbol{A}_{\dashv} = \mathbf{0}, \qquad \boldsymbol{rank}(\boldsymbol{A}_{\dashv}) = \boldsymbol{n} - \boldsymbol{rank}(\boldsymbol{A})$$

Lemma 1.13. Let $Q = Q' \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times n}$. There exists a matrix $X \in \mathbb{R}^{r\times m}$ such that

$$Q + A'X'B + B'XA \prec 0$$

if and only if

$A_{\dashv}'QA_{\dashv} \prec 0$ and $B_{\dashv}'QB_{\dashv} \prec 0$

Reciprocal projection Lemma

Lemma 1.14 (Apkarian 20000). Let $Q = Q' \in \mathbb{R}^{n\times n}$, and $P = P' \in \mathbb{R}^{n\times n}$ be given, with $P \succ 0$. There exists a matrix $S \in \mathbb{R}^{n\times n}$ such that

$$Q + S + S' \prec 0$$

if and only if there exists $W \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} Q + P - (W + W') & S' + W' \\ S + W & -P \end{bmatrix} \prec 0$$

Similarity and congruence transformation

Definition 1.15. Consider $Q = Q' \in \mathbb{R}^{n\times n}$ and a congruence transformation on Q, $T \in \mathbb{R}^{n\times n}$, with $rank(T) = n$. Hence,

$$Q \succ 0 \iff TQT' \succ 0 \qquad (1.15)$$

Definition 1.16. Consider $Q = Q' \in \mathbb{R}^{n\times n}$ and a similarity transformation, $T \in \mathbb{R}^{n\times n}$. If there exists T^{-1}, hence,

$$\lambda(Q) = \lambda(TQT^{-1}) \qquad (1.16)$$

- A congruence transformation preserves the **definiteness** (sign) of Q, but not necessarily its eigenvalues.

- A similarity transformation preserves the eigenvalues of $Q \succ 0$. In this case, symmetry is not ensured anymore.

Elimination Lemma

Lemma 1.17. Let $Q = Q' \in \mathbb{R}^{n\times n}$, and $X \in \mathbb{R}^{n\times m}$ and $Y \in \mathbb{R}^{m\times n}$ be given and let

$$\mathbb{F} = \{F : F \in \mathbb{R}^{m\times m}, F'F \preccurlyeq I\} \qquad \textbf{(1.17)}$$

For any $F \in \mathbb{F}$,

$$Q + XFY + Y'F'X' \prec 0$$

if and only if, there exists a scalar $\delta > 0$, such that

$$Q + \delta XX' + \delta^{-1}Y'Y \prec 0.$$

Lemma 1.18. Consider a partitioned matrix $A = A' \in \mathbb{R}^{(n+m)\times(n+m)}$ as in (1.10). There exists a matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A_{11} - X & A_{12} & X \\ A_{12}' & A_{22} & 0 \\ X & 0 & -X \end{bmatrix} \prec 0$$

if and only if $A \prec 0$.

Other useful Lemmas

Lemma 1.19. Let $X, Y \in \mathbb{R}^{m \times n}$, and $F = F' \in \mathbb{R}^{m \times m}$, $F \succ 0$ and a scalar $\delta > 0$. then

$$X'FY + Y'FX \preccurlyeq \delta X'FX + \delta^{-1} Y'FY$$

Lemma 1.20. Let be given $F \in \mathbb{F}$ as in (1.17). Let $X, Y \in \mathbb{R}^{m \times n}$ and a scalar $\delta > 0$. then

$$X'FY + Y'F'X \preccurlyeq \delta X'X + \delta^{-1} Y'Y$$

Lemma 1.21. Let be given $F \in \mathbb{F}$ as in (1.17). Let $X, Y \in \mathbb{R}^{m \times n}$, $Q \in \mathbb{R}^{n \times n}$, then

$$Q + X'FY + Y'F'X \prec 0, \forall F \in \mathbb{F}$$

if and only if there exists a scalar $\delta > 0$, such that

$$Q + \delta X'X + \delta^{-1} Y'Y \prec 0$$

Problem definition for a norm-based design

- Let us consider both original LTI systems: the discrete-time dynamical system

$$\mathcal{D} := \begin{cases} x[k+1] = Ax[k] + Bu[k] + Ew[k] & , x[0] = x_0 \\ z[k] = C_z x[k] + D_z u[k] \\ y[k] = C_y x[k] \end{cases}$$

and the continuous-time one:

$$\mathcal{S} := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t) & , x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \\ y(t) = C_y x(t) \end{cases}$$

presented before.

- We want to design a control law $u(t)$ such that the performance index, for $\gamma \in \mathbb{R}_+$,

$$\mathcal{J} = \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \tag{1.18}$$

For every $0 \neq w \in L_2/\ell_2$. In this case, the symbol $\|\cdot\|_2^2$ stand for the squared 2-norm of a signal, which is defined as $\|x\|_2^2 = \int_0^\infty x(\tau)'x(\tau)d(\tau)$, with $x \in L_2$, for a continuous-time signal, or $\|x\|_2^2 = \sum_{n=0}^\infty x[n]'x[n]$, with $x \in \ell_2$, for a discrete-time one.

$\mathcal{H}_2$ norm of a system

Definition 1.22. For $H_c(s)$ and A Hurwitz, the $\mathcal{H}_2$ norm is defined by

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2$$

with z_i the impulse response of the system $\mathcal{S}$, with $u \equiv 0$ and $x_0 = 0$, to the impulse $w = e_i\delta(t)$ applied to the i-th channel.

Definition 1.23. For $H_d(z)$ and A Schur, the $\mathcal{H}_2$ norm is defined by

$$\|H_d\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2$$

with z_i the impulse response of the system $\mathcal{D}$, with $u \equiv 0$ and $x_0 = 0$, to the impulse $w = e_i\delta[k]$ applied to the i-th channel.

- The $\mathcal{H}_2$ norm of $\mathcal{S}$ is equivalently calculated through

$$\|H_c\|_2^2 = tr(E'PE)$$

with $A'P + PA = -C_z{}'C_z$, and $P \succ 0$.

- The $\mathcal{H}_2$ norm of $\mathcal{D}$ is equivalently calculated through

$$\|H_d\|_2^2 = tr(E'PE)$$

with $A'PA - P = -C_z{}'C_z$, and $P \succ 0$.

- H_c and H_d are the transfer functions of the systems $\mathcal{S}$ and $\mathcal{D}$, respectively, both with $u \equiv 0$.

$$H_c(s) = C_z(sI - A)^{-1}E \tag{1.19}$$
$$H_d(z) = C_z(zI - A)^{-1}E \tag{1.20}$$

$\mathcal{H}_\infty$ norm of a system

Definition 1.24. For $H_c(s)$ and A Hurwitz, the $\mathcal{H}_\infty$ norm is defined by

$$\|H_c\|_\infty^2 = \sup_{0\neq w\in L_2} \frac{\|z\|_2^2}{\|w\|_2^2}$$

Definition 1.25. For $H_d(z)$ and A Schur, the $\mathcal{H}_\infty$ norm is defined by

$$\|H_d\|_\infty^2 = \sup_{0\neq w\in \ell_2} \frac{\|z\|_2^2}{\|w\|_2^2}$$

- From Bounded Real Lemma

$$\|H_c\|_\infty^2 = \inf\left\{\mu : P \succ 0, \begin{bmatrix} A'P + PA & PE & Cz' \\ \star & -\mu I & 0 \\ \star & \star & -I \end{bmatrix} \prec 0\right\} \tag{1.21}$$

and

$$\|H_d\|_\infty^2 = \inf\left\{\mu : P \succ 0, \begin{bmatrix} A'PA - P & A'PE & Cz' \\ \star & E'PE - \mu I & 0 \\ \star & \star & -I \end{bmatrix} \prec 0\right\} \tag{1.22}$$

are equivalent to definitions 1.24 and 1.25, respectively.

Continuous-time performance index calculation

- For that purpose, for the existence of the continuous-time performance index $\mathcal{J}$, it is necessary that the system $\mathcal{S}$ to be stable. Consider that there exists a Lyapunov function of the form $v(x) = x'Px$, with $0 \prec P \in \mathbb{R}^{n\times n}$, whose derivative is $\dot{v}(x) < 0$. As a consequence, $\mathcal{S}$ is Hurwitz stable and so we can determine exact value of the performance index $\mathcal{J}$ by imposing

$$\dot{v}(x) < -z'z + \gamma^2 w'w \tag{1.23}$$

- Indeed, condition (1.5) remains valid and strictly satisfied for $w \equiv 0$. Moreover, for $x_0 \neq 0$,

$$\int_0^\infty \dot{v}(x)dt < -\mathcal{J} \ \Rightarrow \mathcal{J} < v(x_0) = x_0'Px_0 \tag{1.24}$$

- It is important to notice that for a LMI based norm calculation, the problem to be solved is expressed such as

$$\min_{P>0,}\{x_0'Px_0 \ : \ (1.23)\} \ \ / \min_{P>0,\mu=\gamma^2>0}\{\mu\text{: } (1.23)\} \tag{1.25}$$

Continuous-time state-feedback norm-based design

- Consider the system is feedbacked with a state dependent control law $u(t) = Kx(t)$,

$$\dot{v}(x) < -z'z + \gamma^2 w'w$$
$$\Downarrow$$
$$[(A+BK)x + Ew]'Px + x'P[(A+BK)x + Ew] < -x'(C_z + D_zK)'(C_z + D_zK)x + \gamma^2 w'w$$
$$\Downarrow$$
$$\begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} (A+BK)'P + P(A+BK) + (C_z + D_zK)'(C_z + D_zK) & PE \\ E'P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$
$$\Downarrow$$
$$\begin{bmatrix} (A+BK)'P + P(A+BK) & (C_z + D_zK)' & PE \\ (C_z + D_zK) & -I & 0 \\ E'P & 0 & -\gamma^2 I \end{bmatrix} \prec 0$$

which is a non-linear matrix inequality.

Continuous-time state-feedback norm-based design

- Linearizing the previous inequality by adopting the congruence transformation $diag(P^{-1}, I, I)$ and the one-to-one transformation of variables $W = P^{-1}$ and $Y = KW$, it follows that (1.19) can be rewritten as

$$\begin{bmatrix} WA' + AW + Y'B' + BY & WC_z' + Y'D_z' & E \\ \star & -I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} \prec 0 \tag{1.26}$$

- For the context of the $\mathcal{H}_2$ norm, for which system $\mathcal{S}$ is defined for $x_0 \neq 0$ and $w \equiv 0$, the objective function can be rewritten in the new set of variables as

$$N - E'W^{-1}E \succ 0 \quad \Rightarrow \quad \begin{bmatrix} N & E' \\ \star & W \end{bmatrix} \succ 0 \tag{1.27}$$

- Notice that the response of the systems $\mathcal{S}$ with $x_0 = Ee_i$ and $w \equiv 0$ equals the response of $\mathcal{S}$ with $x_0 = 0$ and $w \equiv \delta(t)$.

Continuous-time state-feedback norm-based design

Theorem 1.26. The close loop system with transfer function $H_c(s)$ is Globally Asymptotically Stable and $\|H_c\|_2^2 = \min_{P,Y,N} tr(N)$, if and only if, there exist matrices W= $W' \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_w}$, and $Y \in \mathbb{R}^{n_u \times n_x}$, such that W$\succ 0$, (1.27), and (1.26) deleting the third column and row. In this case the optimal state-feedback gain is given by $K = YW^{-1}$.

Theorem 1.27. The closed loop system with transfer function $H_c(s)$, is Globally Asymptotically Stable and $\|H_c\|_\infty = \min_{W,\gamma>0} \gamma$, if and only if, there exist matrices W= $W' \in \mathbb{R}^{n_x \times n_x}$ and $Y \in \mathbb{R}^{n_u \times n_x}$, such that W$\succ 0$ and (1.22) hold. In this case the optimal state-feedback gain is given by $K = YW^{-1}$.

Programming with LMI toolboxes

Program 2 — continuous-time state-feedback control design problem

PROG. 2

Continuous-time LTI State-Feedback Control Design

Determine the continuous-time state-feedback gain that minimizes the $\mathcal{H}_2$ norm for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 2u$$

and calculate the $\mathcal{H}_2$ norm.

Now, determine the continuous-time state-feedback gain that minimizes the $\mathcal{H}_\infty$ norm considering the same system. Calculate the $\mathcal{H}_\infty$ norm.

BLOCK 2

02

Discretization and Indirect Design

Step-invariant transformation, the effect of sampling, and controller design in the discrete-time domain

Chen & Francis, Chs. 1–4

Theoretical Exercises

E1) Derive the discrete-time equivalent formulation for Theorem 1.xx.

E2) Obtain an LMI formulation for the inequality

$$(\mathrm{X}^{-1} + Y^{-1})^{-1} \succ Z^{-1} + W'W$$

in the decision variables X, Y, Z and W.

E3) Let A and B be symmetric matrices of the same dimension. Show that

- $A \succ B \Longrightarrow \lambda_{max}(A) > \lambda_{max}(B)$
- $\lambda_{max}(A + B) \leq \lambda_{max}(A) + \lambda_{max}(B)$

Computational Exercises

Discrete-time state-feedback norm-based design

C1) Determine the discrete-time state-feedback gain that stabilizes and minimizes the $\mathcal{H}_2$ norm for the system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0.5 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0.2u$$

Determine also the discrete-time state-feedback gain and calculate the $\mathcal{H}_\infty$ norm that minimizes the $\mathcal{H}_\infty$ norm considering the same system. Present the code, the values of K, $\|\mathcal{D}\|_2$, and $\|\mathcal{D}\|_\infty$, and plot the signals of $x[k]$, $u[k]$, and $z[k]$ for each case.