EE-268

Sampled-Data Systems Through DLMI

Blocks 1 and 2 · Fundamentals and indirect design

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BIBLIOGRAPHY

This content is based on…

GEROMEL (2023) — Differential Linear Matrix Inequalities.

CHEN & FRANCIS (1995) — Optimal Sampled-Data Control Systems.

DUAN & YU (2013) – LMIs in Control Systems: analysis, design and applications.

CHEN, C.-T. (1999) — Linear System Theory and Design.

GOEBEL, TEEL & SANFELICE (2009) — Hybrid Dynamical Systems.

ROAD MAP

What we are going to study...

1

Fundamentals

Lyapunov stability, LMI, LMI solvers and mathematical tools. Stability of LTI systems. Norm definitions. Norm-based control design of CT and DT LTI systems. Algebraic Lyapunov and Riccati Equations (ALE and ARE).

Geromel, Chs. 1–2; Duan & Yu, Chs. 2–4

2

Sampled-data systems

From the continuous-time plant to a discrete-time model. Sampled-data systems in the hybrid approach. Differential Lyapunov and Riccati Equation (DLE and DRE).

Chen & Francis, Chs. 1–4

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BLOCK 1

Fundamentals

Stability of LTI systems. Norm definitions. Norm-based control design of CT and DT LTI systems. Algebraic Lyapunov and Riccati Equations (ALE and ARE).

Geromel, Chs. 1–2; Duan & Yu, Chs. 2–4

Dynamical systems to be studied…

- **Discrete-time Dynamical System:**

$$\mathcal{D} := \begin{cases} \boldsymbol{x[k+1] = Ax[k] + Bu[k] + Ew[k]} & , \boldsymbol{x[0] = x_0} \\ \boldsymbol{z[k] = C_z x[k] + D_z u[k]} \\ \boldsymbol{y[k] = C_y x[k] + D_y u[k]} \end{cases}$$

- **Continuous-time Dynamical System:**

$$\mathcal{S} := \begin{cases} \boldsymbol{\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)} & , \boldsymbol{x(0) = x_0} \\ \boldsymbol{z(t) = C_z x(t) + D_z u(t)} \\ \boldsymbol{y(t) = C_y x(t) + D_y u(t)} \end{cases}$$

These systems evolve from initial condition x_0.

$x(\cdot)/x[\cdot]$: $\mathcal{T} \to \mathbb{R}^{n_x}$ is the state;
$u(\cdot)/u[\cdot]$: $\mathcal{T} \to \mathbb{R}^{n_u}$ is the control input;
$w(\cdot)/w[\cdot]$: $\mathcal{T} \to \mathbb{R}^{n_w}$ is the exogenous input;
$z(\cdot)/z[\cdot]$: $\mathcal{T} \to \mathbb{R}^{n_z}$ is the controlled/filtered output;
$y(\cdot)/y[\cdot]$: $\mathcal{T} \to \mathbb{R}^{n_y}$ is the measured output;

where $\mathcal{T} = \{\mathbb{R}_+, \mathbb{N}\}$.

Stability of dynamical systems is related to the eigenvalues of the system's dynamics (state transition) matrix — which, in turn, encompasses the system's modes of operation: the poles of the transfer function between each output and each input of the minimal realization of a system, determined by $(sI - A)^{-1}$ / $(zI - A)^{-1}$.

Lyapunov stability applied to discrete-time systems

- **Quadratic stability.** In the context of dynamical systems, Lyapunov's direct method plays an important role in stability verification and control or filtering design.

Definition 1.1. x_e is an equilibrium point for $\mathcal{D}$ if $x[k] = x_e$, $\forall\, k \geq k_e \geq 0$, whenever $x[k_e] = x_e$. If x_e is unique, then x_e is the global equilibrium point.

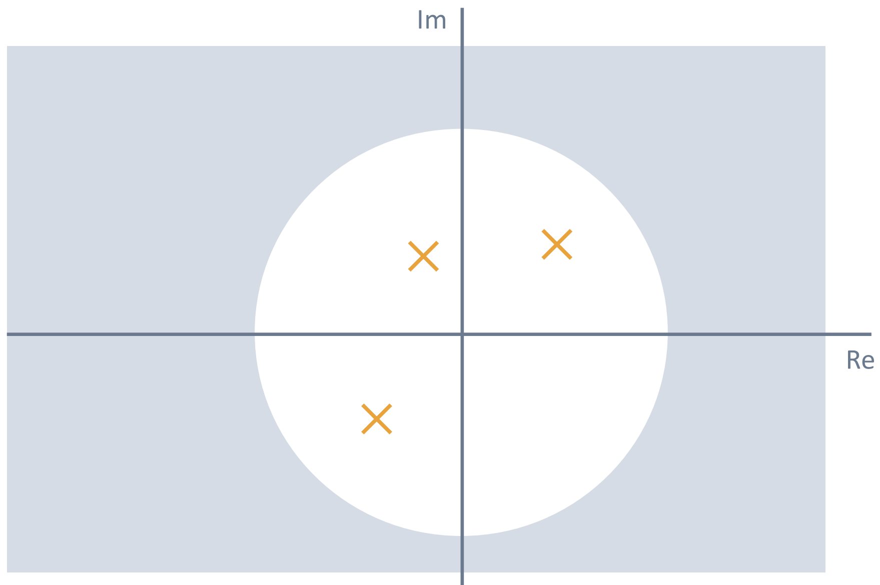
Definition 1.2. Let $V(x[k], k)$ be a real function of $x[k]$. $V(x[k], k)$ is a Lyapunov function if it is such that

$$V(x_e, k) = 0 \text{ and } V(x[k], k) > 0, \text{for } x[k] \neq x_e \qquad (1.1)$$

Moreover, for any $k \in \mathbb{N}$, $V(x[k], k)$ is such that

$$V(x[k+1], k+1) - V(x[k], k) \leq 0. \qquad (1.2)$$

Stability Region (Complex Plane)



Stability Condition: $|\lambda| < 1$

λ(A) — eigenvalues of A

Stability of discrete-time systems via LMIs

- **Lyapunov stability [Gabriel, 2016]**
  - If there exists $V(x[k], k)$, in the sense of Definition 1.2, then x_e is **STABLE**.
  - If there exists $V(x[k], k)$ that strictly satisfies Definition 1.2, then x_e is **ASYMPTOTICALLY STABLE**.
  - If x_e is asymptotically stable and, $\forall k \in \mathbb{N}$, $V(x[k], k)$ is such that $\lim_{|x| \to \infty} V(x, k) \to \infty$, then the equilibrium point is **GLOBALLY ASYMPTOTICALLY STABLE**.

- **LMI Condition:**

> **Theorem 1.3 [Geromel].** Let matrices $B \equiv 0$ and $E \equiv 0$ and $A \in \mathbb{R}^{n \times n}$ in $\mathcal{D}$ be given, along with a symmetric matrix variable $P = P' \in \mathbb{R}^{n \times n}$, such that
> $$A'PA - P \prec 0, \qquad \mathrm{P} \succ 0 \qquad \textbf{(1.3)}$$
> is feasible if and only if A is Schur stable.

- **Schur Stability**: Let matrices $B \equiv 0$ and $E \equiv 0$ be given, in the state transition equation of $\mathcal{D}$. If the eigenvalues of $A \in \mathbb{R}^{n \times n}$ are within the unit circle of the complex plane, then A is Schur stable.

- **LMI (Linear Matrix Inequality):** LMI constraints define a convex region of solutions for optimization problems – This allows stability and performance to be addressed via convex optimization.

Lyapunov stability applied to continuous-time systems

- **Now,** consider the continuous-time system $\mathcal{S}$

Definition 1.4. x_e is an equilibrium point for $\mathcal{S}$ if $x(t) = x_e$, $\forall\, t \geq t_e \geq 0$, whenever $x(t_e) = x_e$. If x_e is unique, then x_e is the global equilibrium point.

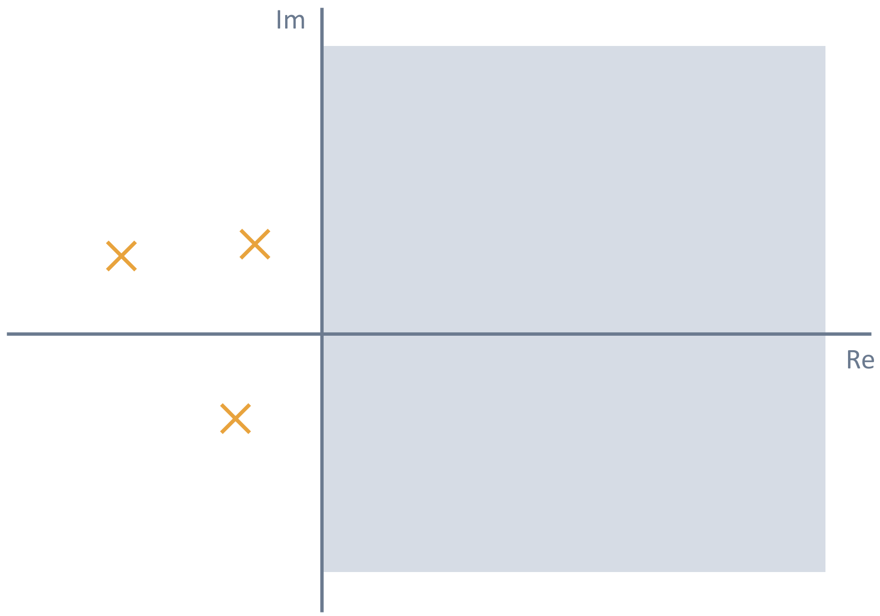
Definition 1.5. Let $V(x(t), t)$ be a real function of $x(t)$. $V(x(t), t)$ is a Lyapunov function if it is such that

$$V(x_e, t) = 0 \text{ and } V(x(t), t) > 0, \text{for } x(t) \neq x_e \qquad (1.4)$$

Moreover, for any t $\in \mathbb{R}_+$, $V(x(t), t)$ is such that

$$\dot{V}(x(t), t) \leq 0. \qquad (1.5)$$

Stability Region (Complex Plane)



Stability Condition: $Re(\lambda) < 0$

λ(A) — eigenvalues of A

Stability of continuous-time systems via LMIs

- **Lyapunov stability [Gabriel, 2016]**
  - If there exists $V(x(t), t)$, in the sense of Definition 1.5, then x_e is **STABLE**.
  - If there exists $V(x(t), t)$ that strictly satisfies Definition 1.5, then x_e is **ASYMPTOTICALLY STABLE**.
  - If x_e is asymptotically stable and, $\forall t \in \mathbb{R}_+$, $V(x(t), t)$ is such that $\lim_{|x| \to \infty} V(x, t) \to \infty$, then the equilibrium point is **GLOBALLY ASYMPTOTICALLY STABLE**.

- **LMI Condition:**

> **Theorem 1.6 [Geromel].** Let matrices $B \equiv 0$ and $E \equiv 0$ and $A \in \mathbb{R}^{n \times n}$ in $\mathcal{S}$ be given, along with a symmetric matrix variable $P = P' \in \mathbb{R}^{n \times n}$, such that
> $$A'P + PA \prec 0, \qquad P \succ 0 \tag{1.6}$$
> is feasible if and only if A is Hurwitz stable.

- **Hurwitz Stability**: Let matrices $B \equiv 0$ and $E \equiv 0$ be given, in the state transition equation of $\mathcal{S}$. If the eigenvalues of $A \in \mathbb{R}^{n \times n}$ have negative real parts, then A is Hurwitz stable.

Matrix definiteness

- **What does** $M \succ 0$ **actually mean?**

Definition 1.7. A symmetric matrix $M = M^T \in \mathbb{R}^{n\times n}$ is positive definite **($M \succ 0$)** if and only if $x'Mx > 0$ for every $x \neq 0$. Similarly, it is positive semidefinite **($M \succeq 0$)** if and only if $x'Mx \geq 0$ for every $x \in \mathbb{R}^n$.

Equivalent characterization. $\boldsymbol{M} \succ \boldsymbol{0}$ if and only if every eigenvalue of $M \in \mathbb{R}^{n\times n}$ is strictly positive: $\lambda_i(M) > 0$ for all $i = 1,2,\dots,n$.
Likewise, $\boldsymbol{M} \succcurlyeq \boldsymbol{0}$ if and only if $\lambda_i(M) \geq 0$ for all $i = 1,2,\dots,n$.

EXAMPLE — Consider $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. *Its eigenvalues are 2 and 3, both positive, so* $M \succ 0$ *(positive definite).*

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EXERCISE – Find a positive definite matrix $P = P' \in \mathbb{R}^{2\times 2}$ *such that*

$$A'P + PA + I = 0;\ A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix};\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

LMI definition

- **LMI.** Matrix inequalities which are linear in the matrix variables. Its general form is

$$\mathcal{L}(X) = \sum_{i=1}^{\ell} (E_i{}' X F_i + F_i{}' X' E_i) + Q \qquad \textbf{(1.7)}$$

$\mathcal{L}(\cdot)$ is a linear operator in the variable $X \in \mathbb{R}^{m\times n}$, with given $Q = Q' \in \mathbb{R}^{n\times n}$, $E_i \in \mathbb{R}^{m\times n}$, and $F_i \in \mathbb{R}^{n\times n}$, for $i \in \{1,2,\ldots,\ell\}$. The constraint "$\mathcal{L}(X) \succ 0$" requires every eigenvalue of the symmetric matrix $\mathcal{L}(X)$ to be strictly positive.

- **Alternatively**, the matrix inequality is expressed as

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \succ 0 \qquad \textbf{(1.8)}$$

where $A_i = A_i{}' \in \mathbb{R}^{n\times n}, i = 0,1,2\ldots,n$. x_i are the decision variables.
Notice all decision variables can be stacked into a vector.

- All sets are convex:

$$\{x : A(x) \succ 0\}$$
$$\{x : A(x) \succcurlyeq 0\}$$
$$\{x : A(x) \prec 0\}$$
$$\{x : A(x) \preccurlyeq 0\}$$

Optimization problem based on LMI

- In general, the main issue regarding convex problems based on LMIs is the feasibility of proposed constraints, which can be expressed through an optimization problem.

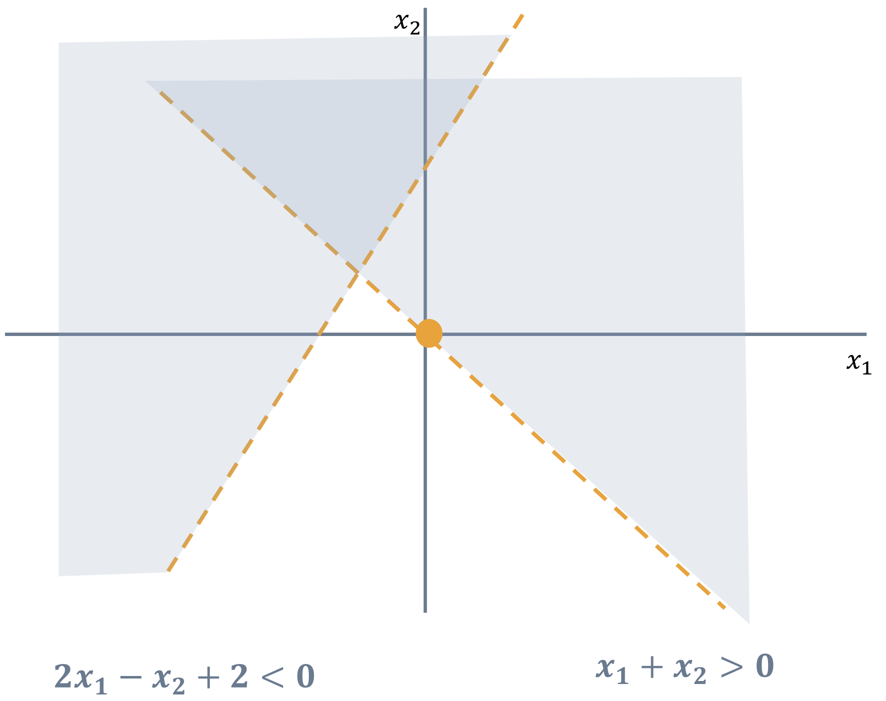
- **A linear optimization problem** is expressed by

$$\begin{cases} \min/\inf & J(X) \\ s.t. & \mathcal{L}(X,Q) \succ 0 \end{cases} \tag{1.9}$$

EXAMPLE — Consider the problem of minimizing x_2, such that $x_1 + x_2 > 0$ and $2x_1 - x_2 + 2 < 0$.

Should the solution be different if we minimize x_1 instead?

Feasible Region (x1, x2 Plane)



Programming LMI problems

Interpreter

A modeling layer that lets you describe an optimization problem in natural, algebra-like syntax — variables, constraints, objective — and automatically translates it into the standard form a solver expects.

e.g., YALMIP, CVX

Solver

The numerical engine that actually solves the resulting convex optimization problem and returns a certificate of feasibility or an optimal solution.

e.g., Mosek, SeDuMi, SDPT3, LMILab

Some LMI solvers and interpreters...

1

Using MATLAB

1. Verify if Robust Control Toolbox is available. It contains the LMILab. (https://matlab.mathworks.com/)

Comercial, LMILab solver

2

Install YALMIP

1. Download YALMIP from yalmip.github.io.
2. Add the extracted folder and subfolders to your MATLAB path.
3. Run yalmiptest in MATLAB to verify.

Free, open-source interpreter

3

Install Mosek

1. Get a free academic license at mosek.com.
2. Install Mosek and add its toolbox to the MATLAB path.
3. Run mosekopt in MATLAB to verify.

Free academic license required solver

Programming with LMI toolboxes

Program 1 — from foundation to code

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PROG. 1

Introduction to LMI solvers and interpreters

Install YALMIP + LMILab/Mosek. Consider the optimization problem $\begin{cases} \min & x_2 \\ s.t.. & x_1 + x_2 > 0 \\ & 2x_1 - x_2 + 2 < 0 \end{cases}$

1) Obtain one feasible solution for (x_1, x_2).

2) Obtain the minimum value of x_2 that solves the problem above.

3) Investigate the solution of the problem $\begin{cases} \min & x_1 \\ s.t.. & x_1 + x_2 > 0 \\ & 2x_1 - x_2 + 2 < 0 \end{cases}$

Schur complement

Definition 1.8. Let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be a partitioned matrix,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \textbf{(1.10)}$$

1. $\boldsymbol{A_{22} - A_{21}A_{11}^{-1}A_{12}}$, with A_{11} nonsingular, is the **Schur complement** of A with respect to block A_{11}.
2. $\boldsymbol{A_{11} - A_{12}A_{22}^{-1}A_{21}}$, with A_{22} nonsingular, is the **Schur complement** of A with respect to block A_{22}

- To verify Lemma 1.9 consider $\tilde{A} = T_1 A T_2$, where

$$T_1 = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}, \qquad T_2 = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

Note: For symmetric A, $\tilde{A}$ and $\hat{A}$ are congruent to A.

Lemma 1.9. Let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10). The following are equivalent.

1. For A_{11} nonsingular,

$$A \longmapsto \tilde{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \qquad \textbf{(1.11)}$$

2. For A_{22} nonsingular,

$$A \longmapsto \hat{A} = \begin{bmatrix} A_{22} & 0 \\ 0 & A_{11} - A_{12}A_{22}^{-1}A_{21} \end{bmatrix} \qquad \textbf{(1.12)}$$

Lemma 1.10. Let $A = A' \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10) with $A_{12} = A_{21}'$. The following hold.

1. $A \succ 0 \Leftrightarrow A_{11} \succ 0, A_{22} - A_{12}'A_{11}^{-1}A_{12} \succ 0$
2. $A \succ 0 \Leftrightarrow A_{22} \succ 0, A_{11} - A_{12}A_{22}^{-1}A_{12}' \succ 0$
3. $A \prec 0 \Leftrightarrow A_{11} \prec 0, A_{22} - A_{12}'A_{11}^{-1}A_{12} \prec 0$
4. $A \prec 0 \Leftrightarrow A_{22} \prec 0, A_{11} - A_{12}A_{22}^{-1}A_{12}' \prec 0$

Inversion Lemma

Woodbury matrix identity. Let $\mathcal{A} \in \mathbb{R}^{n\times n}, \mathcal{B} \in \mathbb{R}^{m\times m}, \mathcal{U} \in \mathbb{R}^{n\times m}, \mathcal{V} \in \mathbb{R}^{m\times n}$, then, for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} + \mathcal{U}\mathcal{B}^{-1}\mathcal{V}$ nonsingular matrices,

$$(\mathcal{A} + \mathcal{U}\mathcal{B}^{-1}\mathcal{V})^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{U}(\mathcal{B} + \mathcal{V}\mathcal{A}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{A}^{-1} \tag{1.13}$$

Lemma 1.11. Now, let $A \in \mathbb{R}^{((n+m)\times(n+m))}$ be the partitioned matrix in (1.10), then

$$A^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \tag{1.14}$$

By invoking the Woodbury identity other formulas can be derived to handle the cases $where$ A_{11} or A_{22} is not invertible.

Projection Lemma

Definition 1.12. Let $A \in \mathbb{R}^{m\times n}$,

1. $A_{\vdash}$ is its left orthogonal complement if
$$A_{\vdash} A = 0, \qquad rank(A_{\vdash}) = m - rank(A)$$

2. $A_{\dashv}$ is its right orthogonal complement if
$$A A_{\dashv} = 0, \qquad rank(A_{\dashv}) = n - rank(A)$$

Lemma 1.13. Let $Q = Q' \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times n}$. There exists a matrix $X \in \mathbb{R}^{r\times m}$ such that

$$Q + A'X'B + B'XA \prec 0$$

if and only if

$A_{\dashv}'QA_{\dashv} \prec 0$ and $B_{\dashv}'QB_{\dashv} \prec 0$

Reciprocal projection Lemma

Lemma 1.14 (Apkarian 2000). Let $Q = Q' \in \mathbb{R}^{n\times n}$, and $P = P' \in \mathbb{R}^{n\times n}$ be given, with $P \succ 0$. There exists a matrix $S \in \mathbb{R}^{n\times n}$ such that

$$Q + S + S' \prec 0$$

if and only if there exists $W \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} Q + P - (W + W') & S' + W' \\ S + W & -P \end{bmatrix} \prec 0$$

Similarity and congruence transformation

Definition 1.15. Consider $Q = Q' \in \mathbb{R}^{n \times n}$ and a congruence transformation on Q, $T \in \mathbb{R}^{n \times n}$, with $rank(T) = n$. Hence,

$$Q \succ 0 \iff TQT' \succ 0 \qquad \textbf{(1.15)}$$

Definition 1.16. Consider $Q = Q' \in \mathbb{R}^{n \times n}$ and a similarity transformation, $T \in \mathbb{R}^{n \times n}$. If there exists T^{-1}, hence,

$$\lambda(Q) = \lambda(TQT^{-1}) \qquad \textbf{(1.16)}$$

- A congruence transformation preserves the **definiteness** (sign) of Q, but not necessarily its eigenvalues.

- A similarity transformation preserves the eigenvalues of $Q \succ 0$. In this case, symmetry is not ensured anymore.

Elimination Lemma

Lemma 1.17. Let $Q = Q' \in \mathbb{R}^{n\times n}$, and $X \in \mathbb{R}^{n\times m}$ and $Y \in \mathbb{R}^{m\times n}$ be given and let

$$\mathbb{F} = \{F : F \in \mathbb{R}^{m\times m}, F'F \preccurlyeq I\} \qquad \textbf{(1.17)}$$

For any $F \in \mathbb{F}$,

$$Q + XFY + Y'F'X' \prec 0$$

if and only if, there exists a scalar $\delta > 0$, such that

$$Q + \delta XX' + \delta^{-1}Y'Y \prec 0.$$

Lemma 1.18. Consider a partitioned matrix $A = A' \in \mathbb{R}^{(n+m)\times(n+m)}$ as in (1.10). There exists a matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A_{11} - X & A_{12} & X \\ A_{12}' & A_{22} & 0 \\ X & 0 & -X \end{bmatrix} \prec 0$$

if and only if $A \prec 0$.

Other useful Lemmas

Lemma 1.19. Let $X, Y \in \mathbb{R}^{m\times n}$, and $F = F' \in \mathbb{R}^{m\times m}$, $F \succ 0$ and a scalar $\delta > 0$. then

$$X'FY + Y'FX \preccurlyeq \delta X'FX + \delta^{-1}Y'FY$$

Lemma 1.20. Let be given $F \in \mathbb{F}$ as in (1.17). Let $X, Y \in \mathbb{R}^{m\times n}$ and a scalar $\delta > 0$. then

$$X'FY + Y'F'X \preccurlyeq \delta X'X + \delta^{-1}Y'Y$$

Lemma 1.21. Let be given $F \in \mathbb{F}$ as in (1.17). Let $X, Y \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{n\times n}$, then

$$Q + X'FY + Y'F'X \prec 0, \forall F \in \mathbb{F}$$

if and only if there exists a scalar $\delta > 0$, such that

$$Q + \delta X'X + \delta^{-1}Y'Y \prec 0$$

Problem definition for a norm-based design

- Let us consider both original LTI systems: the discrete-time dynamical system

$$\mathcal{D} := \begin{cases} x[k+1] = Ax[k] + Bu[k] + Ew[k] \quad , x[0] = x_0 \\ z[k] = C_z x[k] + D_z u[k] \\ y[k] = C_y x[k] + D_y u[k] \end{cases}$$

and the continuous-time one:

$$\mathcal{S} := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \quad , x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \\ y(t) = C_y x(t) + D_y u(t) \end{cases}$$

presented before.

- We want to design a control law $u(t)$ such that the performance index, for $\gamma \in \mathbb{R}_+$,

$$\mathcal{J} = \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \tag{1.18}$$

For every $0 \neq w \in L_2/\ell_2$. In this case, the symbol $\|\cdot\|_2^2$ stands for the squared 2-norm of a signal, which is defined as $\|x\|_2^2 = \int_0^\infty x(\tau)' x(\tau) d(\tau)$, with $x \in L_2$, for a continuous-time signal, or $\|x\|_2^2 = \sum_{n=0}^\infty x[n]' x[n]$, with $x \in \ell_2$, for a discrete-time one.

$\mathcal{H}_2$ norm of a system

Definition 1.22. For $H_c(s)$ and A Hurwitz, the $\mathcal{H}_2$ norm is defined by

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2$$

with z_i the response of the system $\mathcal{S}$, with $u \equiv 0$ and $x_0 = 0$, to the impulse $w = e_i\delta(t)$ applied to the i-th channel.

Definition 1.23. For $H_d(z)$ and A Schur, the $\mathcal{H}_2$ norm is defined by

$$\|H_d\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2$$

with z_i the response of the system $\mathcal{D}$, with $u \equiv 0$ and $x_0 = 0$, to the impulse $w = e_i\delta[k]$ applied to the i-th channel.

- The $\mathcal{H}_2$ norm of $\mathcal{S}$ is equivalently calculated through

$$\|H_c\|_2^2 = tr(E'PE)$$

with $A'P + PA = -C_z{}'C_z$, and $P \succ 0$.

- The $\mathcal{H}_2$ norm of $\mathcal{D}$ is equivalently calculated through

$$\|H_d\|_2^2 = tr(E'PE)$$

with $A'PA - P = -C_z{}'C_z$, and $P \succ 0$.

- H_c and H_d are the transfer functions of the systems $\mathcal{S}$ and $\mathcal{D}$, respectively, both with $u \equiv 0$.

$$H_c(s) = C_z(sI - A)^{-1}E \qquad (1.19)$$
$$H_d(\zeta) = C_z(\zeta I - A)^{-1}E \qquad (1.20)$$

$\mathcal{H}_\infty$ norm of a system

Definition 1.24. For $H_c(s)$ and A Hurwitz, the $\mathcal{H}_\infty$ norm is defined by

$$\|H_c\|_\infty^2 = \sup_{0 \neq w \in L_2} \frac{\|z\|_2^2}{\|w\|_2^2}$$

Definition 1.25. For $H_d(\zeta)$ and A Schur, the $\mathcal{H}_\infty$ norm is defined by

$$\|H_d\|_\infty^2 = \sup_{0 \neq w \in \ell_2} \frac{\|z\|_2^2}{\|w\|_2^2}$$

- From Bounded Real Lemma

$$\|H_c\|_\infty^2 = \inf\left\{\mu : P \succ 0, \begin{bmatrix} A'P + PA & PE & Cz' \\ \star & -\mu I & 0 \\ \star & \star & -I \end{bmatrix} \prec 0\right\} \tag{1.21}$$

and

$$\|H_d\|_\infty^2 = \inf\left\{\mu : P \succ 0, \begin{bmatrix} A'PA - P & A'PE & Cz' \\ \star & E'PE - \mu I & 0 \\ \star & \star & -I \end{bmatrix} \prec 0\right\} \tag{1.22}$$

are equivalent to Definitions 1.24 and 1.25, respectively.

Continuous-time performance index calculation

- For that purpose, for the existence of the continuous-time performance index $\mathcal{J}$, it is necessary for the system $\mathcal{S}$ to be stable. Consider that there exists a Lyapunov candidate function of the form $v(x) = x'Px$, with $0 \prec P \in \mathbb{R}^{n\times n}$, whose derivative is $\dot{v}(x) < 0$. As a consequence, $\mathcal{S}$ is asymptotically stable and so we can determine the exact value of the performance index $\mathcal{J}$ by imposing

$$\dot{v}(x) < -z'z + \gamma^2 w'w \tag{1.23}$$

- Indeed, condition (1.5) remains valid and strictly satisfied for $w \equiv 0$. Moreover, for a given x_0,

$$\int_0^\infty \dot{v}(x)dt < -\mathcal{J} \Rightarrow \mathcal{J} < v(x_0) = x_0'Px_0 \tag{1.24}$$

- It is important to notice that for a LMI based norm calculation, the problem to be solved is expressed such as

$$\min_{P>0,}\{tr(E'PE) : (1.23)\ with\ w \equiv 0\} \ / \min_{P>0,\mu=\gamma^2>0}\{\mu : (1.23)\} \tag{1.25}$$

Continuous-time state-feedback norm-based design

- Consider the system is feedbacked with a state dependent control law $u(t) = Kx(t)$,

$$\dot{v}(x) < -z'z + \gamma^2 w'w$$
$$\Updownarrow$$
$$[(A+BK)x + Ew]'Px + x'P[(A+BK)x + Ew] < -x'(C_z + D_zK)'(C_z + D_zK)x + \gamma^2 w'w$$
$$\Updownarrow$$
$$\begin{bmatrix} x \\ w \end{bmatrix}' \begin{bmatrix} (A+BK)'P + P(A+BK) + (C_z + D_zK)'(C_z + D_zK) & PE \\ E'P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} (A+BK)'P + P(A+BK) & (C_z + D_zK)' & PE \\ (C_z + D_zK) & -I & 0 \\ E'P & 0 & -\gamma^2 I \end{bmatrix} \prec 0 \tag{1.26}$$

which is a non-linear matrix inequality.

Continuous-time state-feedback norm-based design

- Linearizing the previous inequality by adopting the congruence transformation $diag(P^{-1}, I, I)$ and the one-to-one transformation of variables $W = P^{-1}$ and $Y = KW$, it follows that (1.26) can be rewritten as

$$\begin{bmatrix} WA' + AW + Y'B' + BY & WC_z' + Y'D_z' & E \\ \star & -I & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix} \prec 0 \tag{1.27}$$

- For the context of the $\mathcal{H}_2$ norm, for which system $\mathcal{S}$ is defined for $x_0 \neq 0$ and $w \equiv 0$, the objective function can be rewritten in the new set of variables as

$$N - E'W^{-1}E \succ 0 \quad \Leftrightarrow \quad \begin{bmatrix} N & E' \\ \star & W \end{bmatrix} \succ 0 \tag{1.28}$$

- Notice that the response of the systems $\mathcal{S}$ with $x_0 = Ee_i$ and $w \equiv 0$ equals the response of $\mathcal{S}$ with $x_0 = 0$ and $w \equiv \delta(t)$.

Continuous-time state-feedback norm-based design

Theorem 1.26. The closed-loop system with transfer function $H_c(s)$ is Globally Asymptotically Stable and $\|H_c\|_2^2 = \min_{W,Y,N} tr(N)$, if and only if, there exist matrices W$= W' \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_w \times N_w}$, and $Y \in \mathbb{R}^{n_u \times n_x}$, such that W$\succ$ 0, (1.27) deleting the third column and row, and (1.28) hold. In this case the optimal state-feedback gain is given by $K = YW^{-1}$.

Theorem 1.27. The closed-loop system with transfer function $H_c(s)$, is Globally Asymptotically Stable and $\|H_c\|_\infty = \min_{W,Y,\gamma>0} \gamma$, if and only if, there exist matrices W$= W' \in \mathbb{R}^{n_x \times n_x}$ and $Y \in \mathbb{R}^{n_u \times n_x}$, such that W$\succ$ 0 and (1.27) hold. In this case the optimal state-feedback gain is given by $K = YW^{-1}$.

Continuous-time state-feedback norm-based design

Proof (Continuous-time $\mathcal{H}_2$ case). Inequality (1.26) deleting the third row and column is expressed as

$$\begin{bmatrix} WA' + AW + Y'B' + BY & WC_z' + Y'D_z' \\ \star & -I \end{bmatrix} \prec 0$$

which can be rewritten, by applying the Schur complement with respect to the second row and column, as

$$WA' + AW + Y'B' + BY + (WC_z' + Y'D_z')(WC_z' + Y'D_z')' \prec 0$$

Pre- and post-multiplying the previous inequality by P, together with the change of variables $Y = KW$ and $W = P^{-1}$, yields

$$(A + BK)'P + P(A + BK) + (C_z + D_zK)'(C_z + D_zK) \prec 0$$

which is the closed-loop Lyapunov inequality.

Continuous-time state-feedback norm-based design

Proof (Continuous-time $\mathcal{H}_2$ case)… Consider a Lyapunov candidate function of the form $v(x) = x'Px$, without loss of generality. The solution to the previous inequality is also a solution to the inequality $\dot{v}(x) < -z'z$. From this inequality, two conclusions can be drawn:

1. System $\mathcal{S}$ is globally asymptotically stable. Indeed, the existence of a symmetric $P \succ 0$ (from (1.27)) satisfying (1.26) without the third row and column implies that there exists $v(x) > 0, \forall x \neq 0$, and $\dot{v} < 0$.

2. The $\mathcal{H}_2$ norm is exactly calculated due to the equivalence reported on slide 28 and to the infimum problem, which evaluates

$$\|H_c\|_2^2 = \mathcal{J} < \sum_i v(Ee_i) = tr(E'PE)$$

since the Lyapunov inequality admits a minimal matrix element. Since the infimum is attained at the minimal matrix element, no conservatism is introduced and necessity follows.

Programming with LMI toolboxes

Program 2 — continuous-time state-feedback control design problem

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PROG. 2

Continuous-time LTI State-Feedback Control Design

Determine the continuous-time state-feedback gain that minimizes the $\mathcal{H}_2$ norm for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = [1 \quad 0] x + 2u$$

and calculate the $\mathcal{H}_2$ norm.

Now, determine the continuous-time state-feedback gain that minimizes the $\mathcal{H}_\infty$ norm considering the same system. Calculate the $\mathcal{H}_\infty$ norm.

Theoretical Exercises

E1) Prove Theorem 1.27, following similar steps as presented on slides 30-31.

E2) Derive the discrete-time equivalent formulation for Theorem 1.26 and 1.27.

E3) Obtain an LMI formulation for the inequality

$$(\mathrm{X}^{-1} + Y^{-1})^{-1} \succ Z^{-1} + W'W$$

in the decision variables X, Y, Z and W.

E4) Let A and B be symmetric matrices of the same dimension. Show that

- $A \succ B \Longrightarrow \lambda_{max}(A) > \lambda_{max}(B)$
- $\lambda_{max}(A + B) \leq \lambda_{max}(A) + \lambda_{max}(B)$

E5) Write the Woodbury Identity, rewrite the (1.14) considering that a) A_{11} is not invertible, and b) A_{22} is not invertible.

Computational Exercises

Discrete-time state-feedback norm-based design

C1) Determine the discrete-time state-feedback gain that stabilizes and minimizes the $\mathcal{H}_2$ norm for the system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0.5 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0.2u$$

Determine also the discrete-time state-feedback gain and calculate the $\mathcal{H}_\infty$ norm that minimizes the $\mathcal{H}_\infty$ norm considering the same system. Present the code, the values of K, $\|\mathcal{D}\|_2$, and $\|\mathcal{D}\|_\infty$, and plot the signals of $x[k]$, $u[k]$, and $z[k]$ for each case.

BLOCK 2

02

Sampled-Data Systems

From the continuous-time plant to a discrete-time model. Sampled-data systems in the hybrid approach.

Chen & Francis, Chps. 1–4, Geromel, Chp. 3, Goebel, Teel and Sanfelice (2009), Matheus et al. (2014), Briat (2013)

Sampled-data systems: a definition

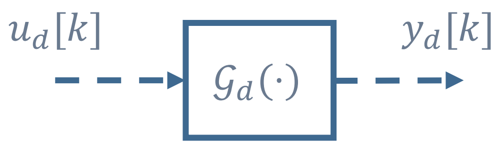


Fig. 2.1 – Discrete-time system

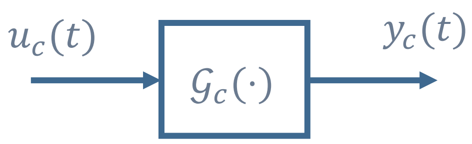


Fig. 2.2 – Continuous-time system

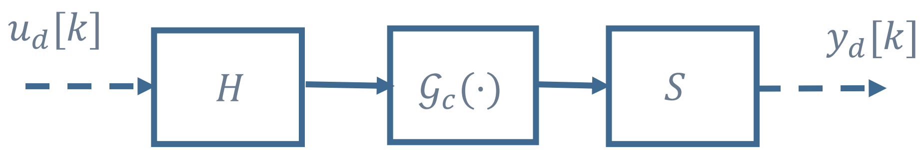


Fig. 2.3 – Sampled-data system

- Control and filtering theory are well established for the contexts of continuous-time or discrete-time systems.
- A continuous-time system has every signal defined in the continuous-time domain $(t \in \mathbb{R}_+)$. Dynamical systems that operate in the continuous-time domain can be described by differential equations. Conversely, a discrete-time system has every signal defined in the discrete-time domain $(k \in \mathbb{N})$. Discrete-time systems can be described by difference equations.
- A **sampled-data system** is a system **whose operation is partially in the continuous-time domain, and partially in the discrete-time domain**. The two domains are bridged by the sampler S and the hold device H (typically a zero-order hold), at the sampling instants $t_k = kh, k \in \mathbb{N}$, and $h > 0$, which allow each part of the system to operate in its own domain. The resulting system is neither a continuous-time nor a discrete-time LTI system, but an **h-periodic system**.

Sampled-data systems: discretization technique

- Consider again the plant $\mathcal{S}$ previously defined, for which $w(t)$ and $z(t)$ have been omitted, for the sake of simplicity:

$$\mathcal{S} \coloneqq \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = C_y x(t) + D_y u(t) \end{cases} \tag{2.1}$$

for which the control input is $u(t) = u[k]$, for all $t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$.

- $\{t_k\}_{k\in\mathbb{N}}$ is the sequence of sampling instants, such that $t_{k+1} - t_k = h > 0$, with $t_0 = 0$ [s].

- The first natural approach is to discretize system $\mathcal{S}$, generating an equivalent system $\mathcal{S}_d$, following the logic behind Figure 3. For that purpose, let us determine the general solution to the dynamical equation of $\mathcal{S}$.

$$x(t_f) = e^{A(t_f - t_i)} x(t_i) + \int_{t_i}^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau \tag{2.2}$$

evaluated in the boundaries of the sampling interval $t_i = t_k$ and $t_f = t_{k+1}$, which leads to

$$x(t_{k+1}) = e^{Ah} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} d\tau \, Bu[k] = e^{Ah} x(t_k) + \int_0^h e^{A\tau} d\tau \, Bu[k] \tag{2.3}$$

where a change of variables has been used to obtain the last integral.

Sampled-data systems: discretization technique

Theorem 2.1 (Chen, Theorem 3.1.1). The step-invariant transformation of the continuous-time system $\mathcal{S}$ into the discrete-time one $\mathcal{S}_d$, maps the state-space matrices

$$(A, B, C_y, D_y) \to \left(e^{Ah}, \int_0^h e^{A\tau} d\tau\, B, C_y, D_y\right)$$

- This map produces the behavior of $\mathcal{S}$ exactly at the sampling instants. The sampling process can destroy controllability at pathological sampling periods, which occurs whenever $\lambda_1 = \lambda_2 + \frac{j2\pi k_i}{h}$, with (λ_1, λ_2) eigenvalues of A.

EXAMPLE — Consider the following map, for $h = \pi$,

$$\left(\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, [0]\right) \to \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, [0]\right)$$

System $\mathcal{S}$ is controllable, but $\mathcal{S}_d$ is not.

EXAMPLE — $\mathbf{A} = \begin{bmatrix} \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-8} & \mathbf{-4} \end{bmatrix}$ are such that

$e^{A\pi} = e^{\tilde{A}\pi}$, for h= π. Notice that the eigenvalues of A are $(-2, -2)$ and the eigenvalues of $\tilde{A}$ are $(-2+2j, -2-2j)$, with $j\omega = \frac{j2\pi}{h} = j2$.

Sampled-data systems: stability issues

- Going further in the analysis, let us consider a simplified system, as proposed in Geromel (2023),

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

consider also a Lyapunov candidate function $v(x) = x'Px$, such that $0 \prec P \in \mathbb{R}^{n\times n}$.

- A is Hurwitz, with eigenvalues $\lambda_i, i \in \{1, \dots, n\}$, if and only if there exists $P \succ 0$ such that $A'P + PA \prec 0$.

- Now, notice that the eigenvalues of the discretized system, e^{Ah}, denoted by μ_i , $i \in \{1, \dots, n\}$, are of the form $\mu_i = e^{\lambda_i h}$, which implies $|\mu_i| = e^{Re(\lambda_i)h}$. As a consequence, A is Hurwitz if and only if $A_d = e^{Ah}$ is Schur. That means there exists $S \succ 0$ such that $e^{A'h}Se^{Ah} - S \prec 0$.

Important! The existence of a positive definite matrix $S \succ 0$ that solves $e^{A'h}Se^{Ah} - S \prec 0$, for some $h > 0$, does not imply that $A'S + SA \prec 0$ is satisfied.

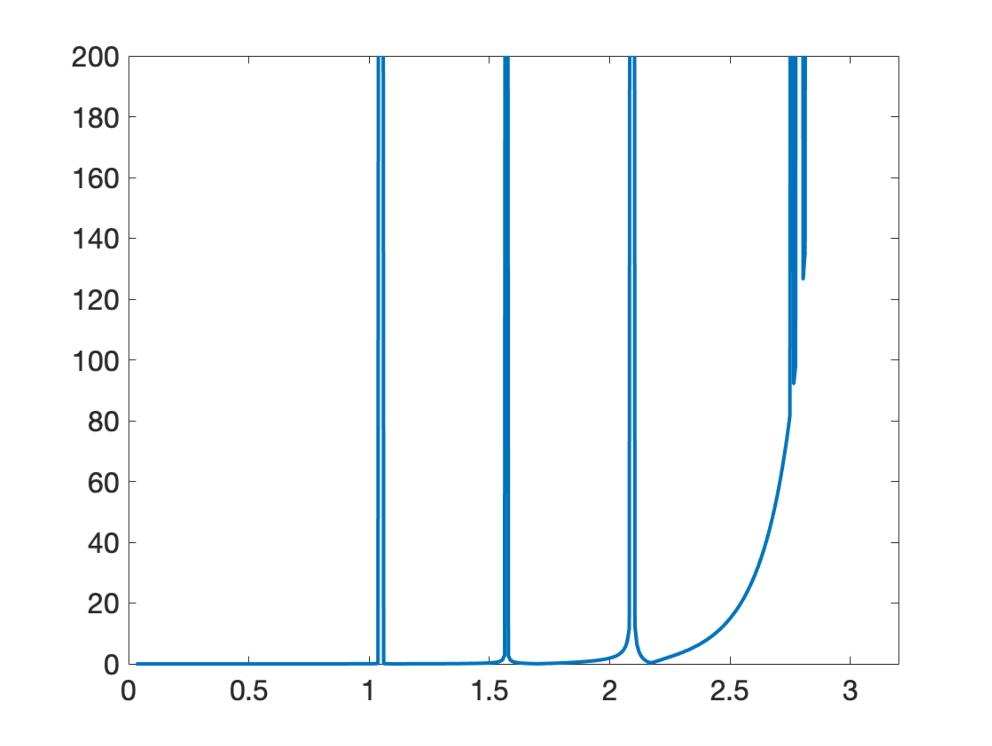
- Notice that the inequality $e^{A'h}Se^{Ah} - S \prec 0$ "sees" A only through e^{Ah} and the exponential map $s \rightarrow e^{sh}$ is periodic of period $j\omega = \frac{j2\pi}{h}$. Hence, there exist matrices $\tilde{A} \neq A$, with $\lambda_i(\tilde{A}) = \lambda_i(A) + \frac{j2\pi k_i}{h}$ for $k_i \in \mathbb{Z}$, such that $e^{Ah} = e^{\tilde{A}h}$.

Sampled-data systems: discretization technique

EXAMPLE — $A = \begin{bmatrix} 4 & -16 & 24 & -20 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; E = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.1 \end{bmatrix};$ and $C = \begin{bmatrix} 0 & 0 & 0 & 0.01 \end{bmatrix}$, D=0. The values of the $\mathcal{H}_2$ norm for different values of $h \in \left[\frac{\pi}{100}, \pi\right]$ are plotted in Figure 2.4. There is some pathological sampling intervals such that the system is not controllable.

The discrete-time equivalent system has been obtained by using c2d command from Matlab.

```
>> Su=ss(A,B,C,D);
>> Sw=ss(A,E,C,0);
>> Sud=c2d(Su, h(i),'zoh’) ;
>> Swd=c2d(Sw, h(i),'zoh');
```



BLOCK 2 — SAMPLED-DATA SYSTEMS

Programming with LMI toolboxes

Program 3 — pathological sampling intervals

PROG. 3

Continuous-time LTI State-Feedback Control Design vs. Discretized LTI State-Feedback Control Design

Determine the discretized system for

$$\dot{x} = \begin{bmatrix} 4 & -16 & 24 & -20 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = \begin{bmatrix} 0 & 0 & 0 & 0.01 \end{bmatrix} x$$

Design a state-feedback sampled-data control for the discretized system, and calculate the $\mathcal{H}_2$ norm for $h \in [\frac{\pi}{100}, \pi]$. Plot (h,$\mathcal{H}_2\ norm$). Analyze and explain the result.

Sampled-data systems: discretization issues

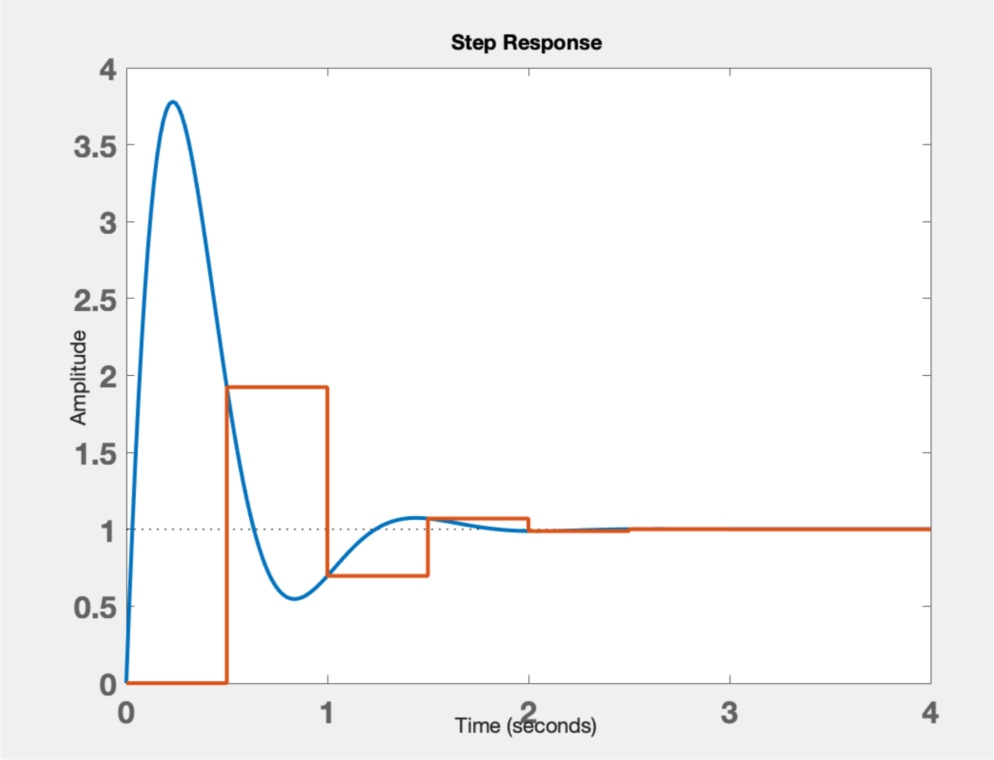
- The time-invariant discretization process over a system $\mathcal{S}$ produces an equivalent system $\mathcal{S}_d$, whose output signal $y[k]$ corresponds to the output signal $y(t)$, from $\mathcal{S}$, at sampling instants, $t = t_k = kh$, for all $k \in \mathbb{N}$. However, **it does not tell us what happens to the system between two sampling instants**.

- Additionally, introducing the hold device H at the output $y[k]$ another signal is produced, $\tilde{y}(t)$, which is an approximation of $y(t)$, where the signal is kept constant after each sample. $\tilde{y}(t) = y(t_k)$ $[t_k, t_{k+1})$.

- An important property is

$$\lim_{h \to 0} \tilde{y}(t) = y(t)$$

since $\|\tilde{y}(t) - y(t)\| \to 0$, as $h \to 0$.



Sampled-data systems: performance calculation

- In the context of discretized systems, let us consider the sampled-data system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), x(0) = 0$$
$$z(t) = C_z x(t) + D_z u(t)$$
$$u(t) = u(t_k) = Kx(t_k), \forall t \in [t_k, t_{k+1})$$

- For an impulsive input $w(t) = \sum_{i=1}^{n_w} e\_i\delta(t)$, its response is equivalent to $w \equiv 0$ and $x[k] = x(t_k) = \sum_{i=1}^{n_w} \left(e^{Ah} + \left(\int_0^h e^{A\tau} B d\tau\, K\right)\right)^k E e_i = \sum_{i=1}^{n_w} x_i(t_k)$. Moreover,

$$x(t) = e^{A(t-t_k)} x(t_k) + \left(\int_0^{t-t_k} e^{A\tau} B d\tau\, K\right) x(t_k)$$

- Then, the output response is

$$z_i(t) = \left(C_z e^{A(t-t_k)} + \left(C_z\left(\int_0^{t-t_k} e^{A\tau} B d\tau\right) + D_z\right) K\right) x_i(t_k)$$

Sampled-data systems: performance calculation

- The $\mathcal{H}_2$ norm associated to the systems is

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2 = \sum_{i=1}^{n_w} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} z_i' z_i dt$$

  where z_i is the output response due to $w(t) = e_i \delta(t)$, $i = 1, \dots n_w$. $\|H_c\|_2^2$ corresponds to the $\mathcal{H}_2$ norm associated to sampled-data system. H_c has no trivial representation in s- or z-domain.

- Considering that the sampled-data system is continuous-time evaluated between two consecutive sampling instants, we can evaluate,

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} x_i(t_k)' \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau \right) + D_z \right) K \right)' \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau \right) + D_z \right) K \right) x_i(t_k) dt$$

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \sum_{k=0}^{\infty} x_i(t_k)' \int_{t_k}^{t_{k+1}} \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau \right) + D_z \right) K \right)' \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau \right) + D_z \right) K \right) dt \, x_i(t_k)$$

$$\|H_c\|_2^2 = \sum_{i=1}^{n_w} \sum_{k=0}^{\infty} x_i(t_k)' (C_{dz} + D_{dz} K)' (C_{dz} + D_{dz}\, K)\, x_i(t_k) = \sum_{i=1}^{n_w} \|z_{di}\|_2^2$$

Sampled-data systems: performance calculation

- Let us consider the system

$$x[k+1] = \left(e^{Ah} + \int_0^h e^{A\tau} B d\tau\, K\right) x[k] + \left(\int_0^h e^{A\tau} E d\tau\right) w[k] = (A_d + B_d K)x[k] + E_d w[k] \tag{2.4}$$

$$z[k] = (C_{dz} + D_{dz}K)x[k] \tag{2.5}$$

with $(C_{dz} + D_{dz}K)\,'(C_{dz} + D_{dz}K) = \int_{t_k}^{t_{k+1}} \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau\right) + D_z\right) K\right)' \left(C_z e^{A(t-t_k)} + \left(C_z \left(\int_0^{t-t_k} e^{A\tau} B d\tau\right) + D_z\right) K\right) dt$

- Consider also the $\mathcal{H}_2$ norm to be determined

$$\|H_d\|_2^2 = \sum_{i=1}^{n_w} \|z_i\|_2^2 \tag{2.6}$$

where $z_i[k]$ is the output due to impulsive signal at each $i-th$ channel of $w_i[k]$, that is, $w_i[k] = e_i \delta[k]$, with $i = 1, \dots, n_w$.

Theorem 2.2 The discretized system (2.4)-(2.5) is asymptotically stable with $\mathcal{H}_2$ norm given by $\inf_{P \succ 0} tr(E_d' P E_d)$ if and only if there exists $P \succ 0$ such that

$$(A_d + B_d K)' P (A_d + B_d K) - P \prec -(C_{dz} + D_{dz}K)'(C_{dz} + D_{dz}K) \tag{2.7}$$

holds.

Sampled-data systems: performance calculation

- **Proof (Sampled-data discretized system)** Consider the closed-loop discrete-time map for the sampled-data system given by $(A_d + B_dK, E_d, C_{dz} + D_{dz}K, 0)$ and a cost-to-go candidate function $v[x] = x'Px$. From (2.7) and (2.4), $v(x[k+1]) - v(x[k]) < -z[k]'z[k]$. Moreover, (2.7) evaluated at the minimal solution gives

$$P = \sum_{k=0}^{\infty} (A_d + B_dK)'^k (C_{dz} + D_{dz}K)'(C_{dz} + D_{dz}K)(A_d + B_dK)^k$$

Since the solution to (2.4) is $x[k] = (A_d + B_dK)^k E e_i$, both are equivalent

$$\|H_d\|_2^2 = \sum_{i=1}^{n_w} \sum_{k=0}^{\infty} x_i(t_k)'(C_{dz} + D_{dz}K)'(C_{dz} + D_{dz}\,K)\,x_i(t_k)$$

and

$$\|H_d\|_2^2 = \inf_{P>0}\{tr(E_d{'}PE_d) \;:\; P = \textstyle\sum_{k=0}^{\infty}(A_d + B_dK)'^k(C_{dz} + D_{dz}K)'(C_{dz} + D_{dz}K)(A_d + B_dK)^k\}$$

- **To obtain the discretized equivalent system is not trivial.**

Hybrid / Impulsive Systems

- Alternatively, we can consider the impulsive / hybrid approach to deal with this kind of systems. Indeed, let us consider a continuous-time plant controlled by a discrete-time controller through a state-feedback law:

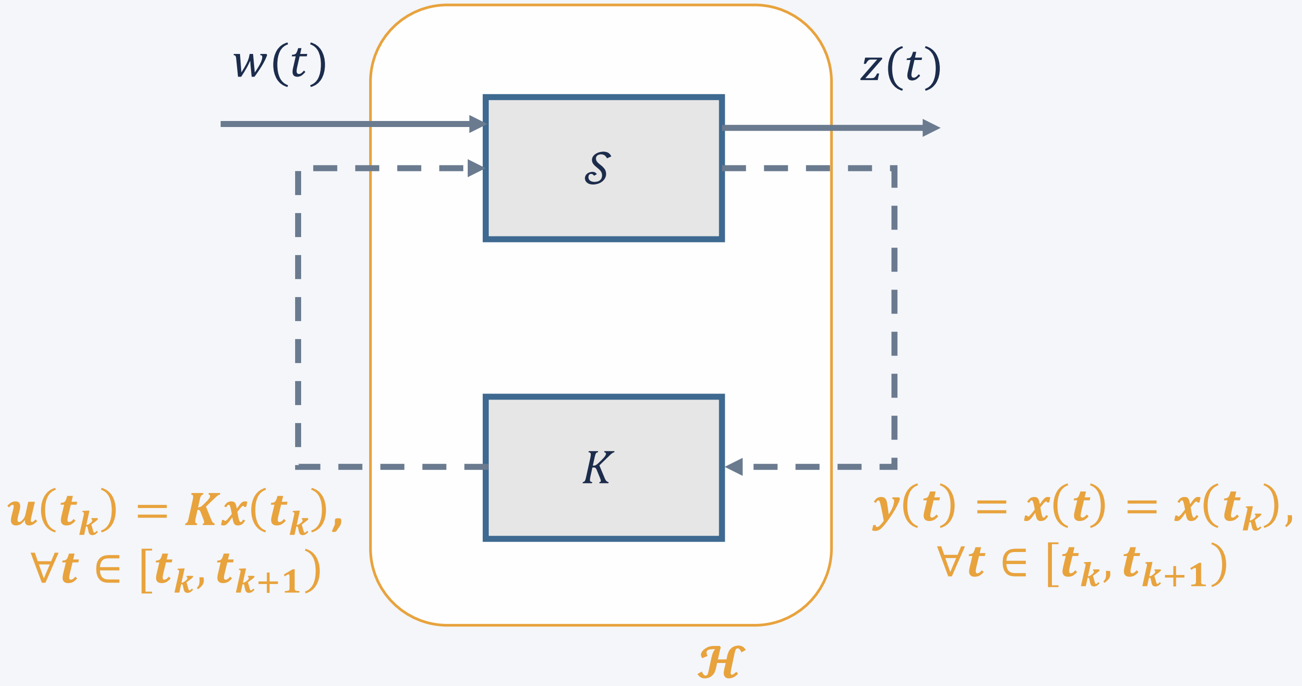
$$\mathcal{S} := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \qquad x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \end{cases} \tag{2.8}$$

$$u(t) = u(t_k) = Kx(t_k), \qquad \forall t \in [t_k, t_{k+1}) \tag{2.9}$$

- It is important to stress that $\{t_k\}_{k\in\mathbb{N}}$ is the sequence of sampling instants such that

$$t_{k+1} - t_k = h \tag{2.10}$$

- System $\mathcal{H}$, composed by equations (2.8) and (2.9), is hybrid since some signals though it is in the continuous-time domain and others in the discrete-time one.



Hybrid / Impulsive Systems

- Following Goebel, Teel and Sanfelice definition of a hybrid system

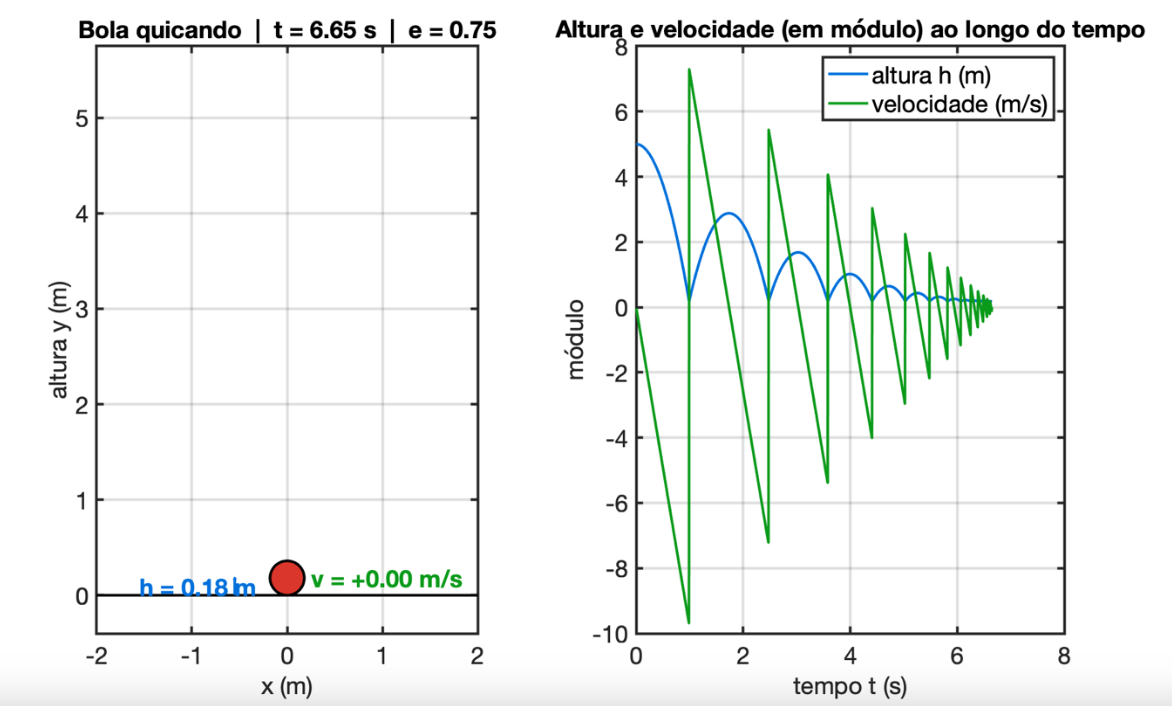
$$\dot{x}(t) = f(x), x \in \mathfrak{C}$$
$$x(t_k)^+ = g(x), x \in \mathfrak{D}$$

- Or in a general formulation

$$\dot{x}(t) \in F(x), x \in \mathfrak{C}$$
$$x(t_k)^+ \in G(x), x \in \mathfrak{D}$$

which maps all admissible paths such that $x \in \mathfrak{C}$ and $x \in \mathfrak{D}$.

- $\dot{x}(t) \in F(x)$ represents the dynamical flow of the systems, whereas $x(t_k)^+ \in G(x)$ describes the jumps that occur at each $t = t_k$.



Hybrid / Impulsive Systems

- Backing to the sampled-data system, a hybrid formulation is such as

$$\mathcal{H} := \begin{cases} \dot{\xi}(t) = F\xi(t), \xi(0) = \xi_0 \\ \xi(t_k^+) = H\xi(t_k^-) \\ z(t) = G\xi(t) \end{cases} \tag{2.11}$$

This system is defined for all $t \in [t_k, t_{k+1})$, for all $k \in \mathbb{N}$, and $w \equiv 0$ in a first analysis. The matrices are

$$F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix};\ H = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}; G' = \begin{bmatrix} C_z' \\ D_z' \end{bmatrix}$$

- Indeed, the first equation of $\mathcal{H}$ recovers the flow equation in (2.8), while the second equation of $\mathcal{H}$ recovers the jump equation in (2.9).

- System in (2.11) is not LTI. Hence consider a Lyapunov candidate function $v(\xi, t) = \xi' P(t^-)\xi$. Ensuring the stability of (2.11) equals to prove that there exists $v(H\xi, t) > 0$, $v(0, t) = 0$, with $\dot{v}(H\xi, t) < 0$ for all $t \in [t_k, t_{k+1})$. In this case, if there exists such a function, the system is stable. Indeed,

$$\dot{v}(H\xi, t) < 0 \xrightarrow{\text{yields}} \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \dot{v}(H\xi, t)dt = \sum_{k=0}^{\infty} [v(\xi, t_{k+1}^-) - v(H\xi, t_k)]$$

- This result tells nothing about the convergence of the system with $t \to \infty$. It is necessary to joint $v(\xi, t_{k+1}^-)$ and $v(H\xi, t_k)$, which is accomplished by the boundary condition

$$P(t_k^-) > H'P(t_k)H \tag{2.12}$$

- Indeed,

$$v(\xi^-, t_k^-) = \xi(t_k^-)'P(t_k^-)\xi(t_k^-) > \xi(t_k^-)'H'P(t_k)H\xi(t_k)$$
$$> \xi(t_k)'P(t_k)\xi(t_k) = v(\xi, t_k)$$

ensures the desired result. The notation $\xi^- = \xi(t_k^-)$ is adopted.

Hybrid / Impulsive Systems

Theorem 2.3 Let be given $h > 0$. The sampled-data system (2.8)-(2.9), rewritten as a hybrid system (2.11), is asymptotically stable with $\mathcal{H}_2$ norm given by $\inf_{P\succ 0} tr(\xi_0{}'H'P(0)H\xi_0)$ if and only if there exists $P(t) \succ 0$ such that the next inequalities hold.

$$\dot{P}(t) + F'P(t) + P(t)F + G'G \prec 0, \qquad P(h) - H'P(0)H \succ 0 \tag{2.13}$$

- **Proof (Sampled-data hybrid system) Necessity…** Consider the sampled-data system given by (2.11) and let $h > 0$ be given. Consider also a cost-to-go candidate function $v(\xi, t) = \xi' P(t)\xi$. From (2.11), the first equation of (2.13), and the time-invariance nature of the problem, it leads to the inequality $\dot{v}(\xi, t) < -z'z$, valid for all $t \in [t_k, t_{k+1})$. Hence,

$$v(\xi, t_{k+1}) - v(\xi, t_k) < v(\xi^-, t_{k+1}^-) - v(\xi, t_k) < -\int_{t_k}^{t_{k+1}} z'zdt \tag{2.14}$$

- Two conclusions can be drawn:

First - The sampled-data system (2.11) is asymptotically stable, since there exists a small enough $\epsilon > 0$ such that $\int_{t_k}^{t_{k+1}} z'zdt \geq \epsilon$, given $t_{k+1} - t_k = h$ and $v(\xi, t_{k+1}) \leq (1-\epsilon)v(\xi, t_k)$. It implies that $v(\xi, t_k) \to 0$ as $k \to \infty$. As a consequence, $v(\xi, t) \to 0$ as $t \to \infty$.

Second – Summing up both sides of the previous inequality, it leads to

$$\xi_0^-{}'H'P(0)H\xi_0^- = -v(\xi, t_0) < -\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} z'zdt = \|H\|_2^2 \tag{2.15}$$

Hybrid / Impulsive Systems

- **Sufficiency…** Conversely, consider the sampled-data system given by (2.11) is asymptotically stable for a given $h > 0$. From Lyapunov's second method of stability, there exists a cost-to-go function such that $v(\xi) > 0, \forall \xi \neq 0$ and $v(0) = 0$, with $\dot{v}(\xi) < 0$. From the fact, the solution to the differential equation determined by (2.13) exists and is uniquely determined by $v(\xi) = \xi' P(t)\xi$, hence also admits a minimal solution the inequality

$$v(\xi, t_{k+1}) < v(\xi, t_k) - \int_{t_k}^{t_{k+1}} z' z dt$$

- From boundary conditions, the inequality (2.13) is readily recovered, whose minimal solution leads to the exact value of the $\mathcal{H}_2$ norm of (2.11), exactly evaluated as $\xi_0^{-\prime} H' P(0) H \xi_0^- = \|H\|_2^2$.

- Notice that the boundary condition in (2.13) implies that $P(t_k^-) > H' P(t_k) H$, which is a jump condition and is a consequence of the time-invariance nature of the problem of calculating the performance index of a sampled-data system.

Hybrid / Impulsive Systems

- Consider now the hybrid system

$$\mathcal{H} := \begin{cases} \dot{\xi}(t) = F\xi(t) + Jw(t), \qquad \xi(0) = \xi_0 \\ \xi(t_k^+) = H\xi(t_k^-) \\ z(t) = G\xi(t) \end{cases} \tag{2.16}$$

with matrices

$$F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}; \;\; H = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}; G' = \begin{bmatrix} C_z' \\ D_z' \end{bmatrix}; J = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

- The same equivalence can be stated for the response of (2.16) with $w = e_i\delta(t)$ and $\xi_0 = 0$ and the response of (2.16) with $w(t) \equiv 0$ and $\xi_0 = Je_i$. As a consequence, the $\mathcal{H}_2$ norm can be calculated as

$$\|H\|_2^2 = \inf_{P \succ 0} tr(J'H'P(0)HJ) : \;\; (2.13) \tag{2.17}$$

Hybrid / Impulsive Systems

Theorem 2.4 Let be given $h > 0$ and $\xi_0 = 0$. The sampled-data system (2.8)-(2.9), rewritten as a hybrid system (2.16), is asymptotically stable with $\mathcal{H}_\infty$ norm given by $\gamma^* = \inf_{P \succ 0, \gamma > 0} \gamma$ if and only if there exists $P(t) \succ 0$ such that the next inequalities hold.

$$\dot{P}(t) + F'P(t) + P(t)F + G'G + \gamma^{-2}P(t)JJ'P(t) \prec 0, \qquad P(h) - H'P(0)H \succ 0 \tag{2.18}$$

- **Proof (Sampled-data hybrid system) Necessity…** Consider the sampled-data system given by (2.8)-(2.9), rewritten as the hybrid system (2.16), and let $h > 0$ be given. Consider also a cost-to-go candidate function $v(\xi, t) = \xi' P(t) \xi$. From (2.16) and the identity

$$\big(\gamma w(t) - \gamma^{-1}J'P(t)\xi(t)\big)'\big(\gamma w(t) - \gamma^{-1}J'P(t)\xi(t)\big) = \gamma^2 w(t)'w(t) + \gamma^{-2}\xi(t)'P(t)JJ'P(t)\xi(t) - w(t)'J'P(t)\xi(t) - \xi(t)P(t)Jw(t)$$

the first inequality of (2.18) can be rewritten as

$$\dot{v}(\xi, t) + \mathrm{z}(\mathrm{t})'z(t) - \gamma^2 w(t)'w(t) + \big(\gamma w(t) - \gamma^{-1}J'P(t)\xi(t)\big)'\big(\gamma w(t) - \gamma^{-1}J'P(t)\xi(t)\big) < 0$$

whose minimal solution is set for $w(t) = \gamma^{-2}J'P(t)\xi(t)$ leading to the inequality $\dot{v}(\xi, t) < -z'z + \gamma^2 w'w$, valid for all $t \in [t_k, t_{k+1})$ (2.19)
due to time-invariance nature of the problem. From the second inequality of (2.18), integrating inside $[t_k, t_{k+1})$

$$v(\xi, t_{k+1}) - v(\xi, t_k) < v(\xi^-, t_{k+1}^-) - v(\xi, t_k) < -\int_{t_k}^{t_{k+1}} z(t)'z(t)dt + \gamma^2 \int_{t_k}^{t_{k+1}} w(t)'w(t)dt \tag{2.20}$$

Hybrid / Impulsive Systems

- Two conclusions can be drawn:

  First - The sampled-data system (2.16) is asymptotically stable, since there exists a small enough $\epsilon > 0$ such that $\int_{t_k}^{t_{k+1}} z'zdt \geq \epsilon$, given $t_{k+1} - t_k = h$ and $v(\xi, t_{k+1}) \leq (1-\epsilon)v(\xi, t_k)$. It implies that $v(\xi, t_k) \to 0$ as $k \to \infty$. Consequently, $v(\xi, t) \to 0$ as $t \to \infty$.
  Second – Summing up both sides of the previous inequality, it leads to

$$0 < -\sum_{k=0}^{\infty}\left[\int_{t_k}^{t_{k+1}} z(t)'z(t)dt - \gamma^2 \int_{t_k}^{t_{k+1}} w(t)'w(t)dt\right] = -\|z\|_2^2 + \gamma^2\|w\|_2^2$$

since (2.19) is evaluated in the limit solution, $\|H\|_\infty = \gamma^*$ is recovered.

- **Sufficiency…** Conversely, consider the hybrid system given by (2.16) is asymptotically stable for a given $h > 0$ and $\xi_0 = 0$. From Lyapunov's second method of stability, there exists a cost-to-go function such that $v(\xi) > 0, \forall \xi \neq 0$ and $v(0) = 0$, with $\dot{v}(\xi) < 0$. From the fact, the solution to the differential equation determined by (2.18) exists and is uniquely determined by $v(\xi) = \xi' P(t)\xi$, hence also admits a minimal solution the inequality (2.20).

- From boundary conditions, and the worst-case noise signal $w(t) = \gamma^{-2}J'P(t)\xi(t)$, the inequality (2.18) is readily recovered, whose minimal solution leads to the exact value of the $\mathcal{H}_\infty$ norm of exactly evaluated as $\gamma^* = \|H\|_\infty$.

Sampled-Data Systems: Discretization vs. Hybrid Approach

- A sampled-data system combines a continuous-time flow dynamics with discrete-time jump conditions. In the context of control and filtering systems, sampled-data systems are connected by samplers and hold systems, usually ZOH.

Discretization

- Discretize the continuous-time system through a **step invariant approach**.
- The design is processed in the discrete-time domain.
- Well established procedure in the literature (Chen & Francis, Chps. 2–7), that leads to **optimal discrete-time design** for the discretized systems.
- **Does not consider the inter-sampling behavior.**

Hybrid Approach

- Modeling using the hybrid approach (**continuous-time flow & discrete-time jump**).
- The $\mathcal{H}_2/\mathcal{H}_\infty$ norm-based performance index formulation takes into account the **inter-sampling behavior of the system**.
- **Represents exactly the physical behavior** with low mathematical complexity

Sampled-data systems: discretizing sampled-data system

EXAMPLE — Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)$$
$$z(t) = C_z x(t)$$

with matrices $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$; $B = E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C_z = [1 \quad 0]$.

Discretizing the sampled-data system at $h = \ln 2$, *leads to*

$$A_d = e^{Ah} = \alpha_0(h)I + \alpha_1(h)A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$B_d = E_d = \int_0^h e^{A\tau} d\tau\, B = A^{-1}(A_d - I)B = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \end{bmatrix}$$

with $\alpha_0(h) = 2e^{-h} - e^{-2h} = \frac{1}{2}$ *and* $\alpha_1(h) = e^{-h} - e^{-2h} = \frac{1}{4}$, *calculated by Cayley-Hamilton, given that* $\lambda = \{-1, -2\}$.

Since A_d *has eigenvalues* $\mu = \left\{\frac{1}{2}, \frac{1}{4}\right\}$, A_d *is Schur stable.*

Indeed, for any $h > 0$, *the condition* $\lambda_1 = \lambda_2 + j\frac{2\pi k}{h}$, *is not attained. Hence, there is no pathological sampling rate.*

The discrete-time Lyapunov equation to the norm calculation is

$$A_d' P A_d - P = -C_z' C_z \quad \rightarrow \quad P = \begin{bmatrix} \frac{64}{35} & \frac{32}{105} \\ \frac{32}{105} & \frac{4}{35} \end{bmatrix} \succ 0$$

$$\|H_d\|_2^2 = tr(E_d' P E_d) = tr\left(\begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \end{bmatrix}' \begin{bmatrix} \frac{64}{35} & \frac{32}{105} \\ \frac{32}{105} & \frac{4}{35} \end{bmatrix} \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \end{bmatrix} \right) = 0{,}0548$$

$$\|H_d\|_2 = 0{,}234$$

Sampled-data systems: discretizing sampled-data system

- Exercise highlights:
  - For **real distinct eigenvalues** of the continuous-time flow characteristic matrix, there is **no pathological sampling rates**. Indeed, $\lambda_i = \lambda_k + j\frac{2\pi n}{h}$ for any $n \in \mathbb{N}$ and any $h > 0$ never will occur.
  - **The discretized system is readily calculated** in a closed form through the exponential map

$$A_d = e^{Ah} \text{ and } B_d = \int_0^h e^{A\tau} d\tau\, B \text{ and } E_d = \int_0^h e^{A\tau} d\tau\, E.$$

  - The norm calculation is immediate through **the solution of a Lyapunov or Riccati algebraic equation**.
  - **Why do we need LMI and DLMI calculation?**

Programming with LMI toolboxes

Program 4 — performance index calculation of sampled-data systems

PROG. 4

Sampled-data $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm calculation

Determine the continuous-time $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm for the system

$$\dot{x} = \begin{bmatrix} 4 & -16 & 24 & -20 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.1 \end{bmatrix} w, \qquad x_0 = 0, \qquad \mathrm{z} = [0 \quad 0 \quad 0 \quad 0.01]x$$

Verify what occurs for $h = [\frac{pi}{100}, pi\,]$.

Theoretical Exercises

E6) Consider the continuous-time harmonic oscillator

$$\dot{x}(t) = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

with natural frequency $\omega > 0$.

- Show that the continuous-time system is fully controllable for any $\omega > 0$
- Compute the exact ZOH-discretized system matrices A_d and B_d as a function of the sampling period $h > 0$.
- Prove that choosing the sampling period as $h = \pi/\omega$ results in a completely uncontrollable discrete-time system.
- Provide a physical and geometric interpretation of this loss of controllability in the state-space phase plane.

E7) Consider the augmented system $\xi(t) = [x(t)' \quad u(t)']'$, the sampled-data system can be summarized as

$$\begin{cases} \dot{\xi}(t) = F\xi(t), & t \in [t_k, t_{k+1}) \\ \xi(t_k^+) = H\xi(t_k^-) \end{cases}$$

According to hybrid theory, a quadratic Lyapunov candidate function $v(\xi, t) = \xi(t)'P(t)\xi(t)$, certifies asymptotic stability if there exists a time-varying $P(t) \succ 0$, solving the differential Lyapunov equation

$$\dot{P}(t) + F'P(t) + P(t)F = -G'G$$

on $t \in [0, h)$, subject to the boundary condition $P(h) - H'P(0)H \succ 0$. Prove that the Hybrid boundary condition is mathematically equivalent to the step-invariant discrete-time Lyapunov inequality

$$(A_d + B_dK)'S(A_d + B_dK) - \mathrm{S} \prec -(\mathrm{C_{dz}} + \mathrm{D_{dz}}\mathrm{K})'(C_{dz} + D_{dz}K)$$

with $S \succ 0$ the top-left block of $P(0)$, and $G_d = [C_{dz} \quad D_{dz}]$ is the integrated output Gramian over the sampling interval $[0, h]$.

Computational Exercises

Discrete-time state-feedback norm-based design

C2) Consider the continuous-time unstable system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -6 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t), \qquad z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

This system is controlled via a ZOH state-feedback control law with $h = 0{,}5\ s$.

- Compute the equivalent discretized matrices A_d and B_d
- Calculate the integrated output Gramian $Q = \int_0^h e^{F't}\, G'Ge^{Ft} dt$ and obtain equivalent discretized matrices C_{dz} and D_{dz}.
- Find the optimal gain K that stabilizes and minimizes the $\mathcal{H}_2$ norm of the discretized equivalent system.
- Verify the closed-loop stability and simulate the state trajectories $x_1(t)$ and $x_2(t)$ as well as the control input considering $x(0) = [1 \quad -1]'$.

NEXT STEPS

In the next blocks, we are going to study...

3

DLMIs

Lyapunov and Riccati differential equations (DLE and DRE). Numerical aspects of the solution of differential linear matrix inequalities (DLMI).

4

Hybrid SD design

Sampled-data system design based on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm optimization problems formulated via DLMI: control and filtering.