EE-268

Sampled-Data Systems Through DLMI

Blocks 3 and 4 · DLMI and Hybrid SD Design

Instituto Tecnológico de Aeronáutica ▫ August, 2026 ▫ Prof. Gabriela Werner Gabriel

This content is based on…

GEROMEL (2023) — Differential Linear Matrix Inequalities: In Sampled-Data Systems, Filtering and Control.

GONÇALVES, GABRIEL & GEROMEL (2019) — Differential Linear Matrix Inequalities Optimization.

BALDISSERA, GALVÃO & GABRIEL (2023) — A polynomial eigenvalue test for checking DLMI constraints.

PEREIRA, GABRIEL, de OLIVEIRA & GEROMEL (2023) — DLMIs in Optimal and Robust Sampled-Data Filtering (IFAC).

BRIAT (2013); ARIOLA et al. (2020) — Convex conditions and numerical solution of DLMIs.

ROAD MAP

What we are going to study...

3

DLMI in SD Systems

Differential Lyapunov/Riccati equations (DLE, DRE). DLMIs as two-point boundary value problems. Numerical methods and certification.

Geromel Ch. 2; Gonçalves 2019; Baldissera 2023

4

Hybrid SD Design

State-feedback $\mathcal{H}_2/\mathcal{H}_\infty$ control, robustness under uncertainty, and optimal sampled-data filtering.

Gonçalves 2019; Pereira 2023; Geromel Ch. 3–4

BLOCK 3

03

DLMI in Sampled-Data Systems

How DLMIs arise, their properties, and how to solve them numerically without conservatism.
How to certify a solution over the whole interval?

Geromel, Ch. 2; Gonçalves et al. 2019; Baldissera et al. 2023; Briat 2013

Why do we need DLMIs?

- **Sampled-data performance is hybrid.** The $\mathcal{H}_2/\mathcal{H}_\infty$ index of a sampled-data system has no trivial transfer-function form in s or ζ — it lives between two consecutive sampling instants.

- **The certificate is time-varying.** Stability and performance require a cost-to-go $v(\xi, t) = \xi' P(t)\xi$ with a matrix function $P(t)$, $\forall t \in [0, h]$ since the analysis is time-interval confined with reset equations.

- **Two-point boundary value problem.** The differential inequality on $[0, h)$ and the jump boundary condition $P(h) \succ H' P(0) H$ must be solved jointly.

THE OBJECT WE MUST HANDLE

$$\mathcal{L}\big(\dot{X}(t), X(t), \mu\big) \prec 0, \qquad \forall t \in [0, h]$$

$$\big(X(0), X(h)\big) \in \mathcal{B}$$

DLMI is a matrix inequality that is linear in the unknown matrix function $X(t)$ and its derivative $\dot{X}(t)$. It is imposed for every t in the time interval, together with a coupling boundary conditions.

This problem is also known as a **Two-Point Boundary Value Problem (TPBVP).**

LMIs and DLMIs come from Lyapunov theory

Both certify behaviour with a Lyapunov function instead of solving the dynamics.

1

Lyapunov theory

A single quadratic $V(x) = x'Px$ certifies stability without integrating the trajectory:

$$A'P + PA \prec 0 \quad \text{(continuous time)}$$
$$A_d{}'P\,A_d - P \prec 0 \quad \text{(discrete time)}$$

Both are what give rise, respectively, to the **DLMI flow condition** and to its **boundary condition**.

Geromel (2023), §1.2, §2.2

2

LMI for LTI context

For **LTI systems** P is constant and the certificate becomes a **convex feasibility problem**. Analysis, control feedback design in its several forms, and filter design all reduce to LMI conditions — a feasible formulation that need not be optimal.

Duan & Yu, Chs. 3–4

3

DLMI for SD context

The system flows between samples with jumps at its boundaries, so constant P could tightly satisfy both, but introducing conservatism. That is, the certificate varies across the sampling interval, tied to the next by a boundary condition.

Geromel (2023), §1.3, Ch. 3

With $h > 0$ constant **only the first interval** needs to be solved in SD: it certifies stability and optimality on every subsequent one.

Formally defining a DLMI

Definition 3.1 [Geromel, 2023]. A DLMI is a matrix inequality, affine in the matrix-valued function $X(t) \in S^{\mathrm{n}}$, its derivative $\dot{\mathrm{X}}(t)$ and an auxiliary vector μ, imposed for all $t\ in\ [0, h]$:

$$\mathcal{L}\big(\dot{X}(t), X(t), \mu\big) \prec 0, \qquad \forall t \in [0, h] \tag{3.1}$$

subject to a boundary condition coupling the endpoints:

$$\big(X(0), X(h)\big) \in \mathcal{B} \tag{3.2}$$

Two-point boundary value problem (TPBVP)

Unlike a classical TBVP, here the flow is governed by a **differential inequality** (a feasible set of trajectories) rather than an equation. Any feasible $X(t)$ that is continuous and satisfies the DLMI almost everywhere on $[0, h)$ is admissible; among them **a minimal element carries the exact performance value**.

From algebraic Lyapunov equation to differential...

The algebraic Lyapunov Equation (ALE) always has a unique solution in the closed form:

$$A'P + PA = -C_z'C_z \quad \rightarrow \quad P = \int_0^{\infty} e^{A'\tau} C_z'C_z \, e^{A\tau} \, d\tau \tag{3.3}$$

$$A_d'PA_d - P = -C_{dz}{}'C_{dz} \quad \rightarrow \quad P = \sum_{k=0}^{\infty} (A_d')^k \, C_{dz}'C_{dz} \, (A_d)^k \tag{3.4}$$

There exists a definite positive $P \succ 0$ if and only if matrix $\boldsymbol{A}$ ($\boldsymbol{A_d}$) is Hurwitz (Schur).

On the other hand, the differential Lyapunov Equation (DLE) is written as

$$P(t) = e^{A'(t_f - t)} P(t_f) e^{A(t_f - t)} + \int_t^{t_f} e^{A'(\tau - t)} C_z'C_z \, e^{A(\tau - t)} d\tau \Leftrightarrow \dot{P} + A'P + PA = -C_z'C_z \tag{3.5}$$

REMARK

A solution always exists with $P(t) \succ 0$ whenever $P(t_f) \succ 0$; Hurwitz stability of $\boldsymbol{A}$ is needed only in algebraic case, which is recovered for $t \rightarrow \infty$.

From algebraic Riccati equation to differential…

- The Algebraic Riccati Equation (ARE) is a nonlinear equation. Hence, the solution may not exist. If it exists, it is not expressed in a closed form, but can be determined numerically.

$$A'P + PA + \gamma^{-2}PEE'P = -C_z{}'C_z \quad \text{(Continuous-time ARE)} \tag{3.6}$$

$$A_d'PA_d - P + A_d'PE(\gamma^2 I - E'PE)^{-1}E'PA_d = -C_{dz}'C_{dz} \text{ (Discrete-time ARE)} \tag{3.7}$$

Unlike the ALE, existence of a solution is not guaranteed: a stabilizing solution requires $\gamma^2 I - E'PE \succ 0$ in the discrete-time case.

- Since ARE is quadratic, it may not admit a solution or a unique solution. In the $\mathcal{H}_\infty$ context a stabilizing $P \succ 0$ appears only once γ^2 exceeds the squared norm of the transfer function.
- The solution is not unique. Among the several symmetric solutions, exactly one is stabilizing — making the closed-loop matrix Hurwitz (Schur) — and it is that one, unique, which carries the performance value.
- The differential Riccati equation (DRE), may admit a definite positive solution $P(t) \succ 0$, if it exists, of the form

$$\dot{P} + A'P + PA + \gamma^{-2}PEE'P = -C_z{}'C_z \Leftrightarrow P(t) = e^{A'(t_f - t)}P(t_f)e^{A(t_f - t)} + \int_t^{t_f} e^{A'(\tau - t)}[C_z{}'C_z + \gamma^{-2}P(\tau)EE'P(\tau)]e^{A(\tau - t)}d\tau \tag{3.8}$$

BLOCK 3 – DIFFERENTIAL LMI

Programming with LMI toolboxes

Prog 5 — Solution of the Riccati equation (1/2)

PROG. 5

Infeasibility of the differential Riccati equation

Consider $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Cz = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $P(t_f) = 0$, with $\mathrm{h} = t_f = 5\ s$.

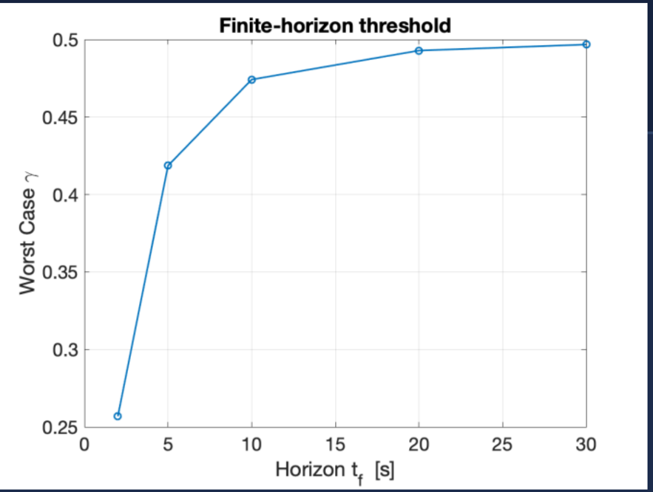
1) For $G(s) = Cz(sI - A)^{-1}E = \frac{1}{s^2 + 3s + 2}$, the algebraic $\mathcal{H}_\infty$ norm is $\|G\|_\infty = \gamma^* = 0.5$.

2) Integrating $\dot{\mathrm{P}} + A'P + PA + \gamma^{-2}PEE'P = -C_z'C_z$ backward from $P(t_f)$, there is a positive definite $P(t)$ such that $\gamma = 0.60$. The solution stays positive definite for all $t \in [0, t_f]$. However, for $\gamma = 0.40$ and $\gamma = 0.30$ the solution is not feasible.

γ	t_f	$P(0)$
0.6	5 s	$P(0) = \begin{bmatrix} 1.086 & 0.320 \\ 0.320 & 0.112 \end{bmatrix}$
0.45	5 s	$P(0) = \begin{bmatrix} 1.758 & 0.607 \\ 0.607 & 0.235 \end{bmatrix}$
0.4	4.34 s	Not Feasible
0.3	2.46 s	Not Feasible

Programming with LMI toolboxes

Prog 5 — Solution of the Riccati equation (2/2)



PROG. 5

Infeasibility of the differential Riccati equation

3) Repeating the backward integration to capture the maximum value of γ (γ^*) for $t_f = \{2, 5, 10, 20, 30\}\ s$, the critical γ approaches $\|G\|_\infty$.

Existence is a joint property of the system data, the horizon and γ. Same result for Bounded Real Lemma from Block 1.

4) Pose the same question as a DLMI and solve the $(2n_\varphi + 1)$ LMIs with YALMIP + MOSEK. Assume that $h = 5\ s$.

γ	P_0 (DLMI)
0.6	$P(0) = \begin{bmatrix} 1.09 & 0.32 \\ 0.32 & 0.11 \end{bmatrix}$
0.45	$P(0) = \begin{bmatrix} 1.76 & 0.61 \\ 0.61 & 0.24 \end{bmatrix}$
0.4	Not Feasible
0.3	Not Feasible

What is a solution of a DLMI?

A DLMI couples a differential inequality on $[0, h)$ with a boundary condition linking the two endpoints.

$$\mathcal{L}(\dot{P}(t), P(t)) \prec 0, \qquad t \in [0, h) \tag{3.9}$$

$$\mathcal{H}(P_0, P_h) \prec 0 \tag{3.10}$$

- $\mathcal{L}$ and $\mathcal{H}$ are linear mappings suitable to be solved by LMI solvers.
- $P(t) : [0, h] \to \mathbb{R}^{nxn}$ is a solution **if it is continuous and satisfies (3.9) almost everywhere on [0,h).**
- A DLMI generally does not admit a unique solution! **We search for a feasible one**.

WHY 'ALMOST EVERYWHERE' MATTERS

Continuity is required; differentiability is not — this admits piecewise-linear solutions

Geromel (2023), Def. 2.1, §2.1

Lyapunov differential inequality

From the Lyapunov differential equation, we relax the equality to an inequality.

$$\dot{P}(t) + A'P(t) + P(t)A + C_z{}'C_z \prec 0, \qquad t \in [0,h) \tag{3.11}$$

Theorem 3.1 [Geromel, Thm 2.1]. Assume the final boundary condition $P(h) = P_h \succ 0$. Then every feasible solution of (3.11) satisfies

$$P(t) \succ 0, \qquad \forall t \in [0,h] P(t) \succ e^{A'(h-t)} P_h \, e^{A(h-t)} + \int_0^{h-t} e^{A's} C_z{}'C_z \, e^{As} \, ds \tag{3.12}$$

READING THE RESULT

The feasible set of the inequality is bounded below by the solution of the Differential Lyapunov Equation (DLE) equation. Anycertificate you compute is never optimistic relative to the exact energy.

Geromel (2023), §2.2, Thm. 2.1

The mirrored form

Reversing the direction of time integration gives the companion inequality, whose solution is computed whenever the boundary data is fixed at $t = 0$.

$$-\dot{P}(t) + A\,P(t) + P(t)A' + BB' \prec 0, \qquad t \in [0,h) \tag{3.13}$$

Theorem 3.2 [Geromel, Thm 2.2]. Assume the initial boundary condition $P(0) = P_0 \succ 0$. Then every feasible solution of (3.13) satisfies

$$P(t) \succ 0, \qquad \forall t \in [0,h] P(t) \succ e^{At}\,P^0 e^{A't} + \int_0^t e^{As}\,BB'\,e^{A's}\,ds \tag{3.14}$$

WHICH FORM TO USE

The final-value form (3.9) appears in filtering problem; the initial-value form (3.13) appears in control problems. Both are DLMIs in the form of (3.9)–(3.10).

Geromel (2023), §2.2, Thm. 2.2

Riccati differential inequality

From differential Riccati equation we relax the equality to an inequality. The quadratic term that encodes feedback action makes this inequality **nonlinear in $P(t)$.**

$$\dot{\mathrm{P}}(t) + A'P(t) + P(t)A + \gamma^{-2}P(t)EE'P(t) + C_z'C_z \prec 0, \qquad t \in [0,h) \tag{3.15}$$

- Due to the nonlinearity, a solution may fail to exist on the interval of interest.
- Scalar counterexample: $\dot{\mathrm{p}}(t) + r\,p(t)^2 = 0$ integrates to $1/p(t) = 1/p(0) + r\,t$.

  For $p(0) = p_0 > 0$ and $r < 0$ the solution escapes to infinity at $t_e = 1/(|r|\,p_0)$, so no solution exists on $[0,h]$ when $h > t_e$.
- For $r \geq 0$ the solution exists on every interval $[0,h]$.

CONTRAST WITH LYAPUNOV

Existence of a solution over the whole horizon is automatic in the Lyapunov case, but it must be established in the Riccati case since the horizon length h is part of the problem data.

Geromel (2023), §2.3

Linearizing Riccati by the Schur complement

The Schur complement converts the nonlinear inequality into an equivalent DLMI on the same interval.

$$\begin{bmatrix} \dot{\mathrm{P}}(t) + A'P(t) + P(t)A & P(t)E & C_z{}' \\ \star & -I\gamma^2 & 0 \\ \star & \star & -I \end{bmatrix} \prec 0, t \in [0, h) \tag{3.16}$$

WHY IS THIS STEP IMPORTANT?

(3.16) is equivalent to (3.15) but is linear in the decision variables. This is the algebraic manipulation that puts the Riccati problem solvable by LMI machinery introduced next.

Geromel (2023), §2.3

Minimality of the feasible solutions

Feasible solutions of the inequality are bounded below by the solution of the corresponding equation.

Theorem 3.3 [Geromel, Thm 2.3]. Assume the final boundary condition $P(h) = P_h \succ 0$. Then every feasible solution of (3.15) satisfies

$$P(t) \succ P_E(t), \qquad \forall t \in [0, h) \tag{3.17}$$

where $P_E(t)$ is the solution of the DRE (3.10) given the same boundary condition. Moreover, $P(t) \succ 0$.

- The mirrored problem can also be stated for the initial-condition form.
- For A Hurwitz, the minimal elements can be taken arbitrarily close to the stabilizing solution of the ARE.

CONSEQUENCE FOR DESIGN

Among all feasible trajectories, a minimal element carries the exact performance value. Minimizing a linear functional of P over the DLMI therefore recovers the optimum solution.

Geromel (2023), §2.3, Thms. 2.3–2.4, Rem. 2.1

Parametrizing $P(t)$: from DLMI to LMI

To solve the DLMI, restrict $P(t)$ to a finite basis of continuous-time functions.

$$P(t) = \Sigma_{i \in \mathcal{N}_\varphi} X_i \, \varphi_i(t), \qquad \mathcal{N}_\varphi = \{0, 1, \dots, n_\varphi\}, \qquad n_\varphi \geq 1 \tag{3.18}$$

- $\{\varphi_i(t)\}$ is the chosen basis; the symmetric matrices $\{X_i\}$ are the variables of the problem.
- Geromel lists 4 natural choices: **piecewise linear**, **Taylor series**, **Fourier series**, and **splines**.

THE QUESTION TO KEEP IN MIND

For which bases does enforcing the constraint at finitely many points guarantee it holds for every t in the interval? The answer differs between the piecewise-linear and polynomial cases.

Geromel (2023), §2.4, eq. (2.41)

The piecewise-linear basis

Split $[0, h)$ into n_φ subintervals of equal length $\eta = h/n_\varphi$ and build the basis:

$$\varphi(t) = 1 - \frac{|t|}{\eta}\ for\ |t| < \eta\,;\quad \varphi(t) = 0\ for\ |t| > \eta\,;\quad \varphi_i(t) = \varphi(t - i\eta) \tag{3.19}$$

Combining all contributions on each segment $t \in [i\eta, (i+1)\eta)$, the parameterization collapses to a straight line between two consecutive matrix variables:

$$P(t) = X_i + \frac{(X_{i+1} - X_i)}{\eta}(t - i\eta) \tag{3.20}$$

REGULARITY

Continuity is preserved, but differentiability generally fails at the segment borders. Moreover, $P(i\eta) = X_i$ and $P\big((i+1)\eta\big) = X_{i+1}$

Geromel (2023), §2.4, eq. (2.42)–(2.43)

The piecewise-linear basis

Theorem 3.4 [Geromel, Thm 2.5]. Let $h > 0$ and $n_\varphi \geq 1$ be given. The piecewise-linear function (3.20) solves the DLMI (3.9) subject to the boundary condition (3.10) IF AND ONLY IF, for all $\boldsymbol{i = 0, 1, \ldots, n_\varphi - 1}$:

$$\begin{aligned} \mathcal{H}\left(X_0, X_{n_\varphi}\right) &\prec 0 \\ \mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_{i+1}\right) &\prec 0 \\ \mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_i\right) &\prec 0 \end{aligned} \tag{3.21}$$

WHAT THIS BUYS

A DLMI is equivalent — necessary AND sufficient — to $2n_\varphi + 1$ LMIs. The two-point boundary value problem is solved in one shot, with no iterative procedure and without adding any kind of conservatism.

Geromel (2023), §2.4, Thm. 2.5

Why the equivalence is exact

The proof is constructed by defining

$$\alpha_i(t) = \frac{t - t_i}{t_{i+1} - t_i} \in [0,1], \qquad P(t) = \big(1 - \alpha_i(t)\big)X_i + \alpha_i(t)X_{i+1} \tag{3.22}$$

- Necessity: (3.21) is $\mathcal{L}(\dot{P}(t), P(t))$ evaluated at the two endpoints t_i and t_{i+1}, so any solution must satisfy it.
- Sufficiency: multiply the first inequality by $(1 - \alpha_i(t))$, the second by $\alpha_i(t)$, and add. Because $\mathcal{L}$ is a LINEAR map, the result recovers $\mathcal{L}(\dot{P}(t), P(t)) \prec 0$ for every $t \in [t_i, t_{i+1}]$.
- Since $P_0 = P(t_0) = X_0$ and $P_h = P(t_{n_\varphi}) = X_{n_\varphi}$, the third condition is identical to the boundary condition (3.10).

IMPORTANT POINT

The constraint is imposed at the nodes only, yet it holds on the whole interval. For the piecewise-linear basis there is NO gap between grid points — the convex combination and the linearity of $\mathcal{L}$ close it.

Geromel (2023), §2.4, proof of Thm. 2.5

Choosing the number of subintervals

With $n_\varphi = 1$ the parameterization is a single straight line over the whole horizon:

$$P(t) = X_0 + \left(\frac{X_h - X_0}{h}\right) t\,, \qquad t \in [0, h) \qquad (3.23)$$

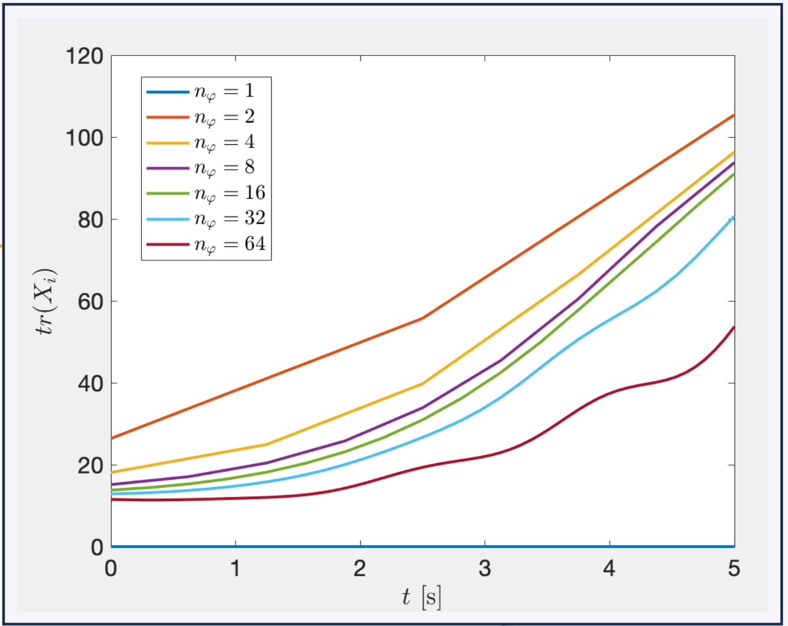
- Any feasible continuous almost-everywhere differentiable solution is reached provided n_φ is large enough.
- For $n_\varphi = b^n$, the convergence to the minimal solution is ensured.

EXAMPLE — Consider

$$F = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

$$G' = \sqrt{10}\, J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



The problem for $h = 5$ [s] is unfeasible for $\gamma = 2.54$ and $n_\varphi = 1$, but feasible for $n_\varphi > 1$.

WHAT DOES n_ϕ ACTUALLY TRADE?

n_φ trades the richness of the solution class against computational burden — not the validity of the certificate. A feasible answer is exact at any n_φ.

Geromel (2023), §2.4; Gonçalves et al. (2019)

Polynomial parametrization and its gap

An alternative basis restricts $P(t)$ to a matrix polynomial of order n_φ, allowing extra optimization variables μ (an upper bound on a cost, for instance).

$$X(t) = \Sigma_{i=0}^{n_\varphi} X_i t^i , \qquad \mathcal{L}(\dot{X}(t), X(t), \mu) \prec 0, \qquad \forall t \in [0, h] \tag{3.24}$$

- Defined in Gonçalves et al. (2019); $\mathcal{L}$ is affine in $(\dot{X}(t), X(t), \mu)$.
- The resulting LMIs are imposed only on a FINITE SET of discrete-time points.
- Consequently, there is no guarantee that the solution satisfies the original continuous-time inequality between those points.

THE ESSENTIAL CONTRAST

This is where the polynomial case differs from Theorem 2.5. For the piecewise-linear basis, node feasibility implies interval feasibility. For the polynomial basis it does not — the gap must be closed by a separate test.

Baldissera, Galvão & Gabriel (2023), §1–2; Briat (2013); Gonçalves et al. (2019)

The polynomial eigenvalue test

Baldissera, Galvão & Gabriel (2023) address the gap the polynomial parameterization leaves open.

- Problem: given a matrix polynomial $P(t)$ obtained from the finite set of LMIs, decide whether it satisfies the original DLMI constraint for EVERY $t \in [0, h]$ in the inter-sampling interval.
- The proposed test is based on the concept of polynomial eigenvalues Higham & Tisseur (2003) and is applied after the optimization procedure.
- For the eigenvalue test, consider

$$P(t) = M_0 + M_1 t + \cdots + M_{n_\varphi} t^{n_\varphi}, \qquad M_i \in \mathbb{R}^{m\times m}, i = 1,2, \dots, n_\varphi$$

Theorem 3.5 [Baldissera, Thm 4]. Consider *that there exists* $P(t') \prec 0$, for some $t' \in [0, h]$. Then, $P(t) \prec 0$ for all $t \in [0, h]$ if and only if $P(t)$ has no real polynomial eigenvalues in the interval $[0, h]$.

- The polynomial eigenvalues of $P(t)$ correspond to the eigenvalues of the matrix pencil $\boldsymbol{tW} + \boldsymbol{Z}$, where

$$W = diag(M_{n_\varphi}, I_m, \dots, I_m) \text{ and } Z = \begin{bmatrix} M_{n_\varphi-1} & M_{n_\varphi-2} & \cdots & M_0 \\ -I_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -I_m & 0 \end{bmatrix}.$$

Baldissera, Galvão & Gabriel (2023)

Other basis: Fourier and splines

The parameterization (3.18) is open to any basis of continuous scalar functions.

- Fourier series — a harmonic basis, natural when the dynamics or the sampling schedule is periodic; the derivative becomes algebraic in the coefficients. For that solution consider $T > h$ and

$$X(t) = \sum_{i \in N_\varphi} X_i \cos\left(\left(\frac{\pi i}{T}\right)\left(t + \frac{T-h}{2}\right)\right)$$

- For any basis: **does feasibility at the enforced instants imply feasibility on the whole interval**? If the answer is no, an independent verification step (as in the polynomial case) is required.

Gonçalves, Gabriel, Geromel (2019)

Solving a DLMI in practice

1. Write the problem as $\mathcal{L}(\dot{P}(t), P(t)) \prec 0$ on $[0, h)$ with the boundary condition $\mathcal{H}(P_0, P_h) \prec 0$.
2. Choose the basis – Piecewise linear is the default, with exact solution to selected n_φ
3. Fix n_φ, set $\eta = h/n_\varphi$, and declare the symmetric variables $X_0, \dots, X_{n_\varphi}$ in the interpreter.
4. Build the $2n_\varphi + 1$ LMIs with any positivity constraints and solve once with the selected solver.

NO ITERATION IS NEEDED

The two-point boundary value problem (TPBVP) is handled in a single optimization shot. The boundary condition is an additional LMI.

5. Recover $P(t)$ and inspect it — plot the minimum eigenvalue over $[0, h]$.
6. Increase n_φ and re-solve. Feasibility and the objective should stabilize.
7. If a polynomial or Fourier basis was used, apply the polynomial eigenvalue test before claiming a certificate.

Geromel (2023), §2.4; Baldissera, Galvão & Gabriel (2023)

Computational Exercises

Hybrid norm calculation

C3) Consider the LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -6 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t), \qquad z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

This system is controlled via a ZOH state-feedback control law with sampling period $h = 0{,}5\ s$.

- Implement a Matlab function that obtain the hybrid system, and calculate the $\mathcal{H}_2$ norm for this system, considering:
  - Taylor parameterization for a given number of polynomial terms n_ϕ;
  - PWL parameterization for a given quantity of subinterval n_ϕ.
- In a same plot, plot the $\mathcal{H}_2$ norm against n_ϕ.

BLOCK 4

04

Hybrid sampled-data design

Sampled-data state-feedback design and filtering design in the hybrid aproach.

Geromel, Chs. 3–4; Gonçalves et al. 2019; Pereira et al. 2023

Sampled-data system definition...

- Consider a continuous-time system in the state-space realization

$$\mathcal{S} \; := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t) & x(0) \\ z(t) = Cz\,x(t) + Dz\,u(t) \end{cases} \tag{4.1}$$

- Consider a control law of the form $u(t) = Kx(t_k)$, where $\{t_k\}_{k\in\mathbb{N}}$ is the sequence of sampling instants, defined such that

$$\boxed{t_{k+1} - t_k = h > 0\,, \qquad t_0 = 0 \quad \text{and} \quad \lim_{k\to\infty} t_k = \infty} \tag{4.2}$$

- The $\mathcal{H}_2$ norm is defined such that $x(0) = 0$ and $\|H\|_2^2 = \int_0^\infty z(t)'z(t)dt = \sum_{i=0}^{n_w} \int_0^\infty z_i(t)'z_i(t)dt$, with z_i the response of the system for $w(t) = e_i\delta(t)$, e_i each of the columns of the identity matrix of dimension n_w.

- For this system we can write the augmented system for $\xi = [x(t)' \quad u(t)']'$

$$\mathcal{H} \; := \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} E \\ 0 \end{bmatrix} w(t) \quad \xi(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix} \\ z(t) = \begin{bmatrix} C & D \end{bmatrix} \xi(t) \\ \xi(t_k) = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \xi(t_k^-) \end{cases}$$

That is,

$$\boxed{\mathcal{H} \; := \begin{cases} \dot{\xi}(t) = F\xi(t) + Jw(t) \quad \xi(0) \\ z(t) = G\xi(t) \\ \xi(t_k) = H\xi(t_k^-) \end{cases}} \tag{4.3}$$

$\mathcal{H}_2$ sampled-data state feedback design ...

- From Theorem 2.3, the following Theorem can be stated

Theorem 4.1 [Gonçalves et al. (2019)]. Let be given $h > 0$ and $x(0) = 0$. The sampled-data system (4.1) with $u(t) = Kx(t_k)$, is asymptotically stable with the square of the $\mathcal{H}_2$ norm given by $\inf_{W \succ 0, V, Y, N} tr(N)$ if and only if there exists matrices $W(t) = W(t)', V = V' \succ 0, Y$, and $N = N \succ 0'$ of compatible dimensions, such that $W(h) \succ 0$ such that the DLMI

$$\begin{bmatrix} -\dot{W} + WF' + FW & WG' \\ \star & -I \end{bmatrix} \prec 0 \tag{4.4}$$

the boundary conditions

$$\begin{bmatrix} W(h) & W(h)\begin{bmatrix} I \\ 0 \end{bmatrix} \\ \star & V \end{bmatrix} \succ 0 \tag{4.5}$$

$$\begin{bmatrix} V & \begin{bmatrix} V & Y' \end{bmatrix} \\ \star & W(0) \end{bmatrix} \succ 0 \tag{4.6}$$

and

$$\begin{bmatrix} N & J' \\ \star & W(h) \end{bmatrix} \succ 0 \tag{4.7}$$

hold. In this case, the control law is $K = V^{-1}Y$.

- The proof of Theorem 4.1 can be found in *Geromel (2023).*

Geromel (2023); Gonçalves et al. (2019)

$\mathcal{H}_2$ sampled-data state feedback design ...

- Considering the second Lyapunov criterium for stability, following the results rom theorem 2.3, it follows that

$$\dot{P} + F'P + PF + G'G \prec 0 \tag{4.8}$$

and

$$P(h) - H'P(0)H \succ 0 \tag{4.9}$$

- hold for stable systems with $\mathcal{H}_2$ norm calculated considering the optimization problem

$$\inf_{P \succ 0} J'H'P(0)HJ \tag{4.10}$$

- Note that (4.9) and (4.10) are nonlinear. Considering (4.9), hence

$$P(h) - \begin{bmatrix} I \\ 0 \end{bmatrix} [I \quad K']'P(0) \begin{bmatrix} I \\ K \end{bmatrix} [I \quad 0] \succ 0 \tag{4.11}$$

which equals to

$$P(h) - \begin{bmatrix} I \\ 0 \end{bmatrix} V^{-1} [I \quad 0] \succ 0 \quad ; \quad V^{-1} - [I \quad K']'P(h) \begin{bmatrix} I \\ K \end{bmatrix} \succ 0 \tag{4.12}$$

$\mathcal{H}_2$ sampled-data state feedback design ...

- Linearizing (4.12) through Schur complement and congruence transform with $diag(P(h)^{-1}, I)$ for the first inequality and $diag(V, I)$ for the second one, yelds

$$\begin{bmatrix} W(h) & W(h)\begin{bmatrix} I \\ 0 \end{bmatrix} \\ \star & V \end{bmatrix} \succ 0 \tag{4.13}$$

$$\begin{bmatrix} V & \begin{bmatrix} V & Y' \end{bmatrix} \\ \star & W(0) \end{bmatrix} \succ 0 \tag{4.14}$$

where have been considered the one-to-one change of variables $P^{-1} = W$ and $Y = KV$. Putting (4.8) and (4.10) in the new set of variables, it follows that

$$\begin{bmatrix} -\dot{W} + FW + WF' & WG' \\ \star & -I \end{bmatrix} \prec 0 \tag{4.15}$$

$$\inf_{W \succ 0, Y, V = V'} N \qquad , \qquad \begin{bmatrix} N & J' \\ \star & W(h) \end{bmatrix} \succ 0 \tag{4.16}$$

Geromel (2023); Gonçalves et al. (2019)

Computational Exercises

Discrete-time state-feedback norm-based design

C4) Consider the LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t), \qquad z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)$$

This system is controlled via a ZOH state-feedback control law with sampling period $\mathrm{h} = 0{,}5\ [s]$.

- Implement a Matlab function that obtain the gain K_d and the $\mathcal{H}_2$ norm considering the discretization of the previous system.
- Implement a Matlab function that obtain the gain K_h and the $\mathcal{H}_2$ considering the hybris system approach design through DLMIs.
- Simulate de hybrid system with K_d and K_h and compare the results. What do you observe.

$\mathcal{H}_\infty$ sampled-data state feedback design ...

- From Theorem 2.4, the following Theorem can be stated

Theorem 4.2 [Geromel (2023). Let be given $h > 0$ and $x(0) = 0$. The sampled-data system (4.1) with $u(t) = Kx(t_k)$, is asymptotically stable with $\mathcal{H}_\infty$ norm given by $\gamma^{*2} = \inf_{W \succ 0, V, Y, \mu > 0} \mu$ if and only if there exists matrices $W(t) = W(t)'$, $V = V' \succ 0$, and Y of compatible dimensions such that $W(h) \succ 0$, and the DLMI

$$\begin{bmatrix} -\dot{W} + WF' + FW & WG' & J \\ \star & -I & 0 \\ \star & \star & -\mu I \end{bmatrix} \prec 0 \tag{4.17}$$

the boundary conditions

$$\begin{bmatrix} W(h) & W(h)\begin{bmatrix} I \\ 0 \end{bmatrix} \\ \star & V \end{bmatrix} \succ 0 \tag{4.18}$$

$$\begin{bmatrix} V & \begin{bmatrix} V & Y' \end{bmatrix} \\ \star & W(0) \end{bmatrix} \succ 0 \tag{4.19}$$

hold. In this case, the control law is $K = V^{-1}Y$. **(4.20)**

- The proof of Theorem 4.2 can be found in *Geromel (2023).*

Geromel (2023)

$\mathcal{H}_\infty$ sampled-data state feedback design ...

- Inequalities (4.19) and (4.20) are obtained as in the $\mathcal{H}_2$ context.

- Inequality (4.17) is similar to (4.4) but with the quadratic term, for which has been considered a one-to-one change of variable $\gamma^2 = \mu > 0$.